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A UNIQUENESS THEOREM FOR SUBMANIFOLDS OF EUCLIDEAN SPACE

BY
JOEL L. WEINER

A uniqueness theorem for a submanifold of Euclidean space gives sufficient conditions in order for the submanifold to be determined up to a rigid motion. Well-known examples are the theorems of Minkowski [5] and Christoffel [3] for convex surfaces in $\mathbf{R}^3$. Each of these theorems states that some curvature as a function of the normal to the convex surface determines the surface up to a translation. These results have been extended to convex hypersurfaces of Euclidean spaces of arbitrary dimension; see, for example, Chern [1]. A few results give extensions to submanifolds of arbitrary codimension; see, for example, C. C. Hsiung [4]. This paper deals with this sort of uniqueness question for submanifolds of arbitrary codimension in Euclidean space.

Let M be a closed C^∞ manifold of dimension n and let M_m denote the tangent space to M at m. The main object of study is a pair of immersions X, $X'\colon M \to \mathbf{R}^{n+k}$, $k \geq 1$, such that for each $m \in M$ the tangent spaces $dX(M_m)$ and $dX'(M_m)$ agree. We impose some convexity conditions on X and X' and assume the equality of some curvature (or curvatures) at corresponding points of X and X' in order to conclude that X and X' differ by a translation. Actually, one condition we impose is the equality of the volume elements induced by X and X' on M. This is equivalent to the assumption that the Lipshitz-Killing curvature induced on the unit normal bundle by X and X' agree since the immersions have the same unit normal bundle with the same metric induced on each fiber of the unit normal bundle. (See [2] for the definition of the Lipshitz-Killing curvature.)

The method of proof is by means of integral formulas in the style of Chern [1]. However, here the integrands are multivector-valued forms on M. Throughout the paper all manifolds and maps are C^∞.

I would like to thank the referee for suggesting ways to clarify the presentation of this paper.

1. Definitions and the statement of the theorem

Let $\mathscr{E}^p(M, \Lambda^r\mathbf{R}^{n+k})$ be the space of p-forms on M whose values are r-vectors in $\mathbf{R}^{n+k}$; if

$$\alpha \in \mathscr{E}^p(M, \Lambda^r\mathbf{R}^{n+k}) \quad \text{and} \quad \beta \in \mathscr{E}^q(M, \Lambda^s\mathbf{R}^{n+k}),$$

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then

$$\alpha \wedge \beta \in \mathscr{E}^{p+q}(M, \Lambda^{r+s}\mathbf{R}^{n+k})$$

is the multi-vector-valued form obtained by simultaneously taking the wedge product of forms on M and multivectors in $\mathbf{R}^{n+k}$. It should be noted that $\alpha \wedge \beta = (-1)^{pq+rs}\beta \wedge \alpha$. For an example of such a product let $X: M \to \mathbf{R}^{n+k}$ be an immersion of an oriented manifold M, then

$$dX \wedge \cdots \wedge dX \ (n \text{ times}) = n!\, \xi \, dV$$

where $\xi(m) = e_1 \wedge \cdots \wedge e_n$ if $e_1, \ldots, e_n$ is a positively oriented orthonormal frame of $dX(M_m)$ and dV is the volume element induced on M by X. We will use the notation ξ and dV introduced above for oriented M throughout the paper. If $X': M \to \mathbf{R}^{n+k}$ is another immersion, let ξ' and dV' denote the corresponding quantities.

If $X, X': M \to \mathbf{R}^{n+k}$ are immersions of M into $\mathbf{R}^{n+k}$ such that $dX(M_m) = dX'(M_m)$, then we say X and X' have the same Gauss map. If M is oriented, this is equivalent to $\xi = \pm\xi'$.

