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REPRESENTATIONS OF INTEGERS BY POSITIVE DEFINITE FORMS OVER ARITHMETIC PROGRESSIONS

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1. In previous works, the authors have analyzed Dirichlet series associated to positive definite integral forms $F(x)$ and applied the results to obtain asymptotic estimates for $\sum_{F(\gamma)\leq y} 1$. In this note, we refine our estimates and analyze the behavior of $F(\gamma)$ as the components of γ vary over arithmetic progressions.

Let F be a positive definite integral form of degree d in n variables and let

$$\zeta(F, \beta, s) = \sum_{\gamma \in Z^n - \{\mathbf{0}\}} F(\gamma)^{-s} e(\langle \beta, \gamma \rangle) \tag{1.1}$$

where $s = \sigma + it$, $\beta \in R^n$, $\langle\ ,\ \rangle$ indicates the standard inner product on R^n and $e(a) = \exp(2\pi i a)$.

In [2] it has been shown that $\zeta(F, \beta, s)$ can be continued analytically as a meromorphic function of s with only a simple pole at $s = n/d$ occurring when $\beta \in Z^n$. It was shown [4] that if $\beta \in Z^n$ and $|t| \geq 2$ then

$$|\zeta(F, \beta, \sigma + it)| \ll \begin{cases} \dfrac{|t|^{n-\sigma d}}{(n - \sigma d)(n - 1 - \sigma d)} & \text{if } \dfrac{n-1}{d} < \sigma < \dfrac{n}{d} - \dfrac{1}{\log |t|} \\ \log |t| & \text{if } \sigma > \dfrac{n}{d} - \dfrac{1}{\log |t|}. \end{cases} \tag{1.2}$$

We shall prove that the restriction on β can be removed.

THEOREM 1. *If $\beta \in R^n$ and $|t| \geq 2$, then* (1.2) *holds.*

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$, $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n) \in Z^n$. Let $\gamma \equiv B(\text{mod } A)$ mean $\gamma_i \equiv B_i(\text{mod } A_i)$ for $i = 1, \ldots, n$. Let $A^* = \prod_{i=1}^n A_i$, $\lambda = \text{Res}_{s=n/d}\, \zeta(F, \mathbf{0}, s)$.

We shall use Theorem 1 to prove the following:

THEOREM 2.

$$\sum_{\substack{F(\gamma)\leq y, \\ \gamma \equiv B(\text{mod } A)}} 1 = \frac{\lambda}{A^*} \frac{d}{n} y^{n/d} + O(y^{(n-1/2)/d} \log y), \quad y > e^d. \tag{1.3}$$

2. Since we know (1.2) holds if $\beta \in Z^n$, we shall assume $\beta \notin Z^n$. Without loss of generality, we assume $0 < \beta_1 < 1$.

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If $x = (x_1, \ldots, x_n) \in R^n$, let $\bar{x} = (x_2, \ldots, x_n)$. Let $K = [|t|]$, $\|\gamma\| = \max |\gamma_i|$ and assume $\sigma > (n-1)/d$. Since the series representation for $\zeta(F, \beta, s)$ is valid for $\sigma > (n-1)/d$ [3], we may write

$$(2.1) \qquad \zeta(F, \beta, s) = \sum_{0 < \|\gamma\| < K} F(\gamma)^{-s} e(\langle \beta, \gamma \rangle) + \sum_{\|\gamma\| \geq K} F(\gamma)^{-s} e(\langle \beta, \gamma \rangle).$$

The first term is bounded by $\sum_{0 < \|\gamma\| < K} F(\gamma)^{-\sigma}$. Since

$$F(\gamma)^{-\sigma} \ll \|\gamma\|^{-\sigma d} \quad \text{and} \quad \sum_{\|\gamma\| = m} 1 \ll m^{n-1}$$

we obtain

$$(2.2) \qquad \sum_{0 < \|\gamma\| < K} F(\gamma)^{-\sigma} \ll \sum_{m < K} m^{-\sigma d + n - 1}$$

which is bounded by the right hand side of (1.2). So we are left to consider the second term of (2.1). To that end, let $C_m = e(m\beta_1)/(e(\beta_1) - 1)$. Thus $e(m\beta_1) = C_{m+1} - C_m$ and $C_m = O(1)$. Since

$$(2.3) \qquad e(\langle \beta, \gamma \rangle) = e(\langle \bar{\beta}, \bar{\gamma} \rangle)(C_{\gamma_1 + 1} - C_{\gamma_1})$$

we can rewrite the second term of (2.1) as

$$(2.4) \qquad \sum_{\max(\|\bar{\gamma}\|, |m|) \geq K} e(\langle \bar{\beta}, \bar{\gamma} \rangle) C_{m+1}(F(m, \bar{\gamma})^{-s} - F(m+1, \bar{\gamma})^{-s})$$
$$+ \sum_{0 < \|\bar{\gamma}\| < K} e(\langle \bar{\beta}, \bar{\gamma} \rangle)(C_{-K+1} F(-K+1, \bar{\gamma})^{-s} - C_{K-1} F(K-1, \bar{\gamma})^{-s}).$$

