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BIG COHEN-MACAULAY MODULES

BY

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0. Since M. Hochster initiated the study of big maximal Cohen-Macaulay modules in [4], these modules have had a wide variety of applications and are rapidly becoming a standard tool for the homological theory of commutative rings. In this note we show that a few of the well known properties of finitely generated Cohen-Macaulay modules can be extended to certain big maximal Cohen-Macaulay modules. Our first result, Theorem 2.1, shows that if M is an R-module with $\dim M = \dim R = d$, then the local cohomology module $H^d_m(M)$ has a secondary representation and

$$\operatorname{Att}\,(H^d_m(M)) \subseteq \{p \in \operatorname{Ass}\,(M) \mid \dim R/p = d\}.$$

In Section 3 we consider some consequences for R of the existence of maximal Cohen-Macaulay modules with nice properties. The exactness of the Cousin complex is also considered.

Throughout this note R denotes a local (noetherian) ring with maximal ideal m and residue field k. The undefined terminology is the same as that in [5], [6].

1. In [9], [10] an R-module M is called *secondary* if for each $x \in R$, multiplication by x on M is either nil-potent or surjective, and in this case $\{x \in R \mid xM \neq M\}$ is a prime ideal which is said to be *attached* to M. It is clear that this in some sense dualizes the notions of primary module and associated prime, and this has been explored by several authors. For example an R-module M is said to have a *secondary representation* if M is a finite sum of secondary submodules, and if this holds then many of the standard results about primary decompositions have analogues for secondary representations [8], [9], [10], [12]. Further, an R-module M has a secondary representation if it is Artinian [8], [9], [10], [12], or injective [18]. In this section we define and give some properties of attached primes of arbitrary R-modules.

If M is an R-module, a prime ideal p of R is said to be *attached* to M if $p = (Q: M)$ for some submodule Q of M. We denote the set of attached primes of M by Att (M). This definition agrees with the usual definition of attached prime if M has a secondary representation [9, Theorem 2.5].

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1.1. LEMMA. *Let M be an R-module.*

(i) $M = 0 \Leftrightarrow \text{Att}(M) = \emptyset$.
(ii) $\cup \text{Att}(M) = \{x \in R \mid xM \neq M\}$.
(iii) *If N is a submodule of M, then* $\text{Att}(M/N) \subseteq \text{Att}(M)$. *Further, if one of the following conditions holds, then* $\text{Att}(M) \subseteq \text{Att}(N) \cup \text{Att}(M/N)$:

  (a) *M has a secondary representation.*
  (b) $\text{Att}(M)$ *consists of maximal ideals.*
  (c) *M is finitely generated.*

Proof. It follows easily that an ideal p of R which is maximal among $\{(Q: M) \mid Q \neq M \text{ is a submodule of } M\}$ is prime, so (i) holds since R is Noetherian. (ii) and the first part of (iii) are immediate from the definition of $\text{Att}(M)$. (iii)(a) follows from [9, Theorems 2.5, 4.1]. As for (b), let $p \in \text{Att}(M)$, say $p = (Q: M)$, Q a submodule of M. If $N + Q = M$, then

$$M/Q = (N + Q)/Q \cong N/(N \cap Q)$$

so $p \in \text{Att}(N)$. If $Q + N \neq M$, then we have $p = (Q: M) \subseteq [(Q + N): M] \neq R$. Thus since p is maximal, $p = (Q + N): M$ and hence $p \in \text{Att}(M/N)$.

Part (c) follows from part (b) since if M is finitely generated and S is a multiplicative subset of R,

$$\text{Att}(S^{-1}M) = \{S^{-1}p \mid p \in \text{Att}(M) \quad \text{and} \quad p \cap S = \emptyset\}$$

as is easily seen.

1.2. LEMMA. *If M is an R-module with* $\text{Att}(M) = \{p\}$ *where p is a minimal prime of R, then M is secondary.*

Proof. It suffices to show that if $x \in p$ then $x^n M = 0$ for some integer $n \geq 1$. But since p is minimal, pR_p is the nilradical of R_p, so there exists $s \in R - p$ and an integer $n \geq 1$ such that $sx^n = 0$. But $s \notin p \Rightarrow sM = M$ and hence $x^n M = x^n s M = 0$.

2. An important application of the theory of attached primes and secondary representations has been to local cohomology modules of finitely generated R-modules [10], [17]. The following result is a generalization of [10, Theorem 2.2] in the case that $\dim M = \dim R$, to R-modules which may not be finitely generated. This applies in particular to the case that M is a maximal Cohen-Macaulay module in which case $d = \dim M = \dim R$ is the unique integer j such that $H^j_m(M) \neq 0$ [3, Lemma 2.1].