If M is not orientable we also use dV and dV' to denote the canonical measures induced on M by X and X', respectively. If $\tilde{M}$ is the orientation covering of M, with a particular orientation, and $\tilde{X}, \tilde{X}': \tilde{M} \to \mathbf{R}^{n+k}$ are immersions covering X and X', respectively, let $\widetilde{dV}$ and $\widetilde{dV}'$ denote the volume elements induced on $\tilde{M}$ by $\tilde{X}$ and $\tilde{X}'$, respectively. Then $dV = dV'$ (as measures) is equivalent to $\widetilde{dV} = \widetilde{dV}'$ (as n-forms).

Let $X: M \to \mathbf{R}^{n+k}$ be an immersion and N be a unit vector normal to X at m. We denote the second fundamental form in the direction N (at m) by II_N, i.e., $II_N = -dN \cdot dX$. The immersion X is said to be locally convex if for each $m \in M$ there exists a unit vector N normal to X at m such that II_N is positive definite. By Hadamard's principle, local convexity is enough in the codimension 1 case to imply that the immersion is star-shaped, i.e., there exist an origin and a unit normal vector field relative to which the support function $p = X \cdot N$ is positive. We need to generalize this concept to higher codimensions. Thus $X: M \to \mathbf{R}^{n+k}$ is called star-shaped relative to the $(n+1)$-dimensional linear subspace L of $\mathbf{R}^{n+k}$ if $P_L \circ X : M \to L$ is a star-shaped immersion (in the codimension 1 sense), where $P_L: \mathbf{R}^{n+k} \to L$ is orthogonal projection onto L. It follows immediately from the definition that if $n \geq 2$ then M is a sphere and X is an embedding. If X is star-shaped relative to L, then we may choose $\lambda \in \Lambda^{n+1}(L)$ and an origin in $\mathbf{R}^{n+k}$ such that $(\lambda, X \wedge \xi) > 0$, where $(\ ,\ )$ denotes the standard inner product on $\Lambda^n(\mathbf{R}^{n+k})$; for the origin in $\mathbf{R}^{n+k}$ we choose a point of L relative to which the support function of $P_L \circ X$ is positive.

Let $X, X': M \to \mathbf{R}^{n+k}$ be two immersions with the same Gauss map. The pair X, X' is said to satisfy condition (A) if there exists a point $m \in M$ and a unit vector N normal to X and X' at m such that both II_N and II'_N are positive definite. Here II'_N is the second fundamental form of X' in the direction N.

Finally we let η and η' denote the mean curvature vector fields along X and X', respectively.

THEOREM. *Let $X, X'\colon M \to \mathbf{R}^{n+k}$ be immersions with the same Gauss map and $dV = dV'$. Suppose X is locally convex and the pair X, X' satisfies condition* (A). *Moreover, if*

(i) *X and X' are both star-shaped relative to the same subspace L, or*
(ii) $\eta = \eta'$,
then X and X' differ by a translation of $\mathbf{R}^{n+k}$.

2. Proof of the theorem

First note that we may reduce the proof to the case where M is oriented. To see this suppose the theorem is true for any oriented M. Let N be a nonorientable n-dimensional manifold and $X, X'\colon N \to \mathbf{R}^{n+k}$ be immersions which satisfy the hypotheses of the theorem. (Of course this may only happen under assumption (ii).) Let $\tilde{N}$ be the orientation covering with a particular orientation and let $\tilde{X}$, $\tilde{X}'\colon \tilde{N} \to \mathbf{R}^{n+k}$ be immersions covering X, X', respectively. Then $\tilde{X}$, $\tilde{X}'\colon \tilde{N} \to \mathbf{R}^{n+k}$ satisfy the hypotheses of the theorem and thus $\tilde{X}$ and $\tilde{X}'$ differ by a translation. Clearly X and X' differ by a translation. For the remainder of the proof, we therefore assume that M is oriented.

