ILLINOIS JOURNAL OF MATHEMATICS
Volume 24, Number 4, Winter 1980

RANDOM FOURIER SERIES FOR CENTRAL FUNCTIONS ON COMPACT LIE GROUPS

BY

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1. Introduction

The theory of random Fourier series of functions defined on the unit circle Π is based on the following two results, due originally to Littlewood (cf. [12, V (8.4)]).

(I) Suppose that for any choice of numbers $\beta_n \in \{1, -1\}$ $\sum \beta_n a_n e^{inx}$ is a Fourier-Stieljes series. Then $\sum |a_n|^2 < \infty$.

(II) Suppose $\sum |a_n|^2 < \infty$. Then for any $p < \infty$, there exist coefficients $\beta_n \in \{1, -1\}$ such that $\sum \beta_n a_n e^{inx}$ is the Fourier series of a function in $L^p(\Pi)$.

Powerful extensions of these results to arbitrary compact groups are due to Figa-Talamanca and Rider [6], [7]. In this case, $f \in L^1(G)$ is uniquely represented by its Fourier series

$$f(x) \sim \sum_{\sigma \in \hat{G}} d_\sigma \operatorname{tr}(A_\sigma \sigma(x)) \quad \text{where } A_\sigma = \int_G f(x)\sigma(x^{-1})\,dx.$$

Here, $\hat{G}$ denotes a maximal set of pairwise inequivalent representations of G.

(I′) Suppose that there exists a set M of positive Haar measure in $\mathscr{G} = \prod_{\sigma \in \hat{G}} \mathscr{U}(d_\sigma)$ such that for $\mathbf{U} = (U_\sigma)_{\sigma \in \hat{G}} \in M$, $\sum_{\sigma \in \hat{G}} d_\sigma \operatorname{tr}(U_\sigma A_\sigma \sigma(x))$ is a Fourier-Stieljes series then $\sum d_\sigma \operatorname{tr}(A_\sigma A_\sigma^*) < \infty$.

(II′) If $\sum_{\sigma \in \hat{G}} d_\sigma \operatorname{tr}(A_\sigma A_\sigma^*) < \alpha$, then for almost every $\mathbf{U} \in \mathscr{G}$,

$$\sum_{\sigma \in \hat{G}} d_\sigma \operatorname{tr}(U_\sigma A_\sigma \sigma(x))$$

is the Fourier series of a function in $\bigcap_{p<\infty} L^p(G)$.

Here, for $\sigma \in \hat{G}$, d_σ denotes the dimension, and $\mathscr{U}(d_\sigma)$ denotes the $d_\sigma \times d_\sigma$ unitary group.

If G is abelian, the group $\mathscr{G} = \prod_{\sigma \in \hat{G}} \mathscr{U}(d_\sigma) = \prod_{\sigma \in \hat{G}} \Pi$ may be replaced by the group $\prod_{\sigma \in \hat{G}} \{1, -1\}$; Figa Talamanca [5] has shown that in the nonabelian case, we may analogously replace $\mathscr{G}$ by $\prod_{\sigma \in \hat{G}} O(d_\sigma)$, where $O(d_\sigma)$ is the real $d_\sigma \times d_\sigma$ orthogonal group. It has been known for some time that one cannot replace $\Pi O(d_\sigma)$ by certain smaller groups (in particular (cf. [5, Section 3]) the

Received September 26, 1978.

product of the groups of orthogonal $d_\sigma \times d_\sigma$ matrices with entries ± 1 does not work).