The second term of (2.4) is clearly $\ll |t|^{n-1-\sigma d}$, so we need concentrate only on the first term, which is bounded by

$$(2.5) \qquad S = \sum_{\max(\|\bar{\gamma}\|, |m|) \geq K} |F(m, \bar{\gamma})^{-s} - F(m+1, \bar{\gamma})^{-s}|.$$

Furthermore,

$$(2.6) \qquad F(m, \bar{\gamma})^{-s} - F(m+1, \bar{\gamma})^{-s} = s \int_m^{m+1} F(u, \bar{\gamma})^{-s-1} \frac{\partial}{\partial u} F(u, \bar{\gamma})\, du.$$

Since

$$|F(u, \bar{\gamma})^{-s-1}| \ll \|(u, \gamma)\|^{(-\sigma-1)d} \quad \text{and} \quad \frac{\partial}{\partial u} F(u, \bar{\gamma}) \ll \|(u, \bar{\gamma})\|^{d-1},$$

we obtain

$$(2.7) \qquad F(m, \bar{\gamma})^{-s} - F(m+1, \bar{\gamma})^{-s} \ll |t| \int_m^{m+1} \|(u, \bar{\gamma})\|^{-\sigma d - 1}\, du.$$

The integral is certainly $\ll \|(m, \bar{\gamma})\|^{-\sigma d - 1}$ yielding

$$(2.8) \qquad S \ll |t| \sum_{\|\gamma\| \geq K} \|\gamma\|^{-\sigma d - 1} \ll |t| \sum_{m > K} m^{n - \sigma d - 2}.$$

Since $K \approx |t|$, the right hand side of (2.8) is $\ll$ the right hand side of (1.2), completing the proof of Theorem 1.

3. Let $A, \beta \in Z^n$ be fixed. We use the following lemma to prove Theorem 2.

LEMMA.

$$\sum_{\substack{F(\gamma)\leq y,\\ \gamma\equiv B(\mathrm{mod}\ A)}} \left(1-\frac{F(\gamma)}{y}\right) = \frac{\lambda y^{n/d}}{\frac{n}{d}\left(\frac{n}{d}+1\right)A^*} + O(y^{(n-1)/d}\log^2 y), \quad y > e^d. \tag{3.1}$$

Proof. Let $\sum'$ represent a sum over all $\alpha \in Q^n$ where $\alpha_i = p_i/A_i$, $p_i \in Z$ and $0 \leq p_i < A_i$. Let

$$\zeta_{B/A}(F, s) = \sum{}' e(-\langle\alpha, B\rangle)\zeta(F, \alpha, s). \tag{3.2}$$

We easily conclude from our knowledge of $\zeta(F, \alpha, s)$ that $\zeta_{B/A}(F, s)$ is meromorphic with only a simple pole of residue λ at $s = n/d$ and that, for $\sigma > (n-1)/d$, $|t| \geq 2$,

$$\zeta_{B/A}(F, s) \ll \begin{cases} \dfrac{|t|^{n-\sigma d}}{(n-\sigma d)(n-1-\sigma d)} & \text{if } \sigma \leq \dfrac{n}{d} - \dfrac{1}{\log|t|} \\ \log|t| & \text{if } \sigma \geq \dfrac{n}{d} - \dfrac{1}{\log|t|}. \end{cases} \tag{3.3}$$

If $\sigma > n/d$ we can write

$$\zeta_{B/A}(F, s) = \sum{}' e(-\langle\alpha, B\rangle) \sum_{\|\gamma\|\neq 0} e(\langle\alpha, \gamma\rangle)F(\gamma)^{-s}. \tag{3.4}$$

Since the series representation for $\zeta(F, \alpha, s)$ converges absolutely if $\sigma > n/d$, we can interchange summations, obtaining

$$\zeta_{B/A}(F, s) = \sum_{\|\gamma\|\neq 0} F(\gamma)^{-s} \sum{}' e(\langle\alpha, \gamma - B\rangle). \tag{3.5}$$

