ILLINOIS JOURNAL OF MATHEMATICS
Volume 24, Number 4, Winter 1980

EXTENDING THE PRODUCT OF TWO REGULAR BOREL MEASURES

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1. Introduction

Let X be a compact Hausdorff space, and let $B(X)$ denote the Borel sets of X. A *Borel measure* on X is a finite, nonnegative, countably additive measure on $B(X)$, and a Borel measure μ on X is *regular* if $\mu(E) = \sup\{\mu(K)\colon K \subset E \text{ and } K$ is compact$\}$ for all E in $B(X)$.

If μ and ν are regular Borel measures on compact Hausdorff spaces X and Y, respectively, then it is well known that the product measure $\mu \times \nu$ on $B(X) \times B(Y)$ has an extension to a regular Borel measure $\mu \otimes \nu$ on $X \times Y$. In this paper we give some partial answers to the following open question: Does $\mu \times \nu$ have only one extension to a Borel measure on $X \times Y$? Equivalently, if ρ is a Borel measure on $X \times Y$ such that $\rho(E \times F) = \mu(E)\nu(F)$ whenever $E \in B(X)$ and $F \in B(Y)$, is ρ regular? In particular, necessary conditions and sufficient conditions are given for the existence of a nonregular Borel extension of $\mu \times \nu$.

We pause to consider an equivalent statement for the condition that $\mu \otimes \nu$ is the only extension of $\mu \times \nu$ to a Borel measure on $X \times Y$.

THEOREM 1.1. *The following are equivalent:*

(1) *If ρ is a Borel measure on $X \times Y$ and $\rho|_{B(X) \times B(Y)} = \mu \times \nu$, then $\rho = \mu \otimes \nu$.*
(2) *If λ is a Borel measure on $X \times Y$ and $\lambda|_{B(X) \times B(Y)}$ is absolutely continuous with respect to $\mu \times \nu$, then λ is absolutely continuous with respect to $\mu \otimes \nu$.*

Proof. In order to show that (1) implies (2), let λ be a Borel measure on $X \times Y$ such that $\lambda|_{B(X) \times B(Y)} \ll \mu \times \nu$. We wish to show that $\lambda \ll \mu \otimes \nu$. Suppose otherwise. Then there exists a Borel set E in $X \times Y$ such that $\mu \otimes \nu(E) = 0$ and $\lambda(E) > 0$. Choose F in $B(X) \times B(Y)$ such that $\lambda(E) = \lambda(E \cap F)$ and such that $\mu \times \nu(F)$ is as small as possible under the requirement that $\lambda(E) = \lambda(E \cap F)$. Then $(\mu \times \nu)_F \ll \lambda$. That is, $(\mu \times \nu)(F \cap G) = 0$ whenever G is in $B(X) \times B(Y)$ and $\lambda(G) = 0$. Otherwise, we would have $\lambda(E) = \lambda(E \cap (F - G))$ and $\mu \times \nu(F - G) < \mu \times \nu(F)$.

Received November 15, 1978.

By the Radon-Nikodým theorem, there exists a nonnegative function f on $X \times Y$ such that f is measurable with respect to $B(X) \times B(Y)$ and $\mu \times \nu_F(H) = \int_H f\, d\lambda$ for all H in $B(X) \times B(Y)$. If $M \in B(X \times Y)$, define

$$\rho(M) = \int_M f\, d\lambda + \mu \otimes \nu(M - F).$$

Then ρ is a Borel measure on $X \times Y$ and $\rho|_{B(X) \times B(Y)} = \mu \times \nu$. Since $\mu \times \nu_F$ and λ_F have the same sets of measure 0 in $B(X) \times B(Y)$, we may assume that the function f is strictly positive on F. Then since $\lambda(E \cap F) > 0$, we have $\rho(E \cap F) = \int_{E \cap F} f\, d\lambda > 0$. Hence, $\rho(E \cap F) > 0$ and $\mu \otimes \nu(E \cap F) = 0$, which violates the hypothesis of (1). Therefore, $\lambda \ll \mu \otimes \nu$.

In order to show that (2) implies (1), let ρ be a Borel measure on $X \times Y$ such that $\rho|_{B(X) \times B(Y)} = \mu \times \nu$. By the hypothesis of (2), ρ is absolutely continuous with respect to the regular Borel measure $\mu \otimes \nu$. Hence, ρ is a *regular* Borel measure on $X \times Y$ [4, Exercise 52.9]. Since ρ extends $\mu \times \nu$, it must therefore be $\mu \otimes \nu$ and we are done.

