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COMPLEMENTS OF CODIMENSION-TWO SUBMANIFOLDS II-HOMOLOGY BELOW THE MIDDLE DIMENSION

BY

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Introduction

This paper studies the modules that can occur as the homology modules below the middle dimension of the complement of a codimension-two imbedding of compact manifolds and it therefore forms a continuation of [26]. Particular attention is focused upon the modules that can occur as the homology modules of a certain covering space of the complement which, in the case of knots, is the infinite cyclic cover.

The only case of this problem that has been studied before is that of high-dimensional knots. In [19], Kervaire characterized the *first nonvanishing* homology module of a knot complement when its fundamental group is $\mathbf{Z}$. This work was continued by Levine in a series of papers that culminated in [20] in which he obtained a complete and simultaneous characterization of the homology of the infinite cyclic cover of a knot complement, except for a slight difficulty in dimension two. The present work studies classes of imbeddings, known as realizations of Poincaré imbeddings (these are defined in Section 1), that include high-dimensional knots as well as other well-known classes of imbeddings such as local knots and knotted lens spaces and obtains complete characterizations, in many cases, of the homology of the complement below the middle dimension. The results of this paper apply equally to smooth, *PL*, and topological imbeddings and manifolds.

In constructing imbeddings with prescribed homology modules in the complement, an algebraic K-theoretic obstruction is encountered, called the χ-invariant in this paper, that takes its value in a *relative* algebraic K-group and which incorporates aspects of both the Wall finiteness obstruction and Whitehead torsion.

In Section 2, where this invariant is discussed, it is shown that all elements of the relative algebraic K-group $K'_0(f)$ (see [4, Chapter 9]) occur as χ-invariants of suitable chain complexes so that we get a *geometric* interpretation of $K'_0(f)$.

In a special case that occurs in the study of knotted lens spaces, this invariant is explicitly calculated; it is shown that, in this case, it can be interpreted as an alternating product of "Alexander polynomials" of complementary homology modules evaluated at a primitive root of unity. Specifically, our result is:

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THEOREM. *Let L_1^{2k-1} and L_2^{2k+1} be homotopy lens spaces of index n (i.e., quotients of spheres by free $\mathbf{Z}_n$-actions), where n is an odd integer, and suppose there exists a locally-flat imbedding of L_1 in L_2. If $f: L_1 \to L_2$ is a locally-flat imbedding of L_1 in L_2 such that the homology modules of the infinite cyclic cover of the complement are $\{H_i\}$; then*

$$\prod_{i=1}^{2k+1} P_{f(H_i)}(\tau)^{(-1)^i} = \Delta(L_1)\,\Delta(L_2)^{-1}(\tau^d - 1)$$

up to multiplication by an nth root of unity. In this formula $\Delta(L_i)$ denotes Reidemeister torsion, τ is a primitive nth root of unity, $d \cdot d(L_1) \equiv d(L_2) \pmod n$ (where $d(L_i)$ is a homotopy invariant defined in 1.9) and $P_{f(H_i)}(*)$ is the "Alexander polynomial" defined in Section 2 of this paper. ∎

Remark. If we regard an imbedding of L_1 in L_2 as *unknotted* when its complementary homology vanishes, this theorem has the interesting consequence for some pairs of homotopy lens spaces (L_1, L_2) that, although there exists a locally-flat imbedding of L_1 in L_2, there *does not* exist an *unknotted* one. The extent to which an imbedding *must* be knotted is *precisely* measured by the χ-invariant. See the discussion following 2.12 for a concrete example of this phenomena.

Section 3 studies the properties of the complementary homology modules, particularly in dimensions 1 and 2 where there is considerable interaction with the fundamental group.

Section 4 contains our main results characterizing complementary homology modules of codimension-two imbeddings, below the middle dimension. Essentially, they show that, in the range from dimension three up to the middle dimension, the homology modules are direct sums of complementary homology modules of a *standard* imbedding with *any* finitely generated modules that become "homologically trivial" over the group ring of the fundamental group of the ambient manifold. See Section 4 for a precise statement.

These results are applied to knotted lens spaces in Theorem 4.9. This theorem paves the way for a result which will appear in a future paper in this series which gives a *complete* and *simultaneous* characterization of the homology modules of the infinite cyclic cover of the complement of knotted lens spaces that is analogous to Levine's results on knot modules in [20].

Future papers in this series will also study the effect of Poincaré duality in the middle dimension and its interaction with the cobordism theory of the imbeddings, and homology above the middle dimension.

Part of this paper is an expansion of results in my doctoral dissertation and I would like to thank my advisor, Professor Sylvain Cappell, for his guidance and assistance.

I would also like to thank the referee for an extremely careful reading of this paper.

1. Codimension-two Poincaré imbeddings

In this section we define a homotopy-theoretic analogue to an imbedding of compact manifolds, known as a *Poincaré imbedding*. This paper will study actual imbeddings of manifolds that are modeled upon a given Poincaré imbedding—these will be called *realizations* of the Poincaré imbedding.

DEFINITION 1.1. Let M^m and V^{m+2} be compact manifolds. Then a Poincaré imbedding $\theta = (E, \xi, h)$ of M in V consists of the following:

(1) a 2-plane bundle ξ over M with associated unit circle and unit disk bundles $S(\xi)$, $T(\xi)$ respectively;

(2) a finite CW-pair $(E, S(\xi))$ and a simple homotopy equivalence

$$h: V \to \bigcup_{S(\xi)} T(\xi)$$

with the homology class im $(h([V])) \in H_{m+2}(E \cup T(\xi), E)$ going by excision to a generator of the top-dimensional homology of $(T(\xi), S(\xi))$; in the non-orientable case we use homology with twisted integer coefficients. ■

Remarks. (1) If the map h is a homotopy equivalence with Whitehead torsion an element, g, of $Wh(\pi_1(V))$ we will call θ a *g-Poincaré imbedding.*

(2) If M and V have boundaries we will assume that E is a *quadrad* and

$$h: (V, \partial V) \to \left(\left(E \bigcup_{S(\xi)} T(\xi)\right), F \bigcup_{S(\xi)|\partial M} T(\xi)\Big|_{\partial M}\right)$$

is a simple homotopy equivalence of pairs.

(3) The definition above is due to Cappell and Shaneson (see [7, Section 5]) and is a specialization of the usual definition found in [29].

(4) Condition 2 above and Proposition 2.7 in [29] imply that $(E, S(\xi))$ is a *Poincaré pair* with local coefficients in $\mathbf{Z}\pi_1(V)$. The Poincaré imbedding θ will be called *regular* if $(E, S(\xi))$ satisfies Poincaré duality with local coefficients in $\mathbf{Z}\pi_1(E)$.

(5) The composite $h^{-1}z: M \to V$, where z is the inclusion of M in $T(\xi)$ as zero-section, will be called the *underlying map* of θ; if this map preserves orientation characters θ will be said to be *orientable.*

(6) Clearly any actual locally-flat imbedding f of M in V induces a Poincaré imbedding $\theta_f = (E, \xi, h)$—$T(\xi)$ is a tubular neighborhood of $f(M)$ and E is its complement.

