ILLINOIS JOURNAL OF MATHEMATICS
Volume 24, Number 4, Winter 1980

INEQUALITIES FOR POTENTIALS OF PARTICLE SYSTEMS[1]

BY

J. R. BAXTER

1. Let $x_1, \ldots, x_n$ be points in $\mathbf{R}^3$. Suppose that a (positive or negative) charge e_i is placed at each x_i. The total energy of the system of charges is $V = \sum_{1 \le i < j \le n} e_i e_j \, |x_1 - x_j|^{-1}$. V may be negative. We are interested in finding lower bounds for V. If the charges approach a smooth distribution f, then V approaches

$$(1/2) \iint f(x) \, |x - y|^{-1} f(y) \ge 0.$$

As a tool in attacking the general case, it is useful to consider an intermediate situation in which the negative charges are replaced by a smooth distribution f, but the positive charges remain discrete, say positive charges $z_1, \ldots, z_m$ at points $y_1, \ldots, y_m$. We may write the energy V in this case as

$$P(f;\, z_1, \ldots, z_m;\, y_1, \ldots, y_m).$$

It is convenient to make a slight generalization and replace both the positive and negative charges by measures $\mu_1, \ldots, \mu_m$ and ν respectively, while still omitting the self energies of the μ_i. Then V can be written as $P(\nu;\, \mu_1, \ldots, \mu_m)$. In Section 2 we prove a decomposition theorem for P and deduce a simple inequality for our original discrete energy V. In Section 3 a version of the no binding theorem [7], [13] is obtained.

2. For any bounded signed measures μ and ν on $\mathbf{R}^3$, let

$$\langle \mu, \nu \rangle = \int |x - y|^{-1} \mu(dx) \nu(dy),$$

provided that the double integral has a well defined finite or infinite value. Define the potential of μ by Pot $\mu(y) = \int |x - y|^{-1} \mu(dx)$. Then of course

$$\langle \mu, \nu \rangle = \int (\text{Pot } \mu) \, d\nu = \int (\text{Pot } \nu) \, d\mu.$$

Received November 20, 1978.

[1] Research supported in part by a National Science Foundation grant.

If μ has a density f we will sometimes write Pot f instead of Pot μ. Suppose ν, $\mu_1, \ldots, \mu_m$ are bounded nonnegative measures with $\langle \mu_i, \nu \rangle$ finite for each i. Let

$$P(\nu; \mu_1, \ldots, \mu_m) = (\tfrac{1}{2})\langle \nu, \nu \rangle - \sum_{i=1}^{m} \langle \mu_i, \nu \rangle + \sum_{1 \le i < j \le m} \langle \mu_i, \mu_j \rangle. \tag{2.1}$$

If ν has a density f we may write $P(\nu; \mu_1, \ldots, \mu_m)$ as $P(f; \mu_1, \ldots, \mu_m)$.

THEOREM 1. *Let ν and μ_i, $i = 1, \ldots, m$, be bounded nonnegative measures on $\mathbf{R}^3$. Suppose that* Pot ν *is finite ν-almost everywhere. Then there exist nonnegative measures ν_1, $i = 1, \ldots, m + 1$, such that:*

$$\sum_{i=1}^{m+1} \nu_i = \nu, \tag{2.2}$$

$$\text{Pot } \nu_i \le \text{Pot } \mu_i \textit{ on } \mathbf{R}^3, \quad i = 1, \ldots, m, \tag{2.3}$$

$$\nu_i(\mathbf{R}^3) \le \mu_i(\mathbf{R}^3), \quad i = 1, \ldots, m, \tag{2.4}$$

$$\text{Pot } \nu_i = \text{Pot } \mu_i, \quad \nu_j\textit{-almost everywhere}, \tag{2.5}$$

$$\textit{for } j = i + 1, \ldots, m + 1, \quad i = 1, \ldots, m.$$

In proving Theorem 1 we need only the following known fact from potential theory: [10], [12], [9], [5]:

THEOREM 2. *Let μ and ν be bounded nonnegative measures on $\mathbf{R}^3$. Suppose that* Pot ν is finite *ν-almost everywhere. Then there exists a nonnegative measure λ, such that:*

$$\lambda \le \nu, \quad \lambda(\mathbf{R}^3) \le \mu(\mathbf{R}^3), \tag{2.6}$$

$$\text{Pot } \lambda \le \text{Pot } \mu \textit{ on } \mathbf{R}^3, \tag{2.7}$$

$$\text{Pot } \lambda = \text{Pot } \mu, \quad (\nu - \lambda)\textit{-almost everywhere}. \tag{2.8}$$

One might say loosely that λ screens μ, as far as $\nu - \lambda$ is concerned.