Notice that $\mu \otimes \nu$ is the only extension of $\mu \times \nu$ to a Borel measure (whether regular or not) on $X \times Y$ if any one of the following equivalent statements holds:

(1) $\mu \times \nu$ and $\mu \otimes \nu$ have the same completion.
(2) Every compact set in $X \times Y$ is $\mu \times \nu$-measurable.
(3) Every compact set in $X \times Y$ with positive $\mu \otimes \nu$-measure contains a set in $B(X) \times B(Y)$ with positive $\mu \times \nu$-measure.
(4) Every Borel set in $X \times Y$ with zero $\mu \otimes \nu$-measure is contained in some set in $B(X) \times B(Y)$ with zero $\mu \times \nu$-measure.

2. Necessary conditions for a nonregular Borel extension of $\mu \times \nu$

THEOREM 2.1. *Suppose some nonregular Borel measure on $X \times Y$ extends $\mu \times \nu$. Then there exists a compact set K in $X \times Y$ and a nonzero Borel measure λ on $X \times Y$ with the following properties:*

(1) $\mu \times \nu_*(K) = 0$ *and* $\mu \times \nu^*(K) > 0$.
(2) *If $S(K)$ is the smallest σ-algebra of subsets of $X \times Y$ containing $B(X) \times B(Y)$ and K and if π is that unique measure on $S(K)$ such that $\pi(K) = 0$ and $\pi(H) = \mu \times \nu^*(H \cap K)$ if $H \in B(X) \times B(Y)$, then $\lambda|_{S(K)} \ll \pi$.*

Proof. Suppose ρ is a nonregular Borel measure on $X \times Y$ which extends $\mu \times \nu$. Since ρ is not absolutely continuous with respect to $\mu \otimes \nu$, there is a Borel set E in $X \times Y$ such that $\rho(E) > 0$ and $\mu \otimes \nu(E) = 0$. Choose a set F in $B(X) \times B(Y)$ such that $\rho(E) = \rho(E \cap F)$ and such that $\mu \times \nu(F)$ is a minimum under the requirement that $\rho(E) = \rho(E \cap F)$. Since $\mu \otimes \nu(E \cap F) = 0$ and $\rho(E \cap F) > 0$ and since $\mu \otimes \nu(F) = \rho(F)$, we have $\mu \otimes \nu(F - E) > \rho(F - E)$. Then since $\mu \otimes \nu$ is regular, there exists a compact set K contained in $F - E$

such that $\mu \otimes \nu(K) > \rho(F - E)$. Necessarily, $\mu \otimes \nu(K) > \rho(K)$. In order to see that $\mu \times \nu_*(K) = 0$, suppose $G \in B(X) \times B(Y)$ and $G \subset K$. Then $G \subset F - E$, so that $E \cap F = E \cap (F - G)$. Hence, $\mu \times \nu(F - G) = \mu \times \nu(F)$. Hence, $\mu \times \nu(G) = \mu \times \nu(F \cap G) = 0$, so that $\mu \times \nu_*(K) = 0$.

Choose a compact G_δ set L containing K such that $\mu \otimes \nu(K) = \mu \otimes \nu(L)$ [1, Theorem 59.1]. Necessarily $L \in B(X) \times B(Y)$. Let $\lambda = \rho_{L-K}$. That is, $\lambda(E) = \rho((L - K) \cap E)$ for all $E \in B(X \times Y)$. Then λ is a Borel measure on $X \times Y$, and λ is nonzero since

$$\lambda(L) = \rho(L - K) = \rho(L) - \rho(K) > \mu \otimes \nu(L) - \mu \otimes \nu(K) = 0.$$

Now let $S(K)$ be the σ-algebra generated by K and the members of $B(X) \times B(Y)$. Each member of $S(K)$ has the form $(P \cap K) \cup (Q - K)$, where P and Q are in $B(X) \times B(Y)$ [4, Exercise 16.2a]. Let π be that unique measure on $S(K)$ such that $\pi(K) = 0$ and

$$\pi(H) = \mu \times \nu^*(H \cap K) \quad \text{if } H \in B(X) \times B(Y).$$

It is easy to see that $\pi((P \cap K) \cup (Q - K))$ must equal $\mu \times \nu^*(Q \cap K)$. Moreover, the formula $\pi((P \cap K) \cup (Q - K)) = \mu \times \nu^*(Q \cap K)$ is well defined, and the resulting set function π is indeed a measure on $S(K)$ [1, Exercise 6.10].

We show that $\lambda|_{S(K)} \ll \pi$. Suppose $\pi((P \cap K) \cup (Q - K)) = 0$, which means that $\mu \times \nu^*(Q \cap K) = 0$. Since $\mu \times \nu(L) = \mu \times \nu^*(K)$, we have

$$\mu \times \nu(Q \cap L) = \mu \times \nu^*(Q \cap K) = 0.$$

Then

$$\lambda((P \cap K) \cup (Q - K)) = \rho((Q \cap L) - K) \leq \rho(Q \cap L) = \mu \times \nu(Q \cap L) = 0.$$

Therefore, $\lambda|_{S(K)} \ll \pi$ and the proof is complete.