If $\gamma - B \equiv 0 \pmod A$ then it is clear that $\sum' e(\langle\alpha, \gamma - B\rangle) = A^*$. Suppose $\gamma - B \not\equiv 0 \pmod A$. We may assume, without loss of generality, that $\gamma_1 - B_1 \not\equiv 0 \pmod{A_1}$. We can then factor out

$$\sum_{\rho_1=0}^{A_1-1} e\left(\frac{\gamma_1 - B_1}{A_1}\rho_1\right) = 0.$$

We thus obtain

$$\zeta_{B/A}(F, s) = A^* \sum_{0\neq\gamma\equiv B(\mathrm{mod}\ A)} F(\gamma)^{-s} \quad \text{if } \sigma > n/d. \tag{3.6}$$

Consider

$$I = \frac{1}{2\Pi i}\int_{\beta-iy}^{\beta+iy} \frac{\zeta_{B/A}(F, s)y^s}{s(s+1)}\,ds \quad \text{where} \quad \beta = \frac{n}{d} + \frac{1}{\log y}. \tag{3.7}$$

Using (3.6) we obtain

$$I = A^* \sum_{0 \neq \gamma \equiv B (\mathrm{mod}\ A)} \frac{1}{2\Pi i} \int_{\beta - iy}^{\beta + iy} \frac{(y/F(\gamma))^s}{s(s+1)}\, ds. \tag{3.8}$$

Since

$$\frac{1}{2\Pi i} \int_{\beta - iy}^{\beta + iy} \frac{z^s}{s(s+1)}\, ds = \begin{cases} O(z^{\beta/y}) & \text{if } z \leq 1 \\ 1 - 1/z + O(z^{\beta}/y) & \text{if } z \geq 1 \end{cases}$$

(cf. [5]), we obtain

$$I = A^* \sum_{\substack{0 \neq \gamma \equiv B (\mathrm{mod}\ A), \\ F(\gamma) \leq y}} \left(1 - \frac{F(\gamma)}{y}\right) + O\left(\sum_{0 \neq \gamma \equiv B (\mathrm{mod}\ A)} y^{\beta - 1}/F(\gamma)^{\beta}\right). \tag{3.9}$$

The error term is

$$\begin{aligned} &\ll y^{n/d} \sum_{\gamma \neq 0} F(\gamma)^{-\beta} \\ &\ll y^{n/d - 1} \sum_{m=1}^{\infty} m^{-\beta d + n - 1} \\ &\ll \frac{y^{n/d - 1}}{n - \beta d} \\ &\ll y^{n/d - 1} \log y \end{aligned}$$

yielding

$$I = A \sum_{\substack{0 \neq \gamma \equiv B (\mathrm{mod}\ A), \\ F(\gamma) \leq y}} \left(1 - \frac{F(\gamma)}{y}\right) + O(y^{n/d - 1} \log y). \tag{3.10}$$

We now estimate I via contour integration. Let $\beta' = (n-1)/d + 1/\log y$, C_1 be the straight line contour from $\beta + iy$ to $\beta' + iy$, C_2 be the straight line contour from $\beta' + iy$ to $\beta' - iy$ and C_3 be the straight line contour from $\beta' - iy$ to $\beta - iy$. Let C_0 be $C_1 + C_2 + C_3 +$ the straight line contour from $\beta - iy$ to $\beta + iy$. Let

$$I_j = \frac{1}{2\Pi i} \int_{C_j} \frac{\zeta_{B/A}(F, s) y^s}{s(s+1)}\, ds \quad \text{for } j = 0, 1, 2, 3. \tag{3.11}$$

Then $I = I_0 - (I_1 + I_2 + I_3)$. Since the only singularity of $[\zeta_{B/A}(F, s) y^s]/[s(s+1)]$ inside C_0 comes from the pole of $\zeta_{B/A}(F, s)$ at $s = n/d$, we obtain

$$I = \frac{\lambda y^{n/d}}{\frac{n}{d}\left(\frac{n}{d} + 1\right)} - (I_1 + I_2 + I_3). \tag{3.12}$$

Along C_1, (3.3) implies that $\zeta_{B/A}(F, s)y^s = O(y^{1+(n-1)/d} \log y)$. Since

$$\frac{1}{s(s+1)} = O\left(\frac{1}{y^2}\right)$$

along C_1, we obtain

$$I_1 = O(y^{(n-1)/d} \log y/y). \tag{3.13}$$

The same estimate clearly holds for I_3. To estimate I_2, we first observe that

$$I_2 = \int_{C_2, |t| \geq 2} \frac{\zeta_{B/A}(F, s)y^s}{s(s+1)}\, ds + O(y^{(n-1)/d}).$$