DEFINITION 1.2. Let $\theta_1 = (E_1, \xi_1, h_1)$, $\theta_2 = (E_2, \xi_2, h_2)$ be Poincaré imbeddings of M^m in V^{m+2} and V'^{m+2}, respectively, where V' is homotopy equivalent to V via $\phi: V \to V'$. Then a map $\theta_1 \to \theta_2$ of Poincaré imbeddings, with respect to ϕ, is a map $f: E_1 \to E_2$ such that $f \mid S(\xi)$ is a bundle isomorphism and $h_2 \cdot \phi = (f \cup 1) \cdot h_1$, up to homotopy. A map $\theta_1 \to \theta_2$ with respect to the identity map of V will simply be called a map of the Poincaré imbeddings. If a map exists between two Poincaré imbeddings, they will be said to be equivalent. ■

Remarks. The map f in the definition above will be called the *complementary map* of the Poincaré imbeddings. It follows by excision and the additivity of Whitehead torsion over finite unions (see [11] or [22]) that, if θ_1 and θ_2 are both g-Poincaré imbeddings, f will induce a simple $\mathbf{Z}\pi_1(V)$-homology equivalence, which will be simple if ϕ is the identity.

DEFINITION 1.3. If θ is a Poincaré imbedding of M^m in V^{m+2} and f: $M \to V'$, where V' is homotopy equivalent to V, via ϕ: $V' \to V$, is an actual imbedding such that there exists a map $\theta_f \to \theta$ with respect to ϕ, f will be said to be a realization of θ. ■

In this paper f will always be assumed to be *locally-flat.*

Note that, in this definition, V' can be any manifold homotopy equivalent to V. Since we will often want to insure that V' is *homeomorphic* to V we make the following definition:

DEFINITION 1.4. Let θ and f be as in 1.3; then f will be called a normal realization if $(V', c \cup 1)$, where c is the complementary map of f, and (V, h) are s-cobordant. ■

One important property of *regular* Poincaré imbeddings is:

PROPOSITION 1.5. *Let c be the complementary map of a realization $f: M^m \to V'^{m+2}$ of a regular Poincaré imbedding $\theta = (E, \xi, h)$ of M into V^{m+2}. Then c induces split surjections in homology and, in particular, if E_f is the complement of $f(M)$ in V',*

$$H_i(E_f; \mathbf{Z}\pi_1(E)) = H_i(E; \mathbf{Z}\pi_1(E)) \oplus K_i \quad \textit{for all } i,$$

where K_i are the homology modules of the mapping cone of f.

Proof. This follows from the fact that, by the remark following 1.2, the complementary map is a $\mathbf{Z}\pi_1(V)$-homology equivalence and therefore, in particular, a *degree*-1 *map.* Since θ is *regular* its complement, E, is a Poincaré complex and the conclusion follows from Lemma 2.2 of [27]. ■

Our main results in Section 4 will actually characterize the *kernel modules* K_i, of the complementary map of realizations of a Poincaré imbedding and the Poincaré imbeddings will be required to be *regular.*

Here are some examples of Poincaré imbeddings and their realizations:

Example 1.6 (*Classical Knots*). *Let $\theta_i = (S^1 \times D^{m+1}, \xi, h)$ be the Poincaré imbedding defined by the standard inclusion of spheres i: $S^m \to S^{m+2}$. It is well-known that all imbeddings of S^m in S^{m+2} are normal realizations of θ_i.*

Example 1.7 (*Local Knots*). *Let $T(\xi)$ be the total space of the unit disk bundle associated to a 2-plane bundle ξ over a manifold M^m, and let $z\colon M \to T(\xi)$ be the inclusion as zero-section. Then Cappell and Shaneson show, in* [6] *that all locally-flat imbeddings of M in $T(\xi)$ homotopic to z are normal realizations of the Poincaré imbedding $\theta_z = (S(\xi) \times I, \xi, h)$ defined by z, where $S(\xi)$ is the unit circle bundle associated to ξ.*

Example 1.8 (*Parametrized Knots*). *Let $f\colon S^n \times M^m \to S^{n+2} \times M^m$ denote the imbedding $i \times 1$, where i is the standard inclusion of S^n in S^{n+2}. Imbeddings homotopic to f were first studied by Cappell and Shaneson in* [6] *in the case where M is simply-connected and closed. The general case was studied by Ocken in his thesis* [24] *under the additional assumptions that the imbedding is homotopic to $i \times 1$ relative to $S^n \times \partial M$. They showed that all imbeddings of this type are normal realizations of the Poincaré imbedding*

$$\theta_f = (D^{n+1} \times M \times S^1, \xi, h),$$

where ξ is a trivial bundle.

Before we can state an example for knotted lens spaces we must discuss some of the algebraic invariants of homotopy lens spaces. Let n be an odd integer and let R_n be the ring of algebraic integers in a cyclotomic field generated by a primitive nth root of unity, τ (which will be fixed for the remainder of this discussion). If L^{2k-1} is a homotopy lens space of index n, $\Delta(L)$ will denote its Reidemeister torsion (see [29] for a definition) and $d(L) \in \mathbf{Z}_n$ will denote its image in I_n^k/I_n^{k+1}, where I_n is the principal ideal of R_n generated by $\tau - 1$; see [29, p. 205] for a proof that $I_n^k/I_n^{k+1} = \mathbf{Z}_n$. Theorem 14E.3 on p. 207 of [29] proves that $d(L)$ determines the *homotopy type* of L in a given dimension and $\Delta(L)$ determines its *simple* homotopy type. The exact sequence on p. 32 of [23] shows that $Wh(\mathbf{Z}_n)$ is isomorphic to the quotient of the subgroup of the group of units of R_n mapping to 1 under $f\colon R_n \to R_n/I_n = \mathbf{Z}_n$ by the subgroup of nth roots of unity, i.e., the Reidemeister torsion of a complex that is acyclic *over* $\mathbf{Z}[\mathbf{Z}_n]$ will be a *unit* of R_n. We will usually regard elements of $W(\mathbf{Z}_n)$ as multipliers of units of R_n by arbitrary nth roots of unity.

Our main result is:

Example 1.9 (*Knotted Lens Spaces*). *Let L_1^{2k-1} and L_2^{2l+1} be homotopy lens spaces of index n, i.e., quotients of spheres by free $\mathbf{Z}_n$-actions, and suppose there exists a locally-flat imbedding of L_1 in L_2. Then all locally-flat imbeddings of L_1 in L_2 are normal realizations of the g-Poincaré imbedding*

$$\theta = (S^1 \times D^{2k}, \xi, h),$$

where $g = \Delta(L_1)(\tau^d - 1)\,\Delta(L_2)^{-1}$, $d \cdot d(L_1) \equiv d(L_2)$ (mod n), *and ξ is the 2-disk bundle over L_1 with Euler class e with* ed $\equiv 1$ (mod n).

Remark. The discussion on p. 205 of [29] implies that the d-invariant of a homotopy lens space is always a *unit* of $\mathbf{Z}_n$ so that e and d are well defined.