2.1. THEOREM. *If M is an R-module with* $\dim M = \dim R = d$, *then $H^d_m(M)$ has a secondary representation and* $\text{Att}\, H^d_m(M) \subseteq \{p \in \text{Ass}(M) \mid \dim R/p = d\}$.

Proof. Let $X = \{p \in \text{Ass}(M) \mid \dim R/p = d\}$ and assume $H_m^d(M) \neq 0$. There exists a submodule N of M such that $\text{Ass}(M/N) = X$ and $\text{Ass}(N) = \text{Ass}(M) - X$ [2, p. 263, Proposition 4]. We get an exact sequence

$$H_m^d(N) \to H_m^d(M) \to H_m^d(M/N) \to H_m^{d+1}(N)$$

and the two modules on the end are zero [15, Theorem 6.1]. Thus $H_m^d(M) \cong H_m^d(M/N)$ and so by considering M/N instead of M, we may assume $\text{Ass}(M) = X$. But then if $x \notin \cup X$ then x is M-regular implies

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

is exact which implies

$$H_m^d(M) \xrightarrow{x} H_m^d(M) \to H_m^d(M/xM)$$

is exact, and since $\dim(M/xM) < d$, $H_m^d(M/xM) = 0$ [15. Theorem 6.1]. This gives $xH_m^d(M) = H_m^d(M)$ and so $x \notin \cup \text{Att}\, H_m^d(M)$. Therefore

$$\cup\, \text{Att}_m^d(M) \subseteq \cup X.$$

But since X is finite, if $p \in \text{Att}\, H_m^d(M)$ then $p \subseteq q$ for some $q \in X$, and hence $p = q \in X$. This shows $\text{Att}\, H_m^d(M) \subseteq X$.

To show that $H_m^d(M)$ has a secondary representation, let

$$X = \{p_1, \ldots, p_n\}.$$

If $n = 1$ then $H_m^d(M)$ is p_1-secondary by the first part of the proof and Lemma 1.2, so we may assume $n > 1$. Let L_i be a submodule of M with $\text{Ass}(L_i) = \{p_i\}$ and $\text{Ass}(M/L_i) = X - \{p_i\}$. Thus by the first part of the proof we have

$$\text{Att}\, H_m^d(L_i) \subseteq \{p_i\} \quad \text{and} \quad \text{Att}\,(H_m^d(M/L_i) \subseteq \text{Ass}(M) - \{p_i\}.$$

Thus $H_m^d(L_i)$ is p_i-secondary or zero, and in the exact sequence

$$H_m^d(L_i) \xrightarrow{\phi} H_m^d(M) \to H_m^d(M/L_i) \to 0,$$

$\phi H_m^d(L_i)$ is p_i-secondary or zero and $H_m^d(M)/\phi H_m^d(L_i) \cong H_m^d(M/L_i)$. Therefore

$$\text{Att}\left[H_m^d(M) \Big/ \sum_{i=1}^n \phi H_m^d(L_i)\right] \subseteq \cap\, \text{Att}\,[H_m^d(M)/\phi H_m^d(L_i)] = \cap\, \text{Att}\,[H_m^d(M/L_i)] = \theta.$$

Thus $H_m^d(M) = \sum_{i=1}^n \phi H_m^d(M)$. Q.E.D.

It can happen that $H_m^d(M) = 0$ where $\dim M = \dim R = d$. For example if $p \in \text{Spec}(R)$ with $\dim R/p = \dim R > 0$, the injective envelope $E = E(R/p)$ of R/p has $\dim E = d$ and $H_m^d(E) = 0$.

Some properties of R-modules which have secondary representations are given in [9]. For example, applying [9, Corollary 2.8] we get the following.

2.2. COROLLARY. *Let M be as in the above theorem, and let I be an ideal of R with $IH_m^d(M) = H_m^d(M)$. Then $xH_m^d(M) = H_m^d(M)$ for some $x \in I$.*

3. Our first three results in this section are similar to results in [13] where it was assumed that R is complete. An R-module M is said to have a basic submodule if M is separated in the m-adic topology and has a pure free submodule F such that $F + m^n M = M$ for all $n > 0$.

3.1. THEOREM. *Let $(x) = (x_1, \ldots, x_d)$ be a system of parameters of R. If there exists an (x)-regular R-module M such that the submodules 0, $x_1 M$, $(x_1, x_2)M, \ldots, (x_1, \ldots, x_d)M$ are closed submodules of M in the m-adic topology. Then* $\dim R/p = \dim R$ *for every $p \in$ Ass (M).*

Proof. We use induction on $d = \dim R$, the assertion being clear for $d = 0$. Assume $d > 0$. Since x_1 is M-regular and M is a separated R-module, it follows as in [11, p. 98, Lemma 1] that $p + x_1 R \subseteq q$ for some $q \in$ Ass $(M/x_1 M)$. But then by the induction hypothesis, $\dim R/q = \dim R/x_1 R = d - 1$, and hence $\dim R/p = d$.