In this article, I am concerned with functions in the convolution centre of L^1—such a function has Fourier series

$$f(x) \sim \sum_{\sigma \in \hat{G}} d_\sigma a_\sigma \chi_\sigma(x),$$

where

$$\chi_\sigma(x) = \operatorname{tr}(\sigma(x)) \quad \text{and} \quad a_\sigma = \frac{1}{d_\sigma} \int_G f(x)\chi_\sigma(x^{-1})\, dx.$$

The main theorems are:

THEOREM 1. *Let G be a compact connected Lie group, and let $(a_\sigma)_{\sigma \in \hat{G}}$ be a sequence of complex numbers. Suppose that there exists a set M of positive measure in $\mathscr{H} = \prod_{\sigma \in \hat{G}} \{1, -1\}$ such that if $\beta = (\beta_\sigma)_{\sigma \in \hat{G}} \in M$, $\sum_{\sigma \in \hat{G}} d_\sigma \beta_\sigma a_\sigma \chi_\sigma(x)$ is a Fourier-Stieljes series. Then $\sum_{\sigma \in \hat{G}} d_\sigma^2 |a_\sigma|^2 < \infty$.*

THEOREM 2. *Let G be a compact connected Lie group and suppose that $(a_{\sigma \in \hat{G}}$ is a sequence of complex numbers such that $\sum_{\sigma \in \hat{G}} d_\sigma^2 |a_\sigma|^2 < \infty$. Then for almost all $\beta = (\beta_\sigma) \in \mathscr{H}$,*

$$\sum_{\sigma \in G} d_\sigma \beta_\sigma a_\sigma \chi_\sigma$$

is the Fourier series of a function in

$$\bigcap_{p < 2 + \varepsilon_G} L^p \quad \textit{where } \varepsilon_G = \frac{2 \operatorname{rank} G}{\dim G - \operatorname{rank} G}.$$

These theorems do not hold for arbitrary compact groups; they have a number of interesting consequences for multipliers and lacunarity, some of which are discussed in Section 3.

2. Proof of the theorems

Our first lemmas state the situation for general compact groups, and isolate the "randomness" techniques.

LEMMA 3. *Let G be a compact group, and let $(a_\sigma)_{\sigma \in \hat{G}}$ be a sequence of complex numbers. Suppose that there exists a set M of positive Haar measure in $\mathscr{H} = \prod_{\hat{G}} \{-1, 1\}$ such that if $\beta = (\beta_\sigma) \in M$, $\sum_{\sigma \in \hat{G}} d_\sigma \beta_\sigma a_\sigma \chi_\sigma$ is the Fourier series of an integrable function. Then*

$$\int_G \left(\sum_{\sigma \in \hat{G}} d_\sigma^2 |a_\sigma|^2 |\chi_\sigma(x)|^2 \right)^{1/2} dx < \infty. \tag{1}$$

Proof. A standard Baire category argument shows that there exists $K \in \mathbf{R}$ such that for all $(\beta_\sigma) \in M$,

$$\int_G \left| \sum_{\sigma \in \hat{G}} d_\sigma \beta_\sigma a_\sigma \chi_\sigma(x) \right| dx < K.$$

Integrating over M and using Fubini's theorem, we obtain

$$\int_G \int_M \left| \sum_{\sigma \in \hat{G}} d_\sigma \beta_\sigma a_\sigma \chi_\sigma(x) \right| d\beta \, dx < K\lambda_{\mathscr{H}}(M). \tag{2}$$

However [8, (36, 2)] the β_σ are a $\Lambda(2)$ set, so that we have

$$\left(\int_M \left| \sum_{\sigma \in \hat{G}} d_\sigma \beta_\sigma a_\sigma \chi_\sigma(x) \right|^2 d\beta \right)^{1/2} \leq 2^{1/2} \int_M \left| \sum_{\sigma \in \hat{G}} d_\sigma \beta_\sigma a_\sigma \chi_\sigma(x) \right| d\beta \tag{3}$$

It is shown in [12, V 8.3] that for any $t > 1$, there is a finite set $F \subseteq \hat{G}$ such that

$$\int_M \left| \sum_{\sigma \in \hat{G} \backslash F} d_\sigma \beta_\sigma a_\sigma \chi_\sigma(x) \right|^2 d\beta \geq \frac{\lambda_{\mathscr{H}}(M)}{t} \sum_{\sigma \in \hat{G} \backslash F} \left| d_\sigma a_\sigma \chi_\sigma(x) \right|^2. \tag{4}$$