The following proof is largely an expansion of a discussion in Section 9 of [6]. I feel that, since Example 1.9 will be used heavily in forthcoming papers in this series, it will be worthwhile to give a more detailed discussion than was done by Cappell and Shaneson.

Proof. Let $f\colon L_1 \to L_2$ be a locally-flat imbedding with normal bundle ξ and suppose T and S are the total spaces, respectively, of the associated unit disk and unit circle bundles. Let $p\colon S^{2k-1} \to L_1$ be the universal covering projection and let $\tilde{S} = p^{-1}(S)$. Consider the low order portion of the exact sequence of the fibration $S^1 \to S \to L_1$:

$$E\colon 0 \to \pi_1(S^1) \xrightarrow{u} \pi_1(S) \xrightarrow{v} \pi_1(L_1) \to 0.$$

Comparison with the universal circle fibration over a $K(\mathbf{Z}_n, 1)$ shows that the *Euler class* of S (and, therefore, the imbedding f) can be identified with the class of the group-extension, E, above; this identification proceeds by the isomorphism $H^2(\mathbf{Z}_n, \mathbf{Z}) \to H^2(L_1, \mathbf{Z})$ induced by the characteristic map of L_1. The proof of proposition 1.1 on p. 64 of [21] shows that this *extension class* can be determined from E as follows:

If $a \in \pi_1(S)$ maps to a generator, $v(a)$, of $\pi_1(L_1)$ let $b \in \pi_1(S^1)$ be $u^{-1}(a^n)$. The image of b in $\mathbf{Z}/n \cdot \mathbf{Z}$ is the class of E in $H^2(\mathbf{Z}_n, \mathbf{Z})$.

Since we can *identify* $\pi_1(S^1)$ with $\pi_1(\tilde{S})$ (via p_*), this procedure is *equivalent* to the *following*:

If $a \in \pi_1(S)$ maps to a generator $\pi_1(L_1)$ and $p_*(b) = a^n$ for some $b \in \pi_1(\tilde{S})$ the image of b in $\pi_1(\tilde{S})/n \cdot \pi_1(\tilde{S})$ is the Euler class of ξ.

Suppose this Euler class is $e \in \mathbf{Z}_n$. If $a \in \mathbf{Z}_n$ is a generator, define a $\mathbf{Z}_n$-action on $S^{2k-1} \times D^2$ by

$$a(s, z) = (a \cdot s, z \cdot \exp(2\pi i e/n)),$$

where the action of S^{2k-1} is defined to be the same as that on the universal cover of L_1.

The discussion above (regarding the Euler class of ξ) shows that $\bar{T} = (S^{2k-1} \times D^2)/\mathbf{Z}_n$ (with the action we have just defined) is the total space of a 2-disk bundle over L_1 with Euler class e. It follows that $\bar{T}$ is isomorphic to T and we have the following.

(1) e must be a *unit* of $\mathbf{Z}_n$. This follows from the fact that $\mathbf{Z}_n$ must act freely on the universal cover of T and, therefore, that of $\bar{T}$. But the definition of $\bar{T}$ implies that this only happens when e is relatively prime to n.

(2) The identity map of L_1 extends to a homeomorphism $T \to \bar{T}$.

Note that $\bar{T}$ is *precisely* the tubular neighborhood of L_1 in L' under the *canonical inclusion*, where L' is the *suspension* of L_1 by the $\mathbf{Z}_n$-action on the complex unit circle defined by multiplication by $\exp(2\pi i e/n)$—see [29, Section 14A]. Since the complement of $\bar{T}$ in L' is a homotopy circle, it follows, by obstruction theory, that the identity map of L_1 extends to a *homotopy equivalence* $L_2 \to L'$, where we regard L_1 as being imbedded in L_2 via the map f (see the beginning of this proof). We claim that the complement of $\bar{T}$ in L' is, *in fact*, $S^1 \times D^{2k}$—this is an immediate consequence of the $\pi - \pi$ theorem, the s-cobordism theorem, and the fact that G/PL, G/TOP and $G/0$ are simply-connected.

We have proved most of the statements made in example 1.9; all that remains to be proved is that d, e, and g have their stated values. The proof of Lemma 14E.1 in [29] shows that $d(L') = d(L_1) \cdot d$, where $d \cdot e = 1 \pmod{n}$ (see Lemma 14E.1 in [29]) and since the d-invariant determines the homotopy type of a homotopy lens space (in a given dimension) it follows that $d(L_2) \equiv d(L_1) \cdot d \pmod{n}$. Proposition 14E.8 in [29] shows that $\Delta(L') = \Delta(L_1)(\tau^d - 1)$ and, since we use the homotopy equivalence $L_2 \to L'$ for the map h in θ, it follows that the Whitehead torsion of h is as stated. ■

We will conclude this section with a description of a special type of Poincaré imbedding that will play an important part in Sections 3 and 4:

DEFINITION 1.10. Let $\theta(E, \xi, h)$ be a Poincaré imbedding of M^m into V^{m+2}, where M and V are compact manifolds. Then θ will be called cyclic if the kernel of the homomorphism of fundamental groups $\pi_1(E) \to \pi_1(E \cup T(\xi))$, induced by inclusion, is a cyclic group. ■

Remarks. (1) This definition is due to Cappell and Shaneson in [6].

(2) The results in Section 4 characterizing some of the modules that can occur as homology modules of realizations of Poincaré imbeddings will only apply to realizations of *cyclic* Poincaré imbeddings.

(3) The theorem in the appendix of [8] shows that, given *any* Poincaré imbedding whose underlying map induces a surjection of fundamental groups, one can attach 2- and 3-cells to form a *cyclic* Poincaré imbedding (also see Proposition 1.6 in [26]). This implies that every codimension-two imbedding of compact manifolds that induces a surjection of fundamental groups is a realization (in fact, even a normal realization) of a cyclic Poincaré imbedding.

Note that all of the examples of Poincaré imbeddings given in this section are *cyclic*.

2. The χ-invariant

In this section we will define an algebraic K-theoretic invariant of chain complexes that incorporates aspects of both the finiteness obstruction of Wall (see [29]) and Whitehead torsion (see [22] and [11]). This invariant will measure the obstruction to prescribing the homology modules of the complement of a realization of a Poincaré imbedding.

Throughout this section the following conventions will be in effect: $f: G \to H$ is a homomorphism of groups with kernel K, $\Lambda = \mathbf{Z}G$ and $\Lambda' = \mathbf{Z}H$, and $F: \Lambda \to \Lambda'$ is the homomorphism of group-rings induced by f. In addition we will assume, unless otherwise stated, that all chain complexes are *bounded from* *below, finite dimensional,* and consist of *finitely generated projective modules,* with the ring acting on the *right.*

DEFINITION 2.1. A Λ-chain complex will be called relatively acyclic if its tensor product with Λ' (with Λ-module structure defined by multiplication by the image under the map F) is acyclic. ■

Recall that the map F induces an exact sequence in algebraic K-theory: $K_1(\Lambda) \to K_1(\Lambda') \to K_0'(F) \to K_0(\Lambda) \to K_0(\Lambda')$, where $K_0'(F)$ is defined (see [4, p. 447]) as the Grothendiek K-group of the semigroup generated by triples of the form (P_1, i, P_2) with P_1 and P_2 projective modules over Λ and where i is a Λ'*-module isomorphism* $i: P_1 \otimes_\Lambda \Lambda' \to P_2 \otimes_\Lambda \Lambda'$, and these triples are subject to the following relations:

(1) $(P_1, ij, P_3) = (P_1, i, P_2) + (P_2, j, P_3)$,
(2) $(P_1 \oplus Q_1, i \oplus j, P_2 \oplus Q_2) = (P_1, i, P_2) + (Q_1, j, Q_2)$,
(3) $(P_1, i, P_2) = 0$ if i is induced by *any* isomorphism *over* Λ.