Proof of Theorem 2. As in [9, Section 2], let σ be the measure, $0 \le \sigma \le \nu$, such that (Pot σ) dm is the réduite of (Pot $(\nu - \mu)$) dm, where m is Lebesgue measure on $\mathbf{R}^3$. Since the value of a potential at a point is the limit of its mean values over balls shrinking to the point, the equation (Pot σ) $dm \ge$ (Pot $(\nu - \mu)$) dm implies Pot $\mu \ge$ Pot λ on $\mathbf{R}^3$, if we define $\lambda = \nu - \sigma$. Thus (2.7) holds. As a consequence of (2.7), or by proposition 4 of [9], $\lambda(\mathbf{R}^3) \le \mu(\mathbf{R}^3)$. Thus (2.6) holds.

Since Pot σ is finite σ-almost everywhere, Lusin's theorem implies that there exists a pairwise disjoint sequence of compact sets K_n such that $\mathbf{R}^3 - \bigcup_{n=1}^{\infty} K_n$ is σ-null and such that Pot σ is continuous on each K_n. Define the measure σ_n by $\sigma_n(A) = \sigma(A \cap K_n)$. Since potentials are lower semicontinuous, each Pot σ_n is continuous on K_n, and hence on $\mathbf{R}^3$ by the theorem of Evans and Vasilesco.

Let a_n be the supremum of Pot σ_n on $\mathbf{R}^3$, $b_n = (2^n(1 + a_n))^{-1}$, $\gamma = \sum_{n=1}^{\infty} b_n \sigma_n$. Then γ is bounded, $\sigma \ll \gamma$, $\gamma \ll \sigma$, and Pot γ is continuous and bounded on $\mathbf{R}^3$. Let $f =$ Pot γ.

By Proposition 6 of [9] there exists a sequence g_n of superharmonic functions such that $0 \leq g_n \leq f$ and $\int g_n d(\nu - \mu) \to \int f\, d\sigma$ as $n \to \infty$. Let γ_n be the measure such that $g_n =$ Pot γ_n. We have

$$\int g_n\, d(\nu - \mu) = \int \text{Pot}\,(\nu - \mu)\, d\gamma_n \leq \int \text{Pot}\,\sigma\, d\gamma_n = \int g_n\, d\sigma \leq \int f\, d\sigma.$$

Thus $\int g_n\, d\sigma \to \int f\, d\sigma$ as $n \to \infty$. Hence $g_n \to f$ in $\mathscr{L}^1(\sigma)$. By choosing a subsequence and relabelling, we may assume $g_n \to f$ σ-almost everywhere, hence $g_n \to f$ γ-almost everywhere.

Fix $x \in \mathbf{R}^3$ and $\varepsilon > 0$. Since $f(x) =$ Pot $\gamma(x) < \infty$, there exists $\delta > 0$ such that if B is any Borel set with $\gamma(\mathbf{R}^3 - B) < \delta$ then Pot $\gamma_B(x) \geq f(x) - \varepsilon$, where γ_B is the measure defined by $\gamma_B(A) = \gamma(A \cap B)$. Choose B compact and $N > 0$ such that $\gamma(\mathbf{R}^3 - B) < \delta$ and $g_n \geq f - \varepsilon$ everywhere on B for all $n \geq N$. By the domination principle $g_n(x) \geq$ Pot $\gamma_B(x) - \varepsilon \geq f(x) - 2\varepsilon$ for $n \geq N$.

Hence $g_n \to f$ pointwise everywhere on $\mathbf{R}^3$. Therefore $\int g_n\, d(\nu - \mu) \to \int f\, d(\nu - \mu)$, so that $\int f\, d(\nu - \mu) = \int f\, d\sigma$, or $\int f\, d\lambda = \int f\, d\mu$, or $\int$ (Pot $\mu -$ Pot λ) $d\gamma = 0$. Hence Pot $\mu =$ Pot λ γ-almost everywhere, so (2.8) holds and Theorem 2 is proved.

Other versions of Theorem 2 are given in [10], [12 Theorem 6], [5 Theorem 2.1], [2]. Theorem 2 actually holds for a wide class of potential kernels, including of course the classical kernel on $\mathbf{R}^N$, $N \geq 3$. We restrict ourselves to $\mathbf{R}^3$ for the sake of simplicity.