Now

$$\int_G \left(\sum_{\sigma \in \hat{G}} \left| d_\sigma a_\sigma \chi_\sigma(x) \right|^2 \right)^{1/2} dx$$
$$\leq \int_G \left(\sum_{\sigma \in F} \left| d_\sigma a_\sigma \chi_\sigma(x) \right|^2 \right)^{1/2} dx + \int_G \left(\sum_{\sigma \in \hat{G} \backslash F} \left| d_\sigma a_\sigma \chi_\sigma(x) \right|^2 \right)^{1/2} dx$$

The first summand on the right hand side of the above expression is the integral of a continuous function and hence is bounded. But (2), (3) and (4) combine to show that the second expression on the right hand side is bounded by

$$\left(\frac{2t}{\lambda_{\mathscr{H}}(M)} \right)^{1/2} K\lambda_{\mathscr{H}}(M). \qquad \blacksquare$$

LEMMA 4. *Let G be any compact group, and let $(a_\sigma)_{\sigma \in \hat{G}}$ be a sequence of complex numbers such that*

$$\int_G \left(\sum_{\sigma \in \hat{G}} |d_\sigma a_\sigma \chi_\sigma(x)|^2 \right)^{p/2} dx < \infty. \tag{5}$$

Then for almost all $\beta = (\beta_\sigma) \in \mathscr{H}$, $\sum_{\sigma \in \hat{G}} d_\sigma \beta_\sigma a_\sigma \chi_\sigma$ is the Fourier series of a function in $L^p(G)$.

Proof. Since [6] the projections form a $\Lambda(p)$ set in $\hat{\mathscr{H}}$, there is a constant $B(p)$ such that for all $x \in G$,

$$\int_{\mathscr{H}} \left| \sum_{\sigma \in \hat{G}} d_\sigma a_\sigma \chi_\sigma(x) \beta_\sigma \right|^p d\beta \leq B(p) \left(\int_{\mathscr{H}} \left| \sum_{\sigma \in \hat{G}} d_\sigma a_\sigma \chi_\sigma(x) \beta_\sigma \right|^2 d\beta \right)^{p/2}$$
$$= B(p) \left(\sum_{\sigma \in \hat{G}} |d_\sigma a_\sigma \chi_\sigma(x)|^2 \right)^{p/2}$$

Thus

$$\int_{\mathscr{H}}\int_G\left|\sum_{\sigma\in\hat{G}} d_\sigma a_\sigma \chi_\sigma(x)\beta_\sigma\right|^p dx\,d\beta \le B(p)\int_G\left(\sum_{\sigma\in\hat{G}}|d_\sigma a_\sigma\beta_\sigma\chi_\sigma(x)|^2\right)^{p/2} dx<\infty.$$

Hence for almost all $\beta\in\mathscr{H}$, $\int_G|\sum_{\sigma\in\hat{G}} d_\sigma\beta_\sigma a_\sigma\chi_\sigma(x)|^p\,dx<\infty$. ∎

Theorems 1 and 2 will result from an examination of conditions (1) and (5) using the Weyl integration formula (cf. [11]): If G is a compact connected Lie group and T is a maximal torus for G, then for any central f, we have

$$\int_G f(x)\,dx=\frac{1}{|W|}\int_T f(t)|q(t)|^2\,dt,$$

where W denotes the Weyl group, and q is a certain trigonometric polynomial on T. It is shown in [1] that $\int_T|q(t)|^{-r}\,dt<\infty$ provided that

$$r<\varepsilon_G=\frac{2\operatorname{rank} G}{\dim G-\operatorname{rank} G}.$$

I shall also need the Weyl character formula—associated to $\sigma\in\hat{G}$ there is a unique character $\chi^{(\sigma)}\in\hat{T}$ such that $q\cdot\chi_\sigma|_T=\sum_{w\in W} w\chi^{(\sigma)}\chi_{(w)}$, where the $\chi_{(w)}$ are certain fixed elements of $\hat{T}$ (independent of σ).