Remarks. (1) The relations above and the fact that $K_0'(F)$ is *abelian* imply that $(F, i, F) = 0$, where F is a *free module* and i is a *simple isomorphism* over Λ'.
(2) If the homomorphism F is *surjective* $K_0'(F) = K_0(F)$—see [4, p. 375].

DEFINITION 2.2. If C_* is a chain complex, C_{odd} will denote the direct sum of the odd-dimensional chain modules and C_{even} will denote the direct sum of the even-dimensional chain modules. ■

PROPOSITION 2.3. *If (C_*, d) is a relatively acyclic chain complex and c is any chain contraction of $C_* \otimes_\Lambda \Lambda'$ such that $c^2 = 0$ (given a chain contraction c', $c = c'dc'$ has the required property),*

$$(d \otimes 1 + c)\colon C_{\text{odd}} \underset{\Lambda}{\otimes} \Lambda' \to C_{\text{even}} \underset{\Lambda}{\otimes} \Lambda'$$

is an isomorphism with inverse

$$(d \otimes 1 + c)\colon C_{\text{even}} \underset{\Lambda}{\otimes} \Lambda' \to C_{\text{odd}} \underset{\Lambda}{\otimes} \Lambda'. \quad ■$$

This follows by composing the maps and recalling the definition of a chain contraction.

DEFINITION 2.4. Let (C_*, d) be a relatively acyclic chain complex. Then define $\chi(C_*)$ to be the element of $K_0'(F)$ defined by the triple $(C_{\text{odd}}, d \otimes c, C_{\text{even}})$, where c is some chain contraction of $C_* \otimes_\Lambda \Lambda'$ such that $c^2 = 0$. ■

Remark. Clearly $-\chi(C_*)$ is the class in $K_0'(F)$ of $(C_{\text{even}}, d \otimes 1 + c, C_{\text{odd}})$.

PROPOSITION 2.5. *The class of $\chi(C_*)$ in $K_0'(F)$, as defined in* 2.4., *is independent of the chain contraction c.*

Proof. Let c and c' be two chain contractions of $C_* \otimes_\Lambda \Lambda'$ whose squares are 0 and let χ_1 and χ_2 be the χ-invariants of C_* computed using c and c' respectively, as the chain contractions (see 2.4). We will use the notation $e = d \otimes 1$, and show that

$$\chi_1 - \chi_2 = (C_{\text{odd}}, (e + c)(e + c'), C_{\text{odd}})$$

represents the zero element of $K_0'(F)$—the fact that $\chi_1 - \chi_2$ has this form follows from the remark following 2.4 and the first relation in the definition of $K_0'(F)$. Let P_* be the acyclic chain complex with chain modules $P_0 = P_1 = P$ and boundary the identity map, and suppose that P is a projective complement of C_{odd}. Let $\bar{c}$ be *any* chain contraction of $P_* \otimes_\Lambda \Lambda'$ and form $C'_* = C_* \oplus P_*$. Then it follows that, if A is the χ-invariant of P_*, calculated with respect to $\bar{c}$ (it is not hard to see that this must actually *vanish*), relation 2 of the definition of $K_0'(F)$ implies that

$$\begin{aligned}\chi_1(C'_*) - \chi_2(C'_*) &= \chi_1(C_* \oplus P_*) - \chi_2(C_* \oplus P_*)\\ &= \chi_1 + A - \chi_2 - A\\ &= \chi_1 - \chi_2\end{aligned}$$

where χ_1 is the χ-invariant calculated by using the chain contraction c on C_* and χ_2 is calculated using c' on C_* and in both cases the chain contraction $\bar{c}$ is used on P_*. If we use e'' to denote the boundary of $C'_* \otimes_\Lambda \Lambda'$ (this will equal $e \oplus 1$) we get

$$\chi_1 - \chi_2 = (C_{\text{odd}} \oplus P, (e'' + c + \bar{c})(e'' + c' + \bar{c}), C_{\text{odd}} \oplus P),$$

where $C_{\text{odd}} \oplus P$ is a *free* module. The conclusion of this theorem now follows from the remark following the definition of $K_0'(F)$ and 15.3 of [11], which implies that the homomorphism

$$(e'' + c + \bar{c})(e'' + c' + \bar{c})$$

is a *simple isomorphism.* ■

Some of the more important properties of the χ-invariant are listed in the following theorem:

THEOREM 2.6. *Let all of the chain complexes in the following statements be relatively acyclic:*

(1) *If* $0 \to C'_* \to C_* \to C''_* \to 0$ *is a short exact sequence of complexes, then* $\chi(C_*) = \chi(C'_*) + \chi(C''_*)$.

(2) *If* C_* *is acyclic, then* $\chi(C_*) = 0$.

(3) *If* C_* *and* C'_* *are chain-homotopy equivalent, then* $\chi(C_*) = \chi(C'_*)$.

(4) *The image of* $\chi(C_*)$ *in* $K_0(\Lambda)$, *under the homomorphism* $K_0'(F) \to K_0(\Lambda)$ *occurring in the exact sequence in K-theory induced by F (see p. 369 of* [4]), *is the Wall finiteness obstruction of the complex* C_* *(see* [30]).

(5) *If the Wall finiteness obstruction of* C_* *is zero, the inverse image of* $\chi(C_*)$ *in* $K_1(\Lambda')/\mathrm{im}\, K_1(\Lambda)$, *under the connecting homomorphism* ∂ *in the exact sequence in K-theory induced by F, is the same as the image of the Whitehead torsion (see* [22] *and* [11]), *over* Λ' *of a complex of free modules stably isomorphic to* C_*.

(6) $\chi(C_*) = 0$ *if and only if* C_* *is chain-homotopy equivalent to a complex of free modules that is simply-acyclic over* Λ' *with respect to some set of bases over* Λ.

(7) *If* $\alpha \in K_0'(F)$ *and F is surjective, there exists a chain complex,* C_*, *with* $\chi(C_*) = \alpha$.

Remark. Statement 7 provides a *geometric interpretation* of $K_0'(F)$ in many cases.

Proof. (1) Clearly $C_{\mathrm{odd}} = C'_{\mathrm{odd}}$ and $C_{\mathrm{even}} = C'_{\mathrm{even}} \oplus C''_{\mathrm{even}}$ *as modules* and 13.2 in [11] implies that $C_* \otimes_\Lambda \Lambda' = C'_* \otimes'_\Lambda \oplus C''_* \otimes_\Lambda \Lambda'$ *as chain complexes.* The result follows from statement 2 in the list of relations satisfied by elements of $K_0'(F)$.