The work of Rost [12] gives a probabilistic interpretation for the measure λ of Theorem 2, in terms of the filling scheme stopping time. The filling scheme was used originally by Chacon and Ornstein in their proof of the ratio ergodic theorem [1].

Proof of Theorem 1. Follows at once by induction from Theorem 2.

Now let ν_i, $i = 1, \ldots, m + 1$ be any system of nonnegative measures satisfying (2.2)–(2.5). Suppose $\langle \mu_i, \nu \rangle$ is finite for $i = 1, \ldots, m$. Clearly, for any l, $1 < l \leq m$,

$$P(\nu; \mu_1, \ldots, \mu_m) = P\left(\sum_{i=1}^{l} \nu_i; \mu_1, \ldots, \mu_l\right) + P\left(\sum_{i=l+1}^{m} \nu_i; \mu_{l+1}, \ldots, \mu_m\right) + Q, \tag{2.9}$$

where the remainder term Q is *nonnegative*. We have

$$Q = \sum_{i=1}^{l} \sum_{j=l+1}^{m} (\langle \mu_i, \mu_j \rangle - \langle \nu_i, \mu_j \rangle) + \tfrac{1}{2}\langle \nu_{m+1}, \nu_{m+1} \rangle. \tag{2.10}$$

Iterating (2.9), we have in particular,

$$P(\nu; \mu_1, \ldots, \mu_m) \geq \sum_{i=1}^{m} P(\nu_i; \mu_i) \geq -\sum_{i=1}^{m} \langle \mu_i, \nu_i \rangle. \tag{2.11}$$

As an application of (2.11), consider negative charges $-q_1, \ldots, -q_n$ at points $x_1, \ldots, x_n$, together with positive charges $z_1, \ldots, z_m$ at points $y_1, \ldots, y_m$. The total energy is

$$V = \sum_{1 \le i < j \le n} q_i q_j |x_i - x_j|^{-1} - \sum_{i=1}^{n} \sum_{j=1}^{m} q_i z_j |x_i - y_j|^{-1} + \sum_{1 \le i < j \le m} z_i z_j |y_i - y_j|^{-1}. \tag{2.12}$$

Suppose $z_j \le z$ for $j = 1, \ldots, m$. Let $R_i = \inf\{|x_i - y_j| : j = 1, \ldots, m\}$.

PROPOSITION 1. *$V \ge -\sum_{i=1}^{n} q_i^2/R_i - \sum_{i=1}^{n} 2q_i z/R_i$. In particular if $q_i = 1 = z$, $R_i \ge R$. $i = 1, \ldots, n$, we have*

$$V \ge -3n/R. \tag{2.13}$$

Proof. For $i = 1, \ldots, n$, let γ_i denote the measure of total mass q_i uniformly distributed on the surface of the sphere with centre x_i and radius $R_i/2$. Let μ_i be the measure with mass z_j concentrated at y_j. Clearly $\text{Pot}\,\gamma_i(y) \le q_i|y - x_i|^{-1}$ for all y in $\mathbf{R}^3$, with equality holding when $|y - x_i| \ge R_i/2$. Hence $\langle \gamma_i, \gamma_j \rangle \le q_i q_j |x_i - x_j|^{-1}$ and $\langle \gamma_i, \mu_j \rangle = q_i z_j |x_i - y_j|^{-1}$. Thus

$$V \ge \sum_{1 \le i < j \le n} \langle \gamma_i, \gamma_j \rangle - \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \gamma_i, \mu_j \rangle + \sum_{1 \le i < j \le m} \langle \mu_i, \mu_j \rangle. \tag{2.14}$$

Let $v = \sum_{i=1}^{n} \gamma_i$, Since $\langle \gamma_i, \gamma_i \rangle = 2q_i^2 R_i^{-1}$, we can rewrite (2.14) as

$$V \ge -\sum_{i=1}^{n} q_i^2/R_i + P(v; \mu_1, \ldots, \mu_m). \tag{2.15}$$

By (2.11), for some $v_i \ge 0$, $i = 1, \ldots, m$, with $\sum_{i=1}^{m} v_i \le v$, we have

$$P(v; \mu_1, \ldots, \mu_m) \ge -\sum_{i=1}^{m} \langle \mu_i, v_i \rangle. \tag{2.16}$$

Let $g(x) = \sup\{\text{Pot}\,\mu_j(x) : j = 1, \ldots, m\}$. Clearly $g(x) \le 2zR_i^{-1}$ when x is in the support of γ_i. We have

$$\sum_{i=1}^{m} \langle \mu_i, v_i \rangle = \sum_{i=1}^{m} \int \text{Pot}\,\mu_i\, dv_i \le \sum_{i=1}^{m} \int g\, dv_i \le \int g\, dv = \sum_{i=1}^{n} \int g\, d\gamma_i \le \sum_{i=1}^{n} 2q_i z R_i^{-1}.$$

This proves Proposition 1.