We introduce an orthonormal moving frame $e_1, \ldots, e_n$ along X and X' so that $\xi = \pm\xi' = e_1 \wedge \cdots \wedge e_n$. Then there exist 1-forms on M, ω_i and ω'_i, $1 \le i \le n$, such that $dX = e\omega$ and $dX' = e\omega'$, where $\omega = (\omega_1, \ldots, \omega_n)^t$, $\omega' = (\omega'_1, \ldots, \omega'_n)^t$, and $e = (e_1, \ldots, e_n)$. There exists an $n \times n$ matrix b such that $\omega = b\omega'$. Note that $\det(b) = \pm 1$ since

$$dV = \omega_1 \wedge \cdots \wedge \omega_n = \det(b)\omega'_1 \wedge \cdots \wedge \omega'_n = \pm \det(b)\, dV' = \pm \det(b)\, dV.$$

Define functions σ_i, $0 \le i \le n$, by

$$\det(b + \lambda 1) = \sum_{i=0}^{n} \binom{n}{i} \sigma_i \lambda^{n-i}$$

where λ is a parameter. The σ_i are the elementary symmetric functions of the eigenvalues of b.

Note that b is determined up to similarity by the choice of the frame $e_1, \ldots, e_n$; in fact it is the matrix of

$$dX \circ (dX')^{-1}\colon dX(M_m) \to dX(M_m)$$

relative to this frame. Hence the eigenvalues of b are invariants of the pair X, X'.

LEMMA 1. *The matrix b is diagonalizable and has only positive eigenvalues on M.*

Proof. Let a_N and a'_N denote the matrices of II_N and II'_N relative to $e_1, \ldots, e_n$, respectively. Since $-dN = ea_N\omega = ea'_N\omega'$, it follows that

$$a'_N = a_N b. \tag{1}$$

Since X is locally convex, we may choose N for each $m \in M$ such that II_N and hence a_N are positive definite. We may simultaneously diagonalize a'_N and a_N and in so doing change a_N to the identity matrix, i.e., a'_N and a_N are cogredient to a real diagonal matrix and the identity matrix, respectively, by means of the same change of basis matrix. Thus we see that b may be diagonalized and has only real eigenvalues.

Using condition (A), there exists $m \in M$ and a normal N to X and X' at m such that both II_N and II'_N are positive definite. Again using (1), we see that b has only positive eigenvalues at m. Since the eigenvalues of b vary continuously on M, $\det(b) \neq 0$, and b has only real eigenvalues on M, b must have only positive eigenvalues on M. ∎

Remark. It is clear from the preceding lemma that if X is locally convex and X, X' satisfies condition (A), then X' is also locally convex. In fact, II_N is positive definite if and only if II'_N is positive definite.

LEMMA 2. $\det(b) = 1$ *and* $\xi = \xi'$.

Proof. This follows immediately from Lemma 1. ∎

The proof of the theorem is now trivial for $n = 1$ and does not require assumptions (i) or (ii). Therefore we suppose $n \geq 2$ for the remainder of the argument.

We are given in condition (i) of the theorem that both X and X' are star-shaped relative to the same linear subspace L. Thus we know there exist an origin in L and $(n+1)$-vectors λ, $\lambda' \in \Lambda^{n+1}(L)$ such that $(X \wedge \xi, \lambda) > 0$ and $(X' \wedge \xi, \lambda') > 0$. We may have to translate X or X' to obtain an origin for which both inner products are positive, but this is immaterial. If we assume $\|\lambda\| = \|\lambda'\| = 1$, then $\lambda' = \pm\lambda$. In fact, the following holds.

LEMMA 3. $\lambda' = \lambda$, *for* $n \geq 2$.

Proof. Suppose $\lambda' = -\lambda$. Set $X_L = P_L \circ X$ and $X'_L = P_L \circ X'$; then X_L and X'_L are star-shaped immersions into L. Let

$$\xi_L = \Lambda^n P_L(\xi)/\|\Lambda^n P_L(\xi)\|;$$

of course, $\xi_L(m)$ is the unit n-vector tangent to X_L and X'_L at m. Define a unit vector field N on M normal to X_L and X'_L by $N \wedge \xi_L = \lambda$. Then

$$X_L \cdot N = (X_L \wedge \xi_L, \lambda) = (X \wedge \xi, \lambda)/\|\Lambda^n P_L(\xi)\| > 0$$