3.2. COROLLARY. *Let M be an R-module as in the above theorem. If M has a basic submodule, then R is Cohen-Macaulay.*

Proof. Since M has a basic R-submodule F, we have Ass $(R) =$ Ass $(F) \subseteq$ Ass (M). If $d = 0$ the result is clear; so assume $d > 0$. Then since $M/x_1 M$ is separated, it follows that $F/x_1 F$ is a basic $R/x_1 R$-submodule of $M/x_1 M$. Using induction on d we have $R/x_1 R$ Cohen-Macaulay. But since x_1 is a regular element of R by Theorem 3.1, then R is Cohen-Macaulay [7, Theorem 156].

3.3. COROLLARY. *If the module M in Theorem* 3.1 *is R-flat (or equivalently has finite projective dimension and is (x)-regular for every system of parameters (x) of R), then R is Cohen-Macaulay.*

Proof. This follows from the above corollary and the result [13, Proposition 3] which says that M is R-flat if and only if M has finite projective dimension and is (x)-regular for every system of parameters of R, and if this holds, M has a basic submodule.

In [16, Theorem 2.4] it was shown that a finitely generated R-module M is Cohen-Macaulay if and only if the Cousin complex $C(M)$ of M is exact. For nonfinitely generated modules we have the following two results.

3.4. THEOREM. *If M is (x)-regular for every system of parameters (x) of R, then the Cousin complex $C(M)$ of M is exact.*

Proof. By [16, Proposition 2.1] it suffices to show that for every $p \in$ Supp (M), $\mathrm{Ext}_R^i(R/p, M) = 0$ whenever $i < ht_M p$. Let $h = ht_M p$. Then $h \leq ht_R p$. Let $x_1, \ldots, x_d$ be a system of parameters of R with $x_1, \ldots, x_h \in p$.

Then for $i < h$ we have

$$\mathrm{Ext}_R^i\ (R/p, M) \cong \mathrm{Hom}_R\ (R/p, M/(x_1, \ldots, x_i)M) = 0$$

[7, p. 101].

3.5. THEOREM. *If M is an R-module with $mM \neq M$ and* $\dim M = \dim R$ *whose Cousin complex $C(M)$ is exact, then M is (x)-regular for some system of parameters (x) of R.*

Proof. It suffices to show that $\mathrm{Ext}_R^i\ (k, M) = 0$ for $i < d = \dim R$ by [3, Corollary 2.2]. But by the partial exact Cousin complex argument [14, Lemma 4.6] $\mathrm{Ext}_R^i\ (k, M) = 0$ if $i < d$ and $\mathrm{Ext}_R^d\ (k, M) \cong \mathrm{Hom}_R(k, M^d)$.

3.6. *Remark.* If M is an R-module which is (x)-regular for every system of parameters (x) of R and $p \in \mathrm{Supp}\ (M)$ is such that $pM_p \neq M_p$, then

$$\mu^i(p, M) = 0 \quad \text{for } i < htp$$

(where $\mu^i(p, M) = \dim_{k(p)} \mathrm{Ext}_R^i\ (R/p, M)_p$ [1]).

Proof. Since $pM_p \neq M_p$ it follows that M_p is a maximal Cohen-Macaulay R_p-module. Thus we have that $\mu^i(pR_p, M_p) = 0$ for $i < htp$. But $\mu^i(pR_p, M_p) = \mu^i(p, M)$ [1, Corollary 2.4] and so the result holds. Q.E.D.

In [19] the Cousin complex of an R-module M was said to *vanish early* if $M^j = 0$ for some $j < \dim M$, and it was shown that if M is finitely generated then $C(M)$ does not vanish early.

3.7. *Remark.* If M is a maximal Cohen-Macaulay R-module with $mM \neq M$, then the Cousin complex $C(M)$ does not vanish early.

Proof. By [3, Lemma 2.1], $H_m^d(M) \neq 0$ where $d = \dim M$, and $H_m^d(M) \cong M^d$ by [19, Theorem] (where the Cousin complex for M is $C(M)\colon 0 \to M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \to \cdots \to M^n \to \cdots$). It then follows from [14, Proposition 2.7(ii)] that $M^j \neq 0$ for $0 \leq i \leq d$.

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