(2) This follows from statement 3 in the list of relations satisfied by elements of $K_0'(F)$ and the fact that the isomorphism $d \otimes 1 + c$ used in the definition of the χ-invariant (see 2.4) may be regarded as being *induced* by an isomorphism $d + c'$ where c' is a chain contraction *of* C_* (and d is the boundary of C_*).

(3) Suppose $g\colon C_* \to C'_*$ is a chain-homotopy equivalence. Then we have an exact sequence of chain complexes

$$0 \to C_* \to M_*(g) \to C'_*(-1) \to 0,$$

where $M_*(g)$ is the algebraic mapping cone of g and $C'_*(-1)$ is a complex identical to C'_* except that the dimensions have been shifted down by 1. This implies that

$$\chi(M_*(g)) = \chi(C_*) + \chi(C'_*(-1)).$$

It is clear, from the definition, that $\chi(C_*(-1)) = -\chi(C_*)$ and the result follows from statement 2 of the present theorem and the fact that $M_*(g)$ is acyclic since g is a chain-homotopy equivalence.

(4) This follows from the definition of the Wall finiteness obstruction (see [30]) and the description of the map $K_0'(F) \to K_0(\Lambda)$ (see p. 269 of [4]).

(5) This follows from the description of the boundary map $K_1(\Lambda) \to K_0'(F)$ (see p. 365 of [4]) and the definition of Whitehead torsion given in Section 15 of [11].

(6) The statement that $\chi(C_*) = 0$ implies, by statement 4 of this theorem that C_* is chain-homotopy equivalent to a *free* complex, and statement 5 of this theorem implies that the Whitehead torsion of the tensor product of this free complex with Λ' (with respect to any set of bases over Λ) lies in the image of $K_1(\Lambda)$ under the map $K_1(\Lambda) \to K_1(\Lambda')$ induced by F. It follows that, after suitably changing bases *over* Λ, the Whitehead torsion can be made to vanish.

(7) Proposition 5.1 on p. 371 of [4] implies that the element $\alpha \in K_0'(F)$ has a representative of the form (P_1, i, P_2) (rather than just a *formal difference* of two such triples). Since P_1 is *projective*, the composite $P_1 \to P_1 \otimes_\Lambda \Lambda' \xrightarrow{i} P_2 \otimes_\Lambda \Lambda'$ *lifts* to a map $j\colon P_1 \to P_2$. The chain complex

$$0 \to P_1 \xrightarrow{i} P_2 \to 0$$

clearly has the required properties. ■

DEFINITION 2.7. A finitely generated right Λ-module A will be said to be relatively acyclic if $\mathrm{Tor}_i^\Lambda(A, \Lambda') = 0$ for all i. ■

Remarks. Unless a statement is made to the contrary, all relatively acyclic modules will *also* be assumed to be of *finite homological dimension* and to have *finitely generated projective resolutions.* This last condition is equivalent, by the corollary to Theorem 1 in [5], to the condition that the functors $\mathrm{Tor}_i^\Lambda(A, *)$ preserve products for all indexing sets and for all i.

DEFINITION 2.8. Let A be a relatively acyclic module. Then $\chi(A)$ is defined to be the χ-invariant of a finitely generated finite dimensional projective resolution of A. ■

Note that this is well defined, by statement 2 of 2.6 and the fact that all projective resolutions of a module have the same chain-homotopy type (see [9]).

THEOREM 2.9. *Let (C_*, d) be a relatively acyclic chain complex and suppose that its homology modules are all relatively acyclic. Then*

$$\chi(C_*) = \sum_{i=0}^{\dim(C_*)} (-1)^i \chi(H_i(C_*)).$$

Remark. This theorem will be used in developing criteria for when a given sequence of modules can be the homology modules of a complex of *free* modules that is *simply* acyclic over Λ'.

Proof. Suppose the first nonvanishing homology module of C_* is in dimension k. Then we can perform an algebraic procedure entirely analogous to the geometric procedure of attaching cells to a CW-complex to kill its first nonvanishing homology module. Let (P_*, p) be a relatively acyclic projective resolution for $H_k(C_*)$ and define the complex (E_*, e) by $E_i = C_i$, $e_i = d_i$, $i \le k$, $E_{k+i} = C_{k+i} \oplus P_{i-1}$, $i \ge 1$, and $e_{k+1} = (d_{k+1}, g)$ where g is a lift of the surjection $P_0 \to H_k(C_*) = Z_k/B_k$ to $Z_k \subset C_k$—it is possible to lift the map above because P_1 is *projective*—and $e_{k+i} = d_{k+1} \oplus p_{i-1}$, $i \ge 2$. We get an exact sequence of chain complexes $0 \to C_* \to E_* \to P_* \to 0$ and the long exact sequence induced in homology by this exact sequence shows that $H_k(E_*) = 0$, $H_{k+i}(E_*) = H_{k+i}(C_*)$, $i \ge 1$, and $\chi(E_*) = \chi(C_*) + (-1)^{k+1}\chi(H_k(C_*))$. We continue this process, killing off homology modules of successively higher dimensions, we will eventually obtain an *acyclic* complex V_* and

$$\chi(V_*) = \chi(C_*) + \sum_{j=0}^{\dim(C_*)} (-1)^{j+1}\chi(H_j(C_*)).$$

The result follows from the fact that the χ-invariant of an acyclic complex vanishes. ■

The following theorem gives a criterion for when Theorem 2.9 is applicable:

THEOREM 2.10. *Suppose the homomorphism $f\colon G \to H$ is surjective (recall the assumptions made at the beginning of this section), $\ker f = K$ is a finitely generated nilpotent group, and H is a finite extension of a polycyclic group. Then a finitely generated chain complex is relatively acyclic if and only if all of its homology modules are relatively acyclic.*

Proof. Statement 1 is a direct consequence of Theorem 1 in [27] which states that under the assumptions above, a chain complex is admissible if and only if its homology modules are *torsion* modules in a suitable sense. Clearly such a condition can be satisfied for the homology modules of a chain complex if and only if it is satisfied for *projective resolutions* of its *homology modules.* ■

We will conclude this section with an *explicit* description of the χ-invariant in a special case:

Example 2.11. Suppose $G = \mathbf{Z}$, $H = \mathbf{Z}_n$ where n is a positive integer. Then the exact sequence in algebraic K-theory is

$$K_1(\mathbf{Z}[\mathbf{Z}]) \xrightarrow{r} K_1(\mathbf{Z}[\mathbf{Z}_n]) \to K_0(F) \to K_0(\mathbf{Z}[\mathbf{Z}]) \xrightarrow{s} K_0(\mathbf{Z}[\mathbf{Z}_n]).$$