The constant 3 in (2.13) is not sharp, as a slight change in the proof shows. On the other hand the best constant cannot be less than 1.5, by a trivial example. It would be of interest to find the best possible value.

Proposition 1 may be compared to an inequality of Onsager [11]. In the notation of Proposition 1, if we let $S_j = \inf\{|x_i - y_j| : i = 1, \ldots, n\}$, Onsager's inequality reads

$$V \geq -\sum_{i=1}^{n} q_i^2/R_i - \sum_{j=1}^{m} z_j^2/S_j. \tag{2.17}$$

A more general inequality is given in [3, Theorem 6].

If we let $q_i = 1 = z_j$, $R_i \geq R$, $S_j \geq R$ for all i and j, (2.17) becomes

$$V \geq -(m+n)/R. \tag{2.18}$$

The bound in (2.18) depends on $m + n$ rather than n as in (2.13), but the constant is smaller.

The proof of (2.17) is similar to that of Proposition 1, except that (2.11) is not used. Instead, the positivity of the self-energy plays a similar role. This positivity may be expressed as follows: if μ and ν are bounded nonnegative measures with $\langle \mu, \nu \rangle$ finite, then

$$\langle \mu - \nu, \mu - \nu \rangle \geq 0. \tag{2.19}$$

We note that (2.11) implies (2.19). Indeed, fix integral $m > 0$, and let $\mu_i = \mu/m$, $i = 1, \ldots, m$. Without loss of generality we may assume that $\langle \mu, \mu \rangle$ and $\langle \nu, \nu \rangle$ are finite. Let ν_i be as in Theorem 1. By (2.11) and (2.3),

$$P(\nu; \mu_1, \ldots, \mu_m) \geq -\sum_{i=1}^{m} \langle \mu_i, \nu_i \rangle \geq -\sum_{i=1}^{m} \langle \mu_i, \mu_i \rangle = -\left(\frac{1}{m}\right)\langle \mu, \mu \rangle. \tag{2.20}$$

But $\langle \mu - \nu, \mu - \nu \rangle = 2P(\nu; \mu_1, \ldots, \mu_m) + \sum_{i=1}^{m} \langle \mu_i, \mu_i \rangle$, so

$$\langle \mu - \nu, \nu - \mu \rangle \geq -(1/m)\langle \mu, \mu \rangle.$$

Letting $m \to \infty$ proves (2.19). Thus one may regard (2.11) as an extension of (2.19).

3. Let $\mathscr{L}$ be a space of Lebesgue integrable functions $f \geq 0$ on $\mathbf{R}^3$ with $\int f < \infty$. (We shall not distinguish functions that differ on a null set.) Suppose $\mathscr{L}$ is closed under addition, and such that if f is in $\mathscr{L}$, g measurable, and $0 \leq g \leq f$ then g is in $\mathscr{L}$. As a consequence $\mathscr{L}$ is also closed under multiplication by nonnegative numbers.

Let $\Phi: \mathscr{L} \to [0, \infty)$ be a nonnegative functional with the property that

$$\Phi(f_1 + f_2) \geq \Phi(f_1) + \Phi(f_2) \quad \text{for all } f_1, f_2 \text{ in } \mathscr{L}.$$

Clearly then $\Phi(0) = 0$ and $\Phi(g) \leq \Phi(f)$ whenever f, g are in $\mathscr{L}$ with $g \leq f$.

We shall assume also that $\langle f, f \rangle$ is finite for all f in $\mathscr{L}$.