Proof of Theorem 2. To prove the theorem, we first show that if $\sum d_\sigma^2|a_\sigma|^2<\infty$, then (a_σ) satisfies (5) for all $2<p<2+\varepsilon_G$. So suppose $p<2+\varepsilon_G$. Then

$$\int_G\left(\sum_{\sigma\in\hat{G}}|d_\sigma a_\sigma\chi_\sigma(x)|^2\right)^{p/2} dx=\frac{1}{|W|}\int_T\left(\sum_{\sigma\in\hat{G}}|d_\sigma a_\sigma q(t)\chi_\sigma(t)|^2\right)^{p/2}|q(t)|^{2-p}\,dt.$$

We apply Hölder's inequality with $1<r<\varepsilon_G/(p-2)$ to see that this expression is dominated by

$$\frac{1}{|W|}\left(\int_T\left(\sum_{\sigma\in\hat{G}}|d_\sigma a_\sigma q(t)\cdot\chi_\sigma(t)|^2\right)^{pr'/2} dt\right)^{1/r'}\left(\int_T|q(t)|^{-r(p-2)}\,dt\right)^{1/r}.$$

Since $r(p-2)<\varepsilon_G$, it suffices to show that $\sum_{\sigma\in\hat{G}}|d_\sigma a_\sigma q\cdot\chi_\sigma|^2\in L^q(T)$ for all $q<\infty$; in fact, this function is in $L^\infty(T)$, for by the Weyl character formula $\|q\cdot\chi_\sigma\|_\infty\le|W|$; thus

$$\left\|\sum|d_\sigma a_\sigma q\cdot\chi_\sigma|^2\right\|_\infty\le|W|^2\sum d_\sigma^2|a_\sigma|^2.$$

It remains to apply lemma 4 to deduce that for almost all $\beta\in\mathscr{H}$,

$$\sum d_\sigma a_\sigma\beta_\sigma\chi_\sigma\in L^p(G);$$

let M_p be the set of measure zero in $\mathscr{H}$ for which $\sum d_\sigma a_\sigma\beta_\sigma\chi_\sigma\notin L^p(G)$ and choose any strictly increasing sequence $p_i\to 2+\varepsilon_G$. For $\beta\notin\bigcup_i M_{p_i}$, $\sum d_\sigma a_\sigma\beta_\sigma\chi_\sigma$ is a member of $L^{p_i}(G)$ for all i, and hence belongs to $\bigcap_{p<2+\varepsilon_G} L^p(G)$. ∎

Proof of Theorem 1. To prove Theorem 1, I will show that if

$$\int_G \left(\sum_{\sigma \in \hat{G}} |d_\sigma a_\sigma \chi_\sigma(x)|^2 \right)^{1/2} dx < \infty$$

then $\sum d_\sigma^2 |a_\sigma|^2 < \infty$.

Again, I apply the Weyl integration formula to write

$$\int_G \left(\sum_{\sigma \in \hat{G}} |d_\sigma a_\sigma \chi_\sigma(x)|^2 \right)^{1/2} dx = \frac{1}{|W|} \int_T \left(\sum_{\sigma \in \hat{G}} |d_\sigma a_\sigma q(t) \chi_\sigma(t)|^2 \right)^{1/2} |q(t)| \, dt.$$

Since $|q|$ is nonzero almost everywhere, this certainly implies that

$$\sum_{\sigma \in \hat{G}} |d_\sigma a_\sigma q(t) \chi_\sigma(t)|^2 < \infty$$

for almost every $t \in T$. From this, it follows that for some set E of positive λ_T measure and for some $M > 0$, we have

$$\sum_{\sigma \in \hat{G}} |d_\sigma a_\sigma q(t) \chi_\sigma(t)|^2 < M \quad \text{for all } t \in E,$$

and integrating over E, I obtain

$$\sum_{\sigma \in \hat{G}} d_\sigma^2 |a_\sigma|^2 \int_E |q(t)\chi_\sigma(t)|^2 \, dt < M\lambda_T(E). \tag{6}$$

An application of the Weyl character formula enables us to write

$$|q(t)\chi_\sigma(t)|^2 = \sum_{w_0 \in W} w_0 \left(\sum_{w \in W} \operatorname{sgn} w \chi^{(\sigma)} \overline{w \chi^{(\sigma)}} \chi_{(w)} \right).$$