Since the homomorphism s is *injective*, it follows that $K_0(F) = \operatorname{coker} r = \mathrm{Wh}\,(\mathbf{Z}_n)$. Proposition 7.3 on p. 623 of [4] implies that $SK_1(\mathbf{Z}[\mathbf{Z}_n]) = 0$ so that $K_1(\mathbf{Z}[\mathbf{Z}_n]) = U(\mathbf{Z}[\mathbf{Z}_n])$, the group of units, and the isomorphism is given by taking the *determinant* of a matrix representing an element of $K_1(\mathbf{Z}[\mathbf{Z}_n])$. This implies that the inclusion $\mathbf{Z}[\mathbf{Z}_n] \to \mathbf{Q}[\mathbf{Z}_n]$ induces an *injection* of K_1 (since a unit of $\mathbf{Z}[\mathbf{Z}_n]$ clearly remains a unit in $\mathbf{Q}[\mathbf{Z}_n]$) so that, before we compute the χ-invariant of a $\mathbf{Z}[\mathbf{Z}]$-module A we may *rationalize*, i.e., we may ignore the $\mathbf{Z}$-torsion component of A. Since $\mathbf{Q}[\mathbf{Z}]$ is a P. I. D. it follows that $A \otimes_{\mathbf{Z}} \mathbf{Q}$ has a short free resolution

$$0 \to F \xrightarrow{B} F \to A \underset{\mathbf{Z}}{\otimes} \mathbf{Q} \to 0,$$

where B is a matrix whose image in the matrix ring, $M(\mathbf{Q}[\mathbf{Z}_n])$, is *invertible*. Let $\mathbf{Z}[\mathbf{Z}] = \mathbf{Z}[t, t^{-1}]$ so that the entries of B will be Laurent polynomials in t. Then $\chi(A)$ may be identified with im $(F(\det(B)))$ in $U(\mathbf{Q}[\mathbf{Z}_n])/\{\pm t^i\}$, where det (B)

maps to a *unit* of $\mathbf{Z}[\mathbf{Z}_n]$. The exact sequence in [23], p. 32 shows that, if $r\colon \mathbf{Q}[\mathbf{Z}_n] \to \mathbf{Q}[\tau]$ (see the discussion preceding 1.9), where τ is a primitive nth root of unity, is the homomorphism mapping a generator of $\mathbf{Z}_n$ to τ, r induces an *injection* of $U(\mathbf{Z}[\mathbf{Z}_n])$ into $U(R_n)$ (recall that R_n is the ring of algebraic integers in $\mathbf{Q}[\tau]$). Thus we can identify $\chi(A)$ with $r(F(\det(B)))$ modulo multiplication by nth roots of unity. If we regard $\det(B)$ as a Laurent polynomial $q(t)$ (recall that $\mathbf{Z}[\mathbf{Z}] = \mathbf{Z}[t, t^{-1}]$) $rF(\det(B))$ is just $q(\tau)$. Furthermore, if $t(A)$ is the $\mathbf{Z}$-torsion submodule and $f(A) = A/t(A)$, the Hilbert Syzygy theorem shows that $f(A)$ has a short free resolution *over* $\mathbf{Z}[\mathbf{Z}]$:

$$0 \to F \xrightarrow{B'} F \to f(A) \to 0,$$

where im B' in $M(\mathbf{Q}[\mathbf{Z}_n]$ is B. Consequently, we *define* $P_{f(A)}(t)$ to be $\det(B')$ and we identify $\chi(A)$ with $P_{f(A)}(\tau)$, modulo multiplication by arbitrary nth roots of unity. Under these circumstances, 2.6 (statement 5), 2.9 and 2.10 combine to give:

LEMMA 2.12. *Let C_* be a finitely generated finite dimensional chain complex over $\mathbf{Z}[\mathbf{Z}] = \mathbf{Z}[t, t^{-1}]$ such that $C_* \otimes_{\mathbf{Z}[\mathbf{Z}]} \mathbf{Z}[\mathbf{Z}_n]$ is acyclic. If $A_i = H_i(C_*)$, the Whitehead torsion of $C_* \otimes_{\mathbf{Z}[\mathbf{Z}]} \mathbf{Z}[\mathbf{Z}_n]$ with respect to any equivalence class of bases over $\mathbf{Z}[\mathbf{Z}]$ is given by*

$$\chi(C_*) = \prod_{i=0}^{\dim(C_*)} P_{f(A_i)}(\tau)^{(-1)^i}$$

where τ is a primitive nth *root of unity and the equality is taken modulo arbitrary* nth *roots of unity—see the description of $Wh(\mathbf{Z}_n)$ preceding* 1.9. ■

Remark. At this point we are in a position to say something about complements of knotted lens spaces. Call a codimension-two imbedding of homotopy lens spaces *unknotted* if the universal covering space of its complement is contractible—for instance, the standard imbedding of a lens space in a suspension (see Section 14A of [29]) is *always* unknotted. Now suppose that in Example 1.9 the Whitehead torsion, g, is *nonzero*—the criteria for the existence of locally-flat imbeddings of homotopy lens spaces in [6] and Theorem 14E.7 of [29] show that this is *often* the case. Then, although there *exists* a locally-flat imbedding of the homotopy lens spaces, the preceding lemma shows that there *doesn't* exist an *unknotted* imbedding—the extent to which an imbedding must be knotted in this case is *precisely measured* by the χ-invariant. A concrete example of this is the classical lens space $L^3(5; 1, 1, 1)$ (see [22]) and the homotopy lens space h-cobordant to $L^5(5; 1, 1, 1, 1, 1)$ via an h-cobordism with Whitehead torsion $\tau^2 - \tau + 1$, where τ is a primitive 5th root of unity.

3. Properties of complementary homology

In this section we will apply the results of Sections 1 and 2 to derive necessary conditions for modules to be complementary homology modules of a realization of a Poincaré imbedding. The following conventions will be in effect throughout the remainder of this paper:

3.1. (1) $\theta = (E, \xi, h)$ is a g-Poincaré imbedding of M^m into V^{m+2}, $m \geq 3$, where an M and V are compact manifolds.

It will also be assumed to *be regular* and *cyclic*—see the fourth remark after 1.1, 1.5, and 1.10.

(2) If $f_\theta\colon M \to V$ is the underlying map of θ, we will assume that f_θ induces an isomorphism of fundamental groups and a surjection of second homotopy groups.

(3) θ possesses a locally-flat realization.

(4) If $f\colon M \to V$ is a locally-flat realization of θ with complementary map $c\colon E' \to E$ then either

(*a*) c induces an isomorphism of fundamental groups, or
(b) $H_2(E, S(\xi); \mathbf{Z}\pi_1(E)) = 0$. ■

Remarks. (1) Note that these conditions are satisfied by all of the examples of Poincaré imbeddings given in Section 1 and their realizations.

(2) Since all of the Poincaré imbeddings we will study will be regular, Proposition 1.5 implies that the complementary map of any *realization* will induce *split surjections* in homology. If $f\colon M \to V$ is a realization of θ with complement E_f, then

$$H_i(E_f; \mathbf{Z}\pi_1(E)) = H_i(E; \mathbf{Z}\pi_1(E)) \oplus K_i,$$

and we will actually study the modules K_i that occur as the homology modules of the algebraic mapping cone of the complementary map (see 1.2 and 1.3);

(3) Assumption 2 makes it possible to use the results of [26] to characterize the complementary fundamental groups of realizations of θ.