and

$$X'_L \cdot N < 0.$$

Note that $\alpha X + \beta X'$ is an immersion for any $\alpha > 0$, $\beta > 0$; this is the case since

$$d(\alpha X + \beta X') \circ (dX')^{-1} = \alpha\, dX \circ (dX')^{-1} + \beta 1$$

and $dX \circ (dX')^{-1}$ is diagonalizable with only positive eigenvalues by Lemma 1. Since for all $\alpha > 0$, $\beta > 0$, $\alpha X + \beta X'$ and X have the same Gauss map

$$\alpha X_L + \beta X'_L = P_L \circ (\alpha X + \beta X')$$

is an immersion. Choose α so large that $X'_L(M)$ is contained in the bounded component of $L - \alpha X_L(M)$; here we are using the fact that X_L is an embedding when $n \geq 2$. Define $\phi\colon M \times [0, 1] \to L$ by

$$\phi(m, t) = (1 - t)\alpha X_L + tX'_L.$$

Note that

$$\frac{\partial \phi}{\partial t} \cdot N = (X'_L - \alpha X_L) \cdot N = X'_L \cdot N - \alpha X_L \cdot N < 0$$

so that

$$\xi_L \wedge \frac{\partial \phi}{\partial t} \neq 0$$

and thus ϕ is a local diffeomorphism.

Let W be the bounded component of $L - X'_L(M)$. Note that

$$X'_L \cdot N < 0 \quad \text{and} \quad \frac{\partial \phi}{\partial t}(\cdot, 1) \cdot N > 0;$$

therefore for values of t close enough to 1, $\phi(m, t) \in W$ for any $m \in M$. In particular $\phi(M \times [0, 1]) \cap W \neq \emptyset$. Let U be the component of $\phi^{-1}(W \cup X'_L(M))$ containing $M \times \{1\}$. Then U is a compact submanifold with boundary of $M \times [0, 1]$ and the boundary of U has more than one component. Consider $\phi \mid U$: $U \to W \cup X'_L(M)$; it is a local diffeomorphism onto $W \cup X'_L(M)$ that maps the boundary of U into the boundary of $W \cup X'(M)$. Thus $\phi \mid U$ is a covering map. But $W \cup X'_L(M)$ is a disk; therefore U must be a disk. However, U has more than one boundary component.

This contradiction implies $\lambda = \lambda'$. ∎

I would like to thank Les Wilson for his assistance in the proof of this lemma.

Let $r \geq 0$, $s \geq 0$ and $r + s = n - 1$ and define

$$\alpha_{r,s} = X \wedge X' \wedge \underbrace{dX \wedge \cdots \wedge dX}_{r} \wedge \underbrace{dX' \wedge \cdots \wedge dX'}_{s}.$$

Then

$$d\alpha_{r,s} = -X' \wedge \underbrace{dX \wedge \cdots \wedge dX}_{r+1} \wedge \underbrace{dX' \wedge \cdots \wedge dX'}_{s}$$

$$+ X \wedge \underbrace{dX \wedge \cdots \wedge dX}_{r} \wedge \underbrace{dX' \wedge \cdots \wedge dX'}_{s+1}$$

Applying Stokes Theorem, we obtain

$$\int_M X' \wedge \underbrace{dX \wedge \cdots \wedge dX}_{r+1} \wedge \underbrace{dX' \wedge \cdots \wedge dX'}_{s} = \int_M X \wedge \underbrace{dX \wedge \cdots \wedge dX}_{r} \wedge \underbrace{dX' \wedge \cdots \wedge dX'}_{s+1}.$$

The substitutions $dX = eb\omega'$ and $dX' = e\omega'$ give

$$\int_M (X \wedge \xi)\sigma_r \, dV = \int_M (X' \wedge \xi)\sigma_{r+1} \, dV.$$

In particular, this implies

$$\begin{aligned} \int_M (X \wedge \xi)(\sigma_0 - \sigma_{n-1}) \, dV &= \int_M (X' \wedge \xi)(\sigma_1 - \sigma_n) \, dV \\ &= \int_M (X' \wedge \xi)(\sigma'_{n-1} - \sigma'_0) \, dV \\ &= -\int_M (X' \wedge \xi)(\sigma'_0 - \sigma'_{n-1}) \, dV, \end{aligned} \tag{2}$$

where σ'_i, $0 \le i \le n$, are elementary symmetric functions of eigenvalues of b^{-1}. Note that $\sigma_i = \sigma'_{n-i}$ since $\det(b) = 1$.