Since the action of W on T preserves the Haar measure, we may suppose that E is invariant under the action of W, so (6) becomes

$$\sum_{\sigma \in \hat{G}} d_\sigma^2 |a_\sigma|^2 \int_E \sum_{w \in W} \operatorname{sgn} w \chi^{(\sigma)}(t) \overline{w \chi^{(\sigma)}}(t) \chi_{(w)}(t) \, dt < \frac{M\lambda_T}{|W|}(E). \tag{7}$$

The remainder of the proof consists in finding $\delta > 0$, such that for σ outside some finite set,

$$\left| \int_E \sum_{w \in W} \operatorname{sgn} w \chi^{(\sigma)}(t) \overline{w \chi^{(\sigma)}}(t) \chi_{(w)}(t) \, dt \right| \geq \delta. \tag{8}$$

For $G = \mathscr{SU}(2)$, $T = \mathbf{T}$, $\sigma = \sigma_n$ (the representation of dimension n), we have

$$\sum_{w \in W} \operatorname{sgn} w \chi^{(\sigma_n)}(t) \overline{w \chi^{(\sigma_n)}(t)} \chi_{(w)}(t) = 1 - e^{2int},$$

and (8) results from the Riemann-Lebesgue lemma. I generalize this proof to an arbitrary compact connected Lie group.

To reduce the technicality of the proof somewhat, I shall assume that G is simply connected. To see that this assumption is justified, recall that we can write $G = A \times G_1$, where $G_1 = [G, G]$ is semisimple, and A is abelian. A maximal torus for G is then $A \times T_1$, where T_1 is a maximal torus for G_1. The Weyl group of G is the same as the Weyl group of G_1 and has the trivial action on A. Thus it suffices to consider G semisimple. Now every compact connected semisimple Lie group G_1 is finitely covered by a compact simply connected Lie group, $\tilde{G}_1$. A maximal torus for G_1 is the image under the quotient map of a maximal torus $\tilde{T}_1$ for $\tilde{G}_1$ (the kernel of the quotient map is contained in $\tilde{T}_1$). Again the Weyl groups of G_1 and $\tilde{G}_1$ are identical, the action on T_1 being the quotient of the action on $\tilde{T}_1$.

Since G is supposed simply connected, there exist characters $\chi_1, \ldots, \chi_l$ of T so that the characters $\chi^{(\sigma)}$ are precisely the characters of the form $\prod_{j=1}^{l} \chi_j^{n_j^\sigma}$, $n_j^\sigma \in \mathbf{N}$. We write $\sigma = \sigma_{(n_1, \ldots, n_l)}$ if $\chi^{(\sigma)}$ has this form.

Now, for $w \in W$, let $\chi_{w,j} = \chi_j w\chi_j$. Then we have $\chi_w = \prod_{j=1}^{n} \chi_{w,j}$; hence for $\sigma = \sigma_{(n_1, \ldots, n_l)}$,

$$\chi^{(\sigma)}\overline{w\chi^{(\sigma)}}\chi_{(w)} = \prod_{j=1}^{l} \chi_j^{n_j+1}\overline{w\chi_j^{n_j+1}} = \prod_{j=1}^{l} \chi_{w,j}^{n_j+1} \tag{9}$$

Note that if $w \neq e$, then at least one of the $\chi_{w,j}$ is a nontrivial character.

I shall now prove:

LEMMA.

$$\inf_{(n_1, \ldots, n_l) \in \mathbf{N}^{+l}} \left| \int_E \sum_{w \in W} \operatorname{sgn} w \prod_{j=1}^{l} \chi_{w,j}^{n_j}(t)\, dt \right| > 0. \tag{10}$$

By (8) and (9), this will be sufficient to conclude the proof of the theorem.

Proof of the lemma. I shall prove (10) by an inductive argument.