Let $\mu_1, \ldots, \mu_m$ be nonnegative bounded measures with compact support such that $\langle \mu_i, \mu_j \rangle$ is finite for $i \neq j$. For any number $a \geq 0$, let

$$F(a; \mu_1, \ldots, \mu_m) = \inf\left\{\Phi(f) + P(f; \mu_1, \ldots, \mu_m) : f \in \mathscr{L}, \int f = a\right\}. \tag{3.1}$$

We will write $F(a) = F(a; \mu_1, \ldots, \mu_m)$. Clearly $-\infty \le F < \infty$. We note:

(3.2) *F is nondecreasing on* $[z, \infty)$, *where* $z = \sum_{i=1}^{m} \mu_i(\mathbf{R}^3)$.

To see this, fix $a \ge z$, and $b \ge a$. Let f be in $\mathscr{L}$ with $\int f = b$. f is the density of a measure ν. Let ν_i, $i = 1, \ldots, m+1$ be the measures of Theorem 1. By (2.4) we can find some c, $0 \le c \le 1$, such that $\sum_{i=1}^{m} \nu_i(\mathbf{R}^3) + c\nu_{m+1}(\mathbf{R}^3) = a$. Let g be the density of $\sum_{i=1}^{m} \nu_i + c\nu_{m+1}$. Then $g \le f$ so $g \in \mathscr{L}$. We have $\int g = a$. Consider (2.9) and (2.10) with ν_{m+1} replaced by $c\nu_{m+1}$ and ν replaced by $\sum_{i=1}^{m} \nu_i + c\nu_{m+1}$. We see at once that

$$P(f; \mu_1, \ldots, \mu_m) = P(g; \mu_1, \ldots, \mu_m) + (1 - c^2)\tfrac{1}{2}\langle \nu_{m+1}, \nu_{m+1} \rangle,$$

and hence that $\Phi(f) + P(f; \mu_1, \ldots, \mu_m) \ge \Phi(g) + P(g; \mu_1, \ldots, \mu_m)$. Hence

$$\Phi(f) + P(f; \mu_1, \ldots, \mu_m) \ge F(a),$$

and thus $F(b) \ge F(a)$. This proves (3.2).

In the Thomas-Fermi case (see (3.4) below), (3.2) is proved in [7].

Now fix l, $1 \le l \le m$. For any $a \ge 0$, let

$$F_1(a) = F(a; \mu_1, \ldots, \mu_l), \quad F_2(a) = F(a; \mu_{l+1}, \ldots, \mu_m).$$

Let $z_1 = \sum_{i=1}^{l} \mu_i(\mathbf{R}^3)$, $z_2 = \sum_{i=l+1}^{m} \mu_i(\mathbf{R}^3)$.

PROPOSITION 2. *For any* $a \ge 0$,

(3.3) $$F(a) \ge \inf\{F_1(x_1) + F_2(x_2) : 0 \le x_1 \le z_1, 0 \le x_2 \le z_2, x_1 + x_2 \le a\}.$$

Proof. Let f be in $\mathscr{L}$ with $\int f = a$. Let ν_i, $i = 1, \ldots, m+1$, be the measures of Theorem 1. Let f_1 be the density of $\sum_{i=1}^{l} \nu_i$, f_2 the density of $\sum_{i=l+1}^{m} \nu_i$. Let $\int f_1 = x_1$, $\int f_2 = x_2$. By (2.4), $x_1 \le z_1$, and $x_2 \le z_2$. By (2.2), $x_1 + x_2 \le a$. By (2.9),

$$P(f; \mu_1, \ldots, \mu_m) \ge P(f_1; \mu_1, \ldots, \mu_l) + P(f_2; \mu_{l+1}, \ldots, \mu_m).$$

Thus

$$\Phi(f) + P(f; \mu_1, \ldots, \mu_m) \ge \Phi(f_1) + P(f_1; \mu_1, \ldots, \mu_l) + \Phi(f_2) + P(f_2; \mu_{l+1}, \ldots, \mu_m),$$

so

$$\Phi(f) + P(f; \mu_1, \ldots, \mu_m) \ge F_1(x_1) + F_2(x_2).$$

Then (3.3) follows at once, so Proposition 2 is proved.

If we take

(3.4) $$\mathscr{L} = \left\{ f : f \text{ measurable}, f \ge 0, \int f^{5/3} \text{ and} \int f \text{ finite} \right\}, \quad \Phi(f) = c \int f^{5/3},$$

then all the assumptions of this section are satisfied. Hölder's inequality gives

$$\text{Pot} f \le (\text{constant}) \left[\int f^{5/3}\right]^{1/2} \left[\int f\right]^{1/6} \text{ on } \mathbf{R}^3. \tag{3.5}$$

By completing a square, (3.5) implies

$$F(a) \ge \sum_{1 \le i < j \le m} \langle \mu_i, \mu_j \rangle - (\text{constant}) z^2 a^{1/3}, \tag{3.6}$$

for all a in $[0, \infty)$.