We begin with the following lemma, whose proof is almost identical to that of Lemma 4.3 of [6]:

LEMMA 3.2. *Let $f\colon M \to V$ be a locally-flat realization of θ. Then f is cobordant to a realization $f'\colon M \to V$ whose complementary map is $[(m+1)/2]$-connected.* ■

An immediate consequence of this is that $\pi_1(E)$ is isomorphic to the fundamental group of the complement of some codimension-two imbedding. We will use this fact and the results of [26] to determine the group $\pi_1(E)$ and to establish some important properties of the groups that occur as fundamental groups of complements of realizations of θ.

DEFINITION 3.3. Let $w\colon \pi_1(M) \to \mathbf{Z}_2 = \{\pm 1\}$ be the homomorphism induced by the first Stiefel-Whitney class of ξ, i.e., $w = w_M \cdot f^* w_V$ where w_M and w_V are the orientation characters of M and V, and let $\mathbf{Z}^w$ be the

$\mathbf{Z}\pi_1(M)$-module of integers twisted by w. Define $C_\theta = \mathbf{Z}^w/(\chi_\theta \cap H_2(M; \mathbf{Z}\pi_1(M))$, where χ_θ is the (twisted) Euler class of ξ. If x is the image of χ_θ under the change of coefficient homomorphism $H^2(M; \mathbf{Z}^w) \to H^2(M; C_\theta)$, then x is in the image of the injection $H^2(\pi_1(M); C_\theta) \to H^2(M; C_\theta)$ induced by the characteristic map of M. Define G_θ to be the group extension of C_θ by $\pi_1(M)$ defined by the inverse image of x in $H^2(\pi_1(M); C_\theta)$. ■

Remark. This is essentially Proposition 1 in [26].

LEMMA 3.4. *Under the hypotheses in effect in this section, the inclusion of $S(\xi)$—the total space of the unit circle bundle associated to ξ—in E induces an isomorphism of fundamental groups $\pi_1(S(\xi)) = \pi_1(E) = G_\theta$. Furthermore, if $f\colon M \to V$ is a realization of θ with complement E_f, the complementary map of f induces a surjection of fundamental groups that is split by the map of fundamental groups induced by the inclusion of $S(\xi)$ in E_f.*

Proof. This is an immediate consequence of 3.1 and of Lemmas 1.1 and 1.5 in [26]. ■

Remarks. (1) In the future, we will identify $\pi_1(E)$ and $\pi_1(S(\xi))$ with G_θ.

(2) Note that the realization $f\colon M \to V$ of θ defines a *canonical inclusion* $i_f\colon G_\theta \to \pi_1(E')$ and *surjection* $j_f\colon \pi_1(E') \to G_\theta$ such that $j_f \circ i_f = 1\colon G_\theta \to G_\theta$—this will prove to be a crucial algebraic property of these fundamental groups.

Proposition 1.5 of [26] shows that $\ker j_f \simeq [K, K]$, where K is the *meridian subgroup* of $\pi_1(E')$.

We will begin by considering the modules that can occur in the *first* and *second* dimensions of the complement of a realization of θ. We treat these dimensions separately because there is considerable interaction between these homology modules and the *fundamental group*. We begin with K_1:

PROPOSITION 3.5. $K_1 = H_1([K, K]; \mathbf{Z})$, *where $\mathbf{Z}G_\theta$ acts on $H_1([K, K]; \mathbf{Z})$ by conjugation of $\pi_1(E')$ by lifts of elements of G_θ over j_f (see Remark 2 following* 3.4).

Proof. This follows upon considering the universal covering space of E or $S(\xi)$, the corresponding covering of E', and the effect of the covering transformations. ■

Now we will turn to the considerably more difficult problem of characterizing K_2. First recall the notion of a presentation of a *pair* of groups (G, F), where F is a subgroup of G—see [16], p. 197. In our case, G is $\pi_1(E')$ and F is $i_f(G_\theta)$—see 3.4 and Remark 2 following it.

DEFINITION 3.6. Consider the hypotheses of this section,

(1) $\tilde{\mathscr{J}}(i_f)$ denotes a relative Jacobian of some presentation of $(\pi_1(E'), i_f(G_\theta))$—see [16, Section 2] for a definition.

(2) Let $\mathscr{J}(i_f) = j_f(\tilde{\mathscr{J}}(i_f))$, i.e., the relative Jacobian at $\mathbf{Z}G_\theta$—see [16, Section 3].

(3) $\tilde{\mathscr{R}}(i_f)$ denotes the kernel of $\tilde{\mathscr{J}}(i_f)$ after regarding this matrix as a homomorphism of free right $\mathbf{Z}\pi_1(E')$-modules (i.e., the matrix left multiplies coordinates).

(4) $\mathscr{R}(i_f)$ denotes the corresponding kernel of $\mathscr{J}(i_f)$. ∎

The following result is probably well known though I have not seen it stated explicitly before (see [28]).

PROPOSITION 3.7. *Let C_*, D_* be cellular chain complexes, over $\mathbf{Z}\pi_1(E')$, of $S(\xi)$ and E', respectively, with C_* a subcomplex of D_*—actually C_* is the cellular chain complex of the inverse image of $S(\xi)$ under the universal covering projection of E'. If $\bar{\partial}_2$ is the boundary map $\bar{\partial}_2\colon D_2/C_2 \to D_1/C_1$ of the relative chain complex, then $\bar{\partial}_2$ is equivalent to $\tilde{\mathscr{J}}(i_f)$—see [17] for the definition of equivalence used here.*

Proof. After collapsing a maximal tree in E' we may assume that the 2-skeleton of $S(\xi)$ is a *cellular model* of a presentation, $p_1 = \langle x; r\rangle$ for $\pi_1(S(\xi)) = G_\theta$—see [14, Section 2]. We may regard the 2-skeleton of E' as being formed from that of $S(\xi)$ by adjoining additional 1-spheres corresponding to additional relations. Let the presentation of $\pi_1(E')$ obtained from that of $\pi_1(S(\xi))$ by this procedure be $p_2 = \langle x, y; r, s\rangle$. If D_* is the cellular chain complex, over $\mathbf{Z}\pi_1(E')$, of E', then $\partial_2\colon D_2 \to D_1$ is a *Jacobian* of p_2—see the discussion preceding proposition 4 in [14]—and this is a matrix of the form

$$\begin{pmatrix} \dfrac{\partial r}{\partial x} & 0 \\[2ex] \dfrac{\partial s}{\partial x} & \dfrac{\partial s}{\partial y} \end{pmatrix}$$

(see [16] for a definition of these "free derivatives"), where the terms $(\partial r_i/\partial y_j)$ are 0 since the relations $\{r_i\}$ do not contain any of the generators $\{y_j\}$. It is not hard to see that $C_2 \subset D_2$ is the submodule generated by the relations $\{r_i\}$ and that

$$\partial_2 | C_2 = \left(\frac{\partial r_i}{\partial x_j}\right),$$

so that $\bar{\partial}_2\colon D_2/C_2 \to D_1/C_1$ is given by the matrix $(\partial s/\partial y)$. The *relative Jacobian* of the presentation p_2, regarded as a presentation of the *pair* $(\pi_1(E'), \pi_1(S(\xi)))$ is the matrix

$$\begin{pmatrix} 0 \\ \dfrac{\partial s}{\partial y} \end{pmatrix}$$

which is clearly equivalent to $\bar{\partial}_2$ by the definition of equivalence given in [17]. ■