As a consequence of Theorem 2.1, we have the following necessary condition for the existence of a nonregular Borel extension of $\mu \times \nu$.

THEOREM 2.2. *Suppose some nonregular Borel measure on $X \times Y$ extends $\mu \times \nu$. Then there exists a compact set K in $X \times Y$ such that $\mu \times \nu_*(K) = 0$ and a Borel measure λ on $X \times Y$ such that $\lambda(H - K) > 0$ if $H \in B(X) \times B(Y)$ and H contains K.*

Proof. Let K be the compact set and λ the Borel measure given in Theorem 2.1. Suppose $H \in B(X) \times B(Y)$ and H contains K. If π is the Borel measure described in Theorem 2.1, then

$$\pi((X \times Y) - H) = \mu \times \nu^* \text{ (empty set)} = 0$$

and $\pi(K) = 0$. Then $\lambda((X \times Y) - H) = 0$ and $\lambda(K) = 0$ since $\lambda|_{S(K)} \ll \pi$. Necessarily, $\lambda(H - K) > 0$ since λ is nonzero.

Recall that a cardinal number is said to be *measurable* if there exists a set Z with that cardinality and a finite, nonzero, countably additive measure on the class of all subsets of Z such that each singleton has measure zero. If such a cardinal exists, it is greater than the first uncountable cardinal ω_1 [6, pp. 141–143].

THEOREM 2.3. *Suppose K is a compact set in $X \times Y$ such that $\mu \times \nu_*(K) = 0$, and suppose λ is a Borel measure on $X \times Y$ such that $\lambda(H - K) > 0$ whenever $H \in B(X) \times B(Y)$ and H contains K. In other words, suppose K and λ satisfy the conclusion of Theorem 2.2. If H is a set in $B(X) \times B(Y)$ such that $K \subset H$, then $H - K$ cannot be expressed as a disjoint union of sets $\{U_i\}_{i \in I}$ such that each U_i is an open F_σ in H unless the cardinality of I is a measurable cardinal.*

Proof. Suppose $H \in B(X) \times B(Y)$, where $K \subset H$ and $H - K$ is a disjoint union of sets $\{U_i\}_{i \in I}$ such that each U_i is an open F_σ in H. Since λ is finite, only countably many of the U_i's have positive λ-measure. By subtracting such U_i's from H, we may assume without loss of generality that each U_i has zero λ-measure. Since each union of U_i's is open in H, each union of U_i's is a Borel set in $X \times Y$. If Z is a set with the same cardinality as I, then there is a natural correspondence between the class of all unions of the U_i's and the class of all subsets of Z. Since $\lambda(H - K) > 0$ and $\lambda(U_i) = 0$ for each $i \in I$, the measure λ induces a finite, nonzero, countably additive measure on the class of all subsets of Z such that each singleton has measure zero. Hence, the cardinality of I is measurable in this case.

Theorems 2.2 and 2.3 can be combined to give the following conditions under which $\mu \otimes \nu$ is the only extension of $\mu \times \nu$ to a Borel measure on $X \times Y$.

COROLLARY. *Suppose for each compact set K in $X \times Y$ there exists a superset $H \in B(X) \times B(Y)$ and a disjoint collection of sets $\{U_i\}_{i \in I}$ such that*

(1) *U_i is an open F_σ in H for each i in I,*
(2) *$H - K = \cup \{U_i: i \in I\}$, and*
(3) *cardinality of I is not a measurable cardinal.*

Then $\mu \otimes \nu$ is the only extension of $\mu \times \nu$ to a Borel measure on $X \times Y$.

The conditions given in the corollary may at first seem contrived and unlikely to occur in practice. Let us therefore look at an example where these conditions are satisfied. Let $X = [-1, 1)$ with the smallest topology containing sets of the form $[-b, b)$ or $X - [-b, b)$, where $0 \le b \le 1$. Then X is a compact Hausdorff space and $B(X) \times B(X)$ is properly contained in $B(X \times X)$ [5, pp. 172–173]. If K is any compact set in $X \times X$, let

$$H = K \cup \{(x, y): (-x, y) \in K\}.$$

It can be seen that H is a compact G_δ containing K and that each vertical cross-section of $H - K$ is an open F_σ in H. That is, $(H - K) \cap (\{x\} \times Y)$ is an open F_σ in H for each x in X.

Under the assumption that c is not a measurable cardinal, we can thus express $H - K$ as a disjoint union of sets $\{U_i\}_{i \in I}$ such that each U_i is an open F_σ in H and such that the cardinality of I is not a measurable cardinal. Hence, if μ and ν are (necessarily regular) Borel measures on X, then $\mu \times \nu$ has only one extension to a Borel measure on $X \times X$.