Let $W_s = \{w \in W: \chi_{w,j} = 1 \text{ if } j > s\}$, $s = 0, \ldots, l$. Then $W_0 = \{e\}$ and $W_l = W$. My induction asserts that

$$\inf_{\mathbf{n} \in \mathbf{N}^{+l}} \left| \int_E \sum_{w \in W_s} \operatorname{sgn} w \prod_{j=1}^{l} \chi_{w,j}^{n_j}(t)\, dt \right| > 0.$$

Since $\chi_{e,1} = 1$ for all j, the first step is trivial. Thus suppose that $0 < s < l$, and that

$$\inf_{(n_1, \ldots, n_l) \in \mathbf{N}^{+l}} \left| \int_E \sum_{w \in W_s} \operatorname{sgn} w \prod_{j=1}^{l} \chi_{w,j}^{n_j}(t)\, dt \right| > \delta_s$$

Then for $(n_1, \ldots, n_l) \in \mathbf{N}^{+l}$,

$$\begin{aligned} &\left| \int_E \sum_{w \in W_{s+1}} \operatorname{sgn} w \prod_{j=1}^{l} \chi_{w,j}^{n_j}(t)\, dt \right| \\ &\quad = \left| \int_E \sum_{w \in W_s} \operatorname{sgn} w \prod_{j=1}^{l} \chi_{w,j}^{n_j}(t)\, dt + \int_E \sum_{w \in W_{s+1} \setminus W_s} \operatorname{sgn} w \prod_{j=1}^{l} \chi_{w,j}^{n_j}(t)\, dt \right| \\ &\quad \geq \delta_s - \left| \int_E \sum_{w \in W_{s+1} \setminus W_s} \operatorname{sgn} w \prod_{j=1}^{l} \chi_{w,j}^{n_j}(t)\, dt \right| \end{aligned}$$

Now by the Riemann-Lebesgue lemma, there exists N such that for any $w \in W_{s+1} \setminus W_s$ and for any $(n_1, \ldots, n_l)$ with $n_k > N$ for some $k \leq s+1$,

$$\left| \int_E \prod_{j=1}^{l} \chi_{w,j}^{n_j}(t)\, dt \right| < \frac{\delta_s}{|W|}.$$

Thus, if $(n_1, \ldots, n_l)$ satisfies $n_k \geq N$ for some $k \leq s+1$, we have

$$\left| \int_E \sum_{w \in W_{s+1}} \operatorname{sgn} w \prod_{j=1}^{l} \chi_{w,j}^{n_j}(t)\, dt \right| \geq \delta_s \left(1 - \frac{|W_{s+1} - W_s|}{|W|} \right) > 0$$

It will thus suffice to show that there exists $\delta'_{s+1} > 0$ such that

$$\left| \int_E \sum_{w \in W_{s+1}} \operatorname{sgn} w \prod_{j=1}^{s+1} \chi_{w,j}^{n_j}(t)\, dt \right| \geq \delta'_{s+1}$$

provided that $n_j \leq N$ for $j = 1, \ldots, s+1$. But

$$\left\{ \sum_{w \in W_{s+1}} \operatorname{sgn} w \prod_{j=1}^{s+1} \chi_{w,j}^{n_j} \colon n_j \leq N \quad j = 1, \ldots, s+1 \right\}$$

is a finite set of trigonometric polynomials; each element of this set must have nonzero integral over E since it is an integral linear combination of distinct characters, one of which is the identity character.

This completes the proof of the Lemma and hence the theorem is proved. ∎

3. Remarks

(a) Theorems 1 and 2 remain valid if we replace $\prod_{\sigma \in \hat{G}} \{1, -1\}$ by $\prod_{\sigma \in \hat{G}} \mathbf{T}$. (To see this, we note that the randomness arguments of Lemmas 3 and 4 are valid for this group.)

(b) If we let ZL^p (resp. $Z\mathbf{M}$) denote the convolution center of L^p (resp. $\mathbf{M}$, the space of measures of G), and $\mathbf{ZE}^q(\hat{G})$ (resp. $\mathbf{ZE}_0(\hat{G})$) the center of the space $\mathbf{E}^q(\hat{G})$ of [8, (28.34)] then it is a consequence of Theorem 1 that if G is a connected Lie group, the set of multipliers $\mathscr{M}(\mathbf{ZE}_0, Z\mathbf{M})$ is equal to $\mathbf{ZE}^2$. (The notation $\mathscr{M}(\quad)$, is explained in [8, (35.1)]).