Proof of the theorem under assumption (i). Since X and X' are both star-shaped relative to the same subspace L, Lemma 3 implies that there is an $(n+1)$-vector λ and an origin such that $(X \wedge \xi, \lambda) > 0$ and $(X' \wedge \xi, \lambda) > 0$. Taking an inner product of each side of (2) with λ, we obtain

$$\int_M (X \wedge \xi, \lambda)(\sigma_0 - \sigma_{n-1}) \, dV = -\int_M (X' \wedge \xi, \lambda)(\sigma'_0 - \sigma'_{n-1}) \, dV. \tag{3}$$

From Lemma 1 and the fact that $\det(b) = 1$, we conclude using Newton's inequality that $\sigma_0 - \sigma_{n-1} = 1 - \sigma_{n-1} \le 0$. Similarly $\sigma'_0 - \sigma'_{n-1} \le 0$. Therefore the left-side of (3) is less than or equal to 0 but the right-side of (3) is greater than or equal to 0. Thus

$$\int_M (X \wedge \xi, \lambda)(\sigma_0 - \sigma_{n-1}) \, dV = 0$$

which implies $\sigma_0 - \sigma_{n-1} = 0$. Hence $b = 1$ by Newton's inequality and thus $dX = dX'$. Therefore X and X' differ by a translation. ▮

Let $v\colon M \to \Lambda^k(\mathbf{R}^{n+k})$ be a map such that $\|v\| = 1$ and $\xi \wedge v$ be the positively oriented unit element of $\Lambda^{n+k}(\mathbf{R}^{n+k})$.

LEMMA 4.

$$\underbrace{dX \wedge \cdots \wedge dX}_{n-1} \wedge dv = (-1)^n n! (* \eta)\, dV,$$

where $*$ denotes Hodge star operator in $\mathbf{R}^{n+k}$.

Proof. We introduce an orthonormal moving frame $e_1, \ldots, e_{n+k}$ along X so that $\xi = e_1 \wedge \cdots \wedge e_n$ and $v = e_{n+1} \wedge \cdots \wedge e_{n+k}$. We then define dual forms ω_i, $1 \leq i \leq n$, by $dX = \sum_{i=1}^n \omega_i e_i$ and connections forms ω_{rs}, $1 \leq r, s \leq n+k$, by $de_r = \sum_{s=1}^{n+k} \omega_{rs} e_s$. Then

$$\begin{aligned} dX \wedge \cdots \wedge dX \wedge dv = (n-1)! \left(\sum_{j=1}^n \omega_1 \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_n e_1 \wedge \hat{e}_j \wedge \cdots \wedge e_n \right) \\ \wedge \left(\sum_{\alpha=n+1}^{n+k} \sum_{i=1}^n \omega_{\alpha i} e_{n+1} \wedge \cdots e_{\alpha-1} \wedge e_i \wedge e_{\alpha+1} \wedge \cdots \wedge e_{n+k} \right) \\ = (n-1)!(-1)^n \sum_{\alpha=n+1}^{n+k} (-1)^{\alpha-1} \left(\sum_{i=1}^n \omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge \omega_{\alpha i} \right. \\ \left. \wedge\, \omega_{i+1} \wedge \cdots \wedge \omega_n \right) e_1 \wedge \cdots \wedge \hat{e}_\alpha \wedge \cdots \wedge e_{n+k} \\ = (-1)^n n! (* \eta)\, dV. \end{aligned}$$

In the preceding computation, a caret above a factor indicates that the factor is to be deleted. ∎