We again use (3.3) to estimate $\zeta_{B/A}(F, s) = O(|t| \log y)$ if $s \varepsilon C_2$, $|t| \geq 2$, obtaining

$$I_2 \ll y^{(n-1)/d} \log y \int_{C_2, |t| \geq 2} \frac{|t|}{|t|^2}\, dt + O(y^{(n-1)/d}), \tag{3.14}$$

so that

$$I_2 = O(y^{(n-1)/d} \log^2 y). \tag{3.15}$$

We combine (3.12), (3.13), and (3.15) to obtain

$$I = \frac{\lambda y^{n/d}}{\frac{n}{d}\left(\frac{n}{d}+1\right)} + O(y^{(n-1)/d} \log^2 y). \tag{3.16}$$

Combining (3.10) and (3.16) completes the proof of the lemma.

4. Let a_k represent the number of solutions to $F(\gamma) = k$ for which $\gamma \equiv B(\text{mod } A)$. Then we may write

$$\sum_{\substack{F(\gamma) \leq y, \\ \gamma \equiv B(\text{mod } A)}} \left(1 - \frac{F(\gamma)}{y}\right) = \sum_{k \leq y} a_k \left(1 - \frac{k}{y}\right). \tag{4.1}$$

Combining (3.1), (4.1) and multiplying by y yields

$$\sum_{k \leq y} a_k (y - k) = \frac{\lambda y^{1+n/d}}{A^* \frac{n}{d}\left(\frac{n}{d}+1\right)} + O(y^{1+(n-1)/d} \log^2 y). \tag{4.2}$$

If we let $A(z) = \sum_{k \leq z} a_k$ and assume y is an integer, then (4.2) becomes

$$\sum_{k < y} A(k) = \frac{\lambda y^{1+n/d}}{A^* \frac{n}{d}\left(\frac{n}{d}+1\right)} + O(y^{1+(n-1)/d} \log^2 y). \tag{4.3}$$

It is clear that (4.3) must also hold if y is not an integer. Now let

$$\alpha = 1 - y^{-1/2d} \log y.$$

Then

$$(4.4)\qquad \sum_{\alpha y \le k < y} A(k) = \frac{\lambda}{A^* \frac{n}{d}\left(1 + \frac{n}{d}\right)} y^{1+n/d}(1 - \alpha^{1+n/d}) + O(y^{1+(n-1)/d} \log^2 y)$$
$$\le (1 - \alpha) y A(y).$$

Since $1 - \alpha^{1+n/d} = (1 + n/d)(1 - \alpha) + O((1 - \alpha)^2)$, if we divide by $1 - \alpha$ we obtain

$$(4.5)\qquad A(y) \ge \frac{\lambda}{A^*} \frac{d}{n} y^{n/d} + O(y^{n/d}(1 - \alpha)) + \frac{O(y^{(n-1)/d} \log^2 y)}{1 - \alpha}.$$

With our choice for α, (4.5) becomes

$$(4.6)\qquad A(y) \ge \frac{\lambda}{A^*} \frac{d}{n} y^{n/d} + O(y^{(n-1/2)/d} \log y).$$

Letting $\beta = 1 + y^{-1/2d} \log y$ and considering $\sum_{y \le k < \beta y} A(k)$ we obtain

$$(4.7)\qquad A(y) \le \frac{\lambda}{A^*} \frac{d}{n} y^{n/d} + O(y^{(n-1/2)/d} \log y).$$

Combining (4.6) and (4.7) yields Theorem 2.

5. We observe the relationship between Theorem 2 and the corresponding result

$$(5.1)\qquad \sum_{F(\gamma) \le y} 1 = \frac{d}{n} \lambda y^{n/d} + O(y^{(n-1/2)/d} \log y).$$

in [4].

Indeed, Theorem 2 essentially combines (5.1) with the fact that $F(\gamma)$ behaves similarly as γ varies over different congruence classes. The latter can be expected since $F(\gamma)/\|\gamma\|^{n/d}$ is bounded. A related question is whether the values of $F(\gamma)$ are evenly distributed over different congruence classes, i.e., is

$$\sum_{\substack{F(\gamma) \le y,\\ F(\gamma) \equiv B (\mathrm{mod}\ A)}} 1 \sim \frac{\lambda}{A^*} \frac{d}{n} y^{n/d}?$$

This leads one to investigate

$$\sum_{0 \ne \gamma} F(\gamma) e\left(\frac{B}{A} F(\gamma)\right).$$

It has been shown [3] that such functions can be continued analytically with at most a simple pole at $s = n/d$, but effective bounds have not yet been computed.

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