DEFINITION 3.8. Two homomorphisms g_i: $A_i \to B$, $i = 1, 2$, of right $\mathbf{Z}G_\theta$ modules will be said to be *s-equivalent* if there exist free modules F_1 and F_2 and an isomorphism h such that the following diagram commutes:

$$\begin{tikzcd} A_1 \oplus F_1 \arrow[rr, "h"] \arrow[dr, "g_1 \oplus 0"'] & & A_2 \oplus F_2 \arrow[dl, "g_2 \oplus 0"] \\ & B & \end{tikzcd}.$$

Note that the homomorphisms g_i have been extended to the F_i by zero maps. ■

DEFINITION 3.9. Under the hypotheses of this section

$$\not{p}(i_f)\colon \mathscr{R}(i_f) \to H_2([K, K]); \mathbf{Z})$$

is defined to be the composite

$$\mathscr{R}(i_f) \to H_2(D_*/C_*; \mathbf{Z}G_\theta) \to H_2(\pi_1(E'); \mathbf{Z}G_\theta) = H_2([K, K]; \mathbf{Z})$$

where C_* and D_* are as in 3.7,

$$\mathscr{R}(i_f) = Z_2(D_*/C_*; \mathbf{Z}G_\theta) = \ker \bar{\partial}_2 \otimes 1\colon (D_2/C_2) \underset{\mathbf{Z}\pi_1(E')}{\otimes} \mathbf{Z}G_\theta \to (D_1/C_1) \underset{\mathbf{Z}\pi_1(E')}{\otimes} \mathbf{Z}G_\theta,$$

and the map $H_2(D_*/C_*) \to H_2(\pi_1(E'); \mathbf{Z}G_\theta)$ is induced by the characteristic map of E'. ■

Remarks. (1) The equality of $H_2(\pi_1(E'); \mathbf{Z}G_\theta) = H_2([K, K]; \mathbf{Z})$ is a consequence of Shapiro's lemma. The $\mathbf{Z}G_\theta$-module structure on $H_2([K, K]; \mathbf{Z})$ is defined exactly like that on K_1 in 3.5—G_θ acts by conjugation of $\pi_1(E')$ by inverse images of elements of G_θ under j_f.

(2) It is not hard to see that $\not{p}(i_f)$ is uniquely determined, up to s-equivalence, by i_f: $G_\theta \to \pi_1(E')$—this is a direct consequence of the definition of equivalence of relative Jacobians and the fact that the equivalence class of Jacobian is determined by the *isomorphism class* of the pair $(\pi_1(E'), i_f(G_\theta))$—see Section 2 of [17].

Our final result on K_2 is:

LEMMA 3.10. *Recall that K_2 is the kernel of the homomorphism in homology with $\mathbf{Z}G_\theta$-coefficients induced by the complementary map of f. Then there exists a right $\mathbf{Z}G_\theta$-module $\mathscr{H}$ and a homomorphism $\mathscr{H} \to K_2$ such that the composite with* *the homomorphism induced by the characteristic map of* E', $g\colon \mathscr{H} \to K_2 \to H_2([K, K]; \mathbf{Z})$ *is s-equivalent to* $\wp(i_f)$.

Proof. Recall assumption 3.1 at the beginning of this section. In case (a) of this assumption, the statements of this lemma become vacuous since $\wp(i_f)$ becomes the *zero map* from a *free module*. We will, therefore, assume that case (b) is in effect—i.e.,

$$H_2(E, S(\xi); \mathbf{Z}G_\theta) = 0.$$

Since $c \mid S(\xi)$ is a *homeomorphism*,

$$K_2 = \ker c_*\colon H_2(E', S(\xi); \mathbf{Z}G_\theta) \to H_2(E, S(\xi); \mathbf{Z}G_\theta) = 0$$

—see Lemma 2.2 of [27], and the conclusion follows upon setting

$$\mathscr{H} = Z_2(E', S(\xi); \mathbf{Z}G_\theta)$$

and by the argument in Remark 2 following 3.9. ■

For the remaining results of this paper, we will make the additional assumption that $\pi_1(V)$ is a *finite extension of a polycyclic group*—thus $\pi_1(V)$ may be any finitely generated abelian group or finite group. Note that, by 3.3, G_θ will also be of this type and, by the Lemma on p. 136 of [25], $\mathbf{Z}G_\theta$ will be a *noetherian* ring.

The following result shows that the condition on K_2 can, in many cases, be simplified considerably:

LEMMA 3.11. *Suppose that* $\dim_{\mathbf{Z}G_\theta}(H_1([K, K]; \mathbf{Z})) \leq 2$ *and* $\dim_{\mathbf{Z}G_\theta}(\mathbf{Z}) \leq 3$. *Then the following two statements about a finitely generated right* $\mathbf{Z}G_\theta$*-module,* K_2, *are equivalent:*

(A) *There exists a surjective homomorphism* $\mathfrak{s}\colon K_2 \to H_2([K, K]; \mathbf{Z})$;
(B) K_2 *satisfies the conditions in Lemma* 3.10.

Proof. First of all, note that (B) implies (A). We will, therefore, assume statement (A) and that $\dim_{\mathbf{Z}G_\theta}(H_1([K, K]; \mathbf{Z})) \leq 2$.

Claim. $\mathscr{R}(i_f)$ (see 3.9) *is a finitely generated module.*

Since G_θ is a finite extension of a polycyclic group and $\mathbf{Z}G_\theta$ is, therefore, a *noetherian* ring, it follows that $\mathscr{R}(i_f)$ is *finitely generated.*

Recall that $\mathscr{R}(i_f)$ is the 2-dimensional cycle module of the relative chain complex (C_*, d) of the 2-dimensional CW-pair realizing $(G, i_f(G_\theta))$. The projectivity of $\mathscr{R}(i_f)$ now follows from a repeated application of Proposition 6.8 on p. 39 of [4].

Let P be a finitely generated projective module stably isomorphic to $\mathscr{R}(i_f)$ such that there exists a surjective homomorphism $P \to K_2$. The statement of the lemma now follows from the form of Schanuel's lemma on p. 193 of [10]. ■

The problem of characterizing the higher-dimensional homology modules is much simpler—most of the work has been done in [27].

DEFINITION 3.12. Under the hypothesis that $\pi_1(V)$ is a finite extension of a polycyclic group, define $\Lambda = \mathbf{Z}G_\theta[S^{-1}]$ as in Section 2 of [27], where S is the multiplicatively closed set of elements of $\mathbf{Z}G_\theta$ of the form $1 + i$, $i \in I$ and I is the kernel of the homomorphism $\mathbf{Z}G_\theta \to \mathbf{Z}\pi_1(V)$ induced by the projection $G_\theta \to G_\theta / C_\theta = \pi_1(V)$ (see 3.3). ■

Remark