Let $r \geq 0$, $s \geq 0$ and $r + s = n - 1$ and define

$$\beta_{r,s} = X \wedge v \wedge \underbrace{dX \wedge \cdots \wedge dX}_{r} \wedge \underbrace{dX' \wedge \cdots \wedge dX'}_{s}.$$

Then

$$d\beta_{r,s} = (-1)^k v \wedge \underbrace{dX \wedge \cdots \wedge dX}_{r+1} \wedge \underbrace{dX' \wedge \cdots \wedge dX'}_{s} + X \wedge dv \wedge \underbrace{dX \wedge \cdots \wedge dX}_{r} \wedge \underbrace{dX' \wedge \cdots \wedge dX'}_{s}.$$

By Stokes Theorem we obtain

$$(4) \qquad (-1)^{k+1} \int_M X \wedge dv \wedge \underbrace{dX \wedge \cdots \wedge dX}_{r} \wedge \underbrace{dX' \wedge \cdots \wedge dX'}_{s} = \int_M v \wedge \underbrace{dX \wedge \cdots \wedge dX}_{r+1} \wedge \underbrace{dX' \wedge \cdots \wedge dX'}_{s}.$$

Proof of the theorem under assumption (ii) By Lemma 4, the assumption $\eta = \eta'$ implies

$$\underbrace{dX \wedge \cdots \wedge dX}_{n-1} \wedge dv = \underbrace{dX' \wedge \cdots \wedge dX'}_{n-1} \wedge dv.$$

Multiplying both sides by X and integrating over M, we obtain

$$(5) \qquad \int_M X \wedge dv \wedge \underbrace{dX \wedge \cdots \wedge dX}_{n-1} - X \wedge dv \wedge \underbrace{dX' \wedge \cdots \wedge dX'}_{n-1} = 0.$$

Using (4) for $r = 0$ and $n - 1$ in (5), and the fact that

$$\underbrace{dX \wedge \cdots \wedge dX}_{n} = \underbrace{dX' \wedge \cdots \wedge dX'}_{n}$$

we get

$$\int_M v \wedge [\underbrace{dX' \wedge \cdots \wedge dX'}_{n} - dX \wedge \underbrace{dX' \wedge \cdots \wedge dX'}_{n-1}] = 0$$

This implies

$$(6) \qquad \int_M (v \wedge \xi)(\sigma_0 - \sigma_1)\, dV = 0.$$

Since $v \wedge \xi$ is constant and $\sigma_0 - \sigma_1 = 1 - \sigma_1 \leq 0$, by Newton's inequality and the fact that $\det(b) = 1$, we obtain from (6) that $1 - \sigma_1 = 0$. Again Newton's inequality implies $b = 1$ and thus X and X' differ by a translation. ∎

Remark 1. In the proof of the theorem under assumption (ii), it is enough to assume that $|\det(b)| = \sigma_n \leq 1$ rather than $dV = dV'$, i.e., $|det(b)| = 1$. Lemma 1 still holds; the proof only uses $\det(b) \neq 0$. In this case equation (6) would become

$$\int_M (v \wedge \xi)(\sigma_n - \sigma_1)\, dV' = 0.$$

Using Newton's inequality, we obtain $\sigma_1 \geq \sigma_n^{1/n} \geq \sigma_n$ or $\sigma_n - \sigma_1 \leq 0$. Again we conclude $\sigma_1 = \sigma_n$ and thus $b = 1$.

Remark 2. The theorem essentially holds for manifolds with (nonempty) boundary. We know of no analogue for Lemma 3 when M is not closed. Therefore condition (i) must be changed to read: M is oriented and there exists an origin of $\mathbf{R}^{n+k}$ and an $(n+1)$-vector $\lambda \in \Lambda^{n+1}(\mathbf{R}^{n+k})$ such that $(X \wedge \xi, \lambda) > 0$ and $(X' \wedge \xi, \lambda) > 0$. Leaving the other hypotheses of the theorem as they are and assuming that X and X' differ by a translation on each component of the boundary (and not necessarily the same translation on each boundary component) we may conclude that X and X' differ by a translation. The remarks in Remark 1 apply here too.