(c) It is not hard, using (b) together with the argument of [8, (36.15)] to see that if $1 \leq p < 2$, then $\mathscr{M}(ZC(G), \mathbf{ZE}^p(\hat{G})) = \mathbf{ZE}^{2p/(2-p)}(\hat{G})$.

(d) A *central p-Sidon* set is a subset R of $\hat{G}$ for which $ZC_R(G) \subseteq \mathbf{ZE}^p(G)$, where $ZC_R(G)$ denotes the convolution center of the space of continuous functions whose Fourier transform is zero off R. These sets have been studied in [2], [3].

LEMMA 5. *If G is a compact group for which Theorem* 1 *holds then:*

(i) *Every central Sidon* ($=$ 1-*Sidon*) *set satisfies* $(ZL_R^1) \subseteq \mathbf{E}^2$ (*i.e. is central* $\Lambda(2)$).

(ii) *Every central p-Sidon set satisfies* $(ZL_R^1) \subseteq \mathbf{E}^{2p/(2-p)}(\hat{G})$ (*i.e. is of type central* $V(\infty, 2p/(3p-2))$ *in the sense of* [2]).

Proof. Since (i) is the case $p = 1$ of (ii), I prove only (ii). Since Theorem 1 holds for G, we have $\mathcal{M}(ZC, \mathbf{ZE}^p) = \mathbf{ZE}^{2p/(2-p)}(\hat{G})$.

Suppose $f \in ZL^1_R(G)$. Then for any $g \in ZC(G)$, $f * g \in ZC_R(G)$. Thus since R is central p-Sidon, $f * g \in \mathbf{ZE}^p(G)$. We have shown that

$$f \in \mathcal{M}(ZC, \mathbf{ZE}^p) = \mathbf{ZE}^{2p/(2-p)}(\hat{G}),$$

and so we have finished. ∎

Example 6. The following construction generalizes that of [10, Example 1] to give a class of compact groups for which Theorem 1 does not hold.

Let H be any compact group, and suppose that H has an irreducible representation η such that $|\chi_\eta|$ is not constant. (Almost every compact nonabelian group has such a representation.) Let $G_n = H \times \cdots \times H$ (n times), and let $G = \prod_{n=1}^\infty G_n$. Then $\eta_n = \eta \otimes \cdots \otimes \eta$ is an irreducible representation of G_n, and if $\pi_n\colon G \to G_n$ is the projection, then $\sigma_n = \eta_n \circ \pi_n$ is an irreducible representation of G. It is shown by Parker [9] that $R = \{\sigma_n\colon n \in \mathbf{N}\}$ is a central Sidon set in $\hat{G}$. On the other hand, if R were a central $\Lambda(2)$ set, there would exist a constant K so that $K \leq \|\chi_{\sigma_n}\|_1$. Now $\|\chi_{\sigma_n}\|_1 = \|\chi_\eta\|_1^n$ and since $|\chi_\eta|$ is nonconstant, we have $\|\chi_\eta\|_1 < \|\chi_\eta\|_2 = 1$, and this is a contradiction.

We may choose H connected, so that G is connected.

The following theorem was proved in [4] for $p = 1$, but is new for $p > 1$ (cf. also [3]).

COROLLARY 7. *If G is a compact connected group, then $\hat{G}$ is not a central p-Sidon set for any $p < 2$.*

Proof. By the well known structure theorem for compact connected groups, we may write $G = G_1 \times G_2$ where G_1 is a compact connected Lie group. If $\hat{G}$ is central p-Sidon, then so is $\hat{G}_1$. But this implies, by (ii) of Lemma 5, that $\hat{G}_1$ is of type central $V(\infty, 2p/(3p-2))$, which contradicts [2, (6.12)]. ∎

I would like to thank Professors Robert Edwards and Garth Gaudry, and Dr. John Price for helpful discussions concerning this work.

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