The referee has observed that if the set H in the preceding corollary is compact, then $H - K$ is weakly θ-refinable. In other words, each open cover of $H - K$ can be refined by a sequence of families $V(n)$ of open sets such that if $x \in H - K$, then there exists $n(x)$ such that x is in some member of $V(n(x))$ and only in a finite number of members of $V(n(x))$. Of course, the members of each $V(n)$ can be arranged to be contained in one of the U_i's of the corollary. By the reasoning of [3, Theorem 3.9], each locally zero Borel measure on $H - K$ is 0 on $H - K$. Now choose a set M in $B(X) \times B(Y)$ such that $K \subset M \subset H$ and such that $\mu \otimes \nu(M - K) = 0$. If ρ is an extension of $\mu \times \nu$ to a Borel measure, than ρ_M is locally zero on $M - K$. Hence $\rho(M - K) = 0$ so that $\rho(K) = \mu \otimes \nu(K)$. These ideas lead to the following strengthening of the corollary to Theorem 2.3.

THEOREM 2.4. *Suppose for each compact set K in $X \times Y$ there exists a superset $H \in B(X) \times B(Y)$ such that $H - K$ is weakly θ-refinable and such that the cardinality of each discrete subspace of $H - K$ is not a measurable cardinal. Then $\mu \otimes \nu$ is the only extension of $\mu \times \nu$ to a Borel measure on $X \times Y$.*

3. Sufficient conditions for a nonregular Borel extension of $\mu \times \nu$

THEOREM 3.1. *Suppose K is a compact set in $X \times Y$ such that $\mu \times \nu_*(K) = 0$ and $\mu \times \nu^*(K) > 0$. Let $S(K)$ be the smallest σ-algebra of subsets of $X \times Y$ containing $B(X) \times B(Y)$ and K. Let π be that unique measure on $S(K)$ such that $\pi(K) = 0$ and such that $\pi(H) = \mu \times \nu^*(H \cap K)$ if $H \in B(X) \times B(Y)$. If there exists a nonzero Borel measure λ on $X \times Y$ such that $\lambda|_{S(K)} \ll \pi$, then $\mu \times \nu$ can be extended to a nonregular Borel measure on $X \times Y$.*

Proof. Suppose λ is a nonzero Borel measure on $X \times Y$ such that $\lambda|_{S(K)} \ll \pi$. If $F \in B(X) \times B(Y)$ and $\mu \times \nu(F) = 0$, clearly $\pi(F) = 0$ so that $\lambda(F) = 0$. Hence, $\lambda|_{B(X) \times B(Y)} \ll \mu \times \nu$. Choose a set G in $B(X) \times B(Y)$ such that $K \subset G$ and such that $\mu \times \nu^*(K) = \mu \times \nu(G)$. Then $\pi(X \times Y - G) = \mu \times \nu^*$ (empty set) $= 0$, so that $\lambda(X \times Y - G) = 0$. Hence, $\lambda(G) > 0$ since λ is nonzero. Since $\pi(K) = 0$ and since $\lambda|_{S(K)} \ll \pi$, we have $\lambda(K) = 0$. Hence, $\lambda(G - K) > 0$. However, $\mu \otimes \nu(G - K) = 0$, so that λ is not absolutely continuous with respect to $\mu \otimes \nu$. From Theorem 1.1, we see that $\mu \times \nu$ can be extended to a nonregular Borel measure on $X \times Y$.

Under what conditions can there exist a nonzero Borel measure λ on $X \times Y$ such that $\lambda|_{S(K)} \ll \pi$, where $S(K)$ and π are the σ-algebra and measure given in Theorem 3.1? If π can be extended to a Borel measure on $X \times Y$, then that extension will serve as the measure λ. Such an extension is possible for example, if the domain of completion of π includes the Borel sets of $X \times Y$. More generally, we have the conditions of the following theorem:

THEOREM 3.2. *Let π be the measure described in Theorem 3.1. Let π^S be the smallest (countably additive) measure on $B(X \times Y)$ such that $\pi^*(E) \leq \pi^S(E)$ for all Borel sets E in $X \times Y$. If $0 < \pi^S(D) < \infty$ for some Borel set D, then there exists a nonzero Borel measure λ on $X \times Y$ such that $\lambda|_{S(K)} \ll \pi$. Hence, $\mu \times \nu$ can be extended to a nonregular Borel measure on $X \times Y$ in this case.*

Proof. Of course, $\pi^S|_{S(K)} \ll \pi$. Now suppose $0 < \pi^S(D) < \infty$ for some Borel set D. Let $\lambda = \pi^S_D$. That is, $\lambda(E) = \pi^S(D \cap E)$ for all E in $B(X \times Y)$. Then λ is a nonzero Borel measure on $X \times Y$ such that $\lambda|_{S(K)} \ll \pi$.

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