3. An example

In the codimension 1 case it is enough to assume $\xi = \pm\xi'$, $dV = dV'$, X is locally convex, and X, X' satisfies condition (A) in order to conclude X and X' differ by a translation. These are precisely the hypotheses for the Minkowski problem. Clearly X and X' map M onto a convex hypersurface. Also if K and K' denote the Gauss-Kronecker curvature of X and X', respectively, then these conditions imply $K = K' > 0$ as a function of the same normal (inward versus outward pointing normal). Without assuming that X, X' satisfies condition (A) it is possible that $K(N) = K'(-N)$ for all inward pointing normals N.

For higher codimensions some additional assumptions, such as (i) and (ii), need to be made. In fact we present a continuous family of locally convex immersions $X_\theta\colon S^1 \times S^1 \to \mathbf{R}^4$ all of which have the same Gauss map and induced volume element. That any pair X_θ, X'_θ, satisfies condition (A) follows from continuity. But no immersions in the family induce the same metric on M; hence no two may differ by a translation.

Let S^1 be the circle of radius $1/\sqrt{2}$ centered at the origin of $\mathbf{R}^2$ and $X\colon S^1 \times S^1 \to \mathbf{R}^4$ be the embedding given by taking the product of the inclusion $i\colon S^1 \to \mathbf{R}^2$ with itself. Let $Y\colon S^1 \times S^1 \to \mathbf{R}^4$ satisfy the condition that X, Y is an orthonormal moving frame of the normal bundle of the embedding X. For $-\pi/4 < \theta < \pi/4$, define

$$X_\theta = \frac{1}{\sqrt{\cos 2\theta}}(\cos\theta \cdot X + \sin\theta \cdot Y).$$

If ω_1, ω_2 are 1-forms such that $dX = \omega_1 e_1 + \omega_2 e_2$, then $dY = \omega_1 e_1 - \omega_2 e_2$. Thus

$$dX_\theta = \frac{1}{\sqrt{\cos 2\theta}}((\cos\theta + \sin\theta)\omega_1 e_1 + (\cos\theta - \sin\theta)\omega_2 e_2).$$

Clearly $dX_\theta \wedge dX_\theta = dX \wedge dX$ so that $\xi_\theta = \xi$ and $dV_\theta = dV$ for all θ. Since each immersion X_θ has its image in a 3-sphere, it necessarily follows that each immersion is locally convex. It is clear, moreover, that no two of the metrics induced on M by the X_θ are equal.

4. On condition (A)

Let X, $X'\colon M \to \mathbf{R}^{n+k}$ be immersions with the same Gauss map. We are concerned with finding properties of M or X which imply that the pair X, X' satisfies condition (A) when X is locally convex. Then under certain circumstances we could eliminate the assumption that the pair X, X' satisfies condition (A) from the hypotheses of the theorem.

In codimension 1 if X is locally convex then X embeds M as a convex hypersurface of $\mathbf{R}^{n+1}$. If X' has the same Gauss map as X, then X' also embeds M as a convex hypersurface [2, Theorem 3]. So for any unit normal N we must have either II_N and II'_N definite in the same sense, i.e., condition (A) holds, or II_N and II'_N are definite in the opposite sense and thus X and $-X'$ satisfies condition (A).

We say an immersion $X\colon M \to \mathbf{R}^{n+k}$ is firm if the following holds: if $X'\colon M \to \mathbf{R}^{n+k}$ has the same Gauss map as X, then either X, X' or $X, -X'$ satisfies condition (A). We restate this definition from another point of view. The immersion X is firm if and only if for every immersion $X'\colon M \to \mathbf{R}^{n+k}$ with the same Gauss map as X there exists a unit vector z in $\mathbf{R}^{n+k}$ and a point $m \in M$ such that the height functions $h_z = X \cdot z$ and $h'_z = X' \cdot z$ are non-degenerate and have a local extreme at m. This follows by observing that the hessians of h_z and h'_z at their critical points are II_z and II'_z, respectively.