Considering $f = \alpha\chi_B$, where B is a large ball, gives

$$F(a) \le \sum_{1 \le i < j \le m} \langle \mu_i, \mu_j \rangle. \tag{3.7}$$

(3.6) and (3.7) show F is continuous at 0. Since Φ and P are convex in f, F is convex on $[0, \infty)$, and hence F is continuous on $(0, \infty)$. Thus F is continuous on $[0, \infty)$. Since F is convex and bounded above, F is nonincreasing on $[0, \infty)$. Thus, by (3.2), F is constant on $[z, \infty)$.

Since F_1 and F_2 in (3.3) are now continuous, one can rewrite (3.3) as

$$F(a) \ge F_1(a_1) + F_2(a_2), \tag{3.8}$$

where a_1 and a_2 are chosen to minimize $F(x_1) + F(x_2)$, $0 \le x_1 \le z_1$, $0 \le x_2 \le z_2$, $0 \le x_1 + x_2 \le a$. (If $a = z$ then $a_1 = z_1, a_2 = z_2$ by monotonicity.)

If we assume (3.4), and in addition assume that the measures μ_i are point measures, we are dealing with the Thomas-Fermi atomic model. The μ_i are nuclei, f is the electronic charge density, $P(f; \mu_1, \ldots, \mu_m)$ is the classical electrostatic energy, and $\Phi(f)$ is an approximation to the kinetic energy of the electrons. Then $F(a)$ is an approximation to the lowest energy level under the constraint that the total electronic charge is a. See [6], [7], [8], [13]. Proposition 2 is a version of the no binding theorem [7], [13] for the Thomas-Fermi model. Many other properties of this model, in addition to those mentioned here, are given in [7].

One reason for interest in the no binding theorem is that it was used by Lieb and Thirring, together with their estimate for the average kinetic energy of a system of fermions, to give an elegant proof of the stability of matter [8]. A discussion of the relations between the stability theorem of Lieb and Thirring and the original stability theorem of Dyson and Lenard [3], [4] is given in [8].

REFERENCES

1. R. V. CHACON and D. S. ORNSTEIN, *A general ergodic theorem*, Illinois J. Math., vol. 4 (1960), pp. 153–160.
2. A. CORNEA and G. LICEA, *Order and potential, resolvent families of kernels*, Lecture Notes in Mathematics no. 494, Springer-Verlag, Berlin 1975.
3. F. J. DYSON and A. LENARD, *Stability of matter I*, J. Math. Phys., vol. 8 (1967), pp. 423–434.
4. A. LENARD and F. J. DYSON, *Stability of matter II*, J. Math. Phys., vol. 9 (1968), pp. 698–711.

5. H. LEWY and G. STAMPACCHIA, *On the smoothness of superharmonics which solve a minimum problem*, J. Analyse Math., vol. 23 (1970), pp. 227–236.
6. E. H. LIEB, *The stability of matter*, Rev. Modern Phys. vol. 48 (1976), pp. 553–569.
7. E. H. LIEB and B. SIMON, *The Thomas-Fermi theory of atoms, molecules, and solids*, Advances in Math., vol. 23 (1977), pp. 22–116.
8. E. H. LIEB and W. E. THIRRING, *A bound for the kinetic energy of fermions which proves the stability of matter*, Phys. Rev. Lett., vol. 35 (1975), pp. 687–689.
9. P. A. MEYER, *Le schéma de remplissage en temps continu*, Séminaire de Probabilitiés V, Lecture Notes in Mathematics no. 258, Springer-Verlag, Berlin 1972, pp. 130–150.
10. G. MOKOBODZKI, *Densité relative de deux potentiels comparables*, Séminaire de Probabilitiés IV, Lecture Notes in Mathematics no. 124, Springer-Verlag, Berlin 1970, pp. 170–194.
11. L. ONSAGER, *Electrostatic interaction of molecules*, J. Phys. Chem., vol. 43 (1939), pp. 189–196.
12. H. ROST, *The stopping distributions of a Markov process*, Invent. Math., vol. 14 (1971), pp. 1–16.
13. E. TELLER, *On the stability of molecules in the Thomas-Fermi theory*, Rev. Mod. Phys., vol. 34 (1962), pp. 627–631.

UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MINNESOTA