In codimension 1 every locally convex immersion is firm. It is natural to ask whether every locally convex immersion is firm. The answer is no as the following example illustrates. Let $X_i\colon M_i \to \mathbf{R}^{q_i}$, $i = 1, 2$, be locally convex immersions. Let $X = X_1 \times X_2$ and $X' = X_1 \times (-X_2)$. Both X and X' have the same Gauss map and, in fact, both are locally convex but X is not firm since neither X, X' nor $X, -X'$ satisfies condition (A).

We now give some sufficient conditions on M or X that imply X is firm. In the next proposition the trivial subbundles are the 0 and n dimensional ones.

PROPOSITION 1. *If the tangent bundle of M has no nontrivial subbundles (and hence M is even dimensional). then every locally convex immersion $X\colon M \to \mathbf{R}^{n+k}$ is firm.*

Proof. Suppose there exists an immersion $X'\colon M \to \mathbf{R}^{n+k}$ with the same Gauss map as X but for which X and $\pm X'$ do not satisfy condition (A). The proof of Lemma 1 may be slightly modified to show that b has a constant number q (counting multiplicities) of negative eigenvalues and $q \neq 0, n$. This implies that for each $m \in M$ there exists the same number of linearly independent vectors $v \in M_m$ such that $dX(v) = \lambda\, dX'(v)$ with $\lambda < 0$; these vectors v correspond to the eigenvectors of b with negative eigenvalues. The span of these vectors in each M_m defines a q-dimensional subbundle of the tangent bundle of M. This contradiction proves the proposition. ∎

Note that any surface other than a torus or a Klein bottle is an example of a manifold whose tangent bundle has no nontrivial subbundles. This condition is also satisfied if $\chi(M) \neq 0$ and $H^i(M, Z) = 0$, $0 < i < n$, in particular, if M is an even dimensional sphere.

PROPOSITION 2. *Let $X\colon M \to \mathbf{R}^{n+k}$ be an immersion (which is not necessarily* *locally convex). Suppose there exists a non-degenerate height function* $h_z = X \cdot z(\|z\| = 1)$ *with less than* 4 *critical points, then* X is firm.

Proof. Let $X': M \to \mathbf{R}^{n+k}$ be an immersion with the same Gauss map as X. The height function $h'_z = X \cdot z$ has the same critical points as h_z. Two of the (at most) three critical points are extremes of h_z; two of the (at most) three critical points are extremes of h'_z. Therefore there must be one critical point which is an extreme of both h_z and h'_z. ∎

PROPOSITION 3. *Let* $M = S^n$ *with* $n \equiv 1 \bmod 4$ *and* $X: M \to \mathbf{R}^{n+k}$ *be a locally convex immersion. If there exists a non-degenerate height function with no critical points of index* 1 *or* $n - 1$, *then* X *is firm.*

Proof. This proposition follows from the fact that the tangent bundle of these spheres have nontrivial subbundles only in dimensions 1 and $n - 1$ [6, p. 144]. ∎

There are many unanswered questions. For example, we know that some locally convex immersions of the torus are not firm, but are there any locally convex immersions that are firm? Are all locally convex immersions of the Klein bottle firm? Are all locally convex immersions of S^3 firm; to find a counterexample we must look among those immersions for which almost all height functions have at least 4 critical points. In general, what are necessary and sufficient conditions on locally convex immersions X for X to be firm?

Finally suppose M or X satisfy conditions which imply that X is firm if X is locally convex, e.g., M is an even dimensional sphere. Then the assumption that X, X' satisfy condition (A) may be deleted from the statement of the theorem and the theorem still essentially holds. For now we may conclude that either X and X' or X and $-X'$ differ by a translation under assumption (i) while X and X' still must differ by a translation under assumption (ii).

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UNIVERSITY OF HAWAII AT MANOA
HONOLULU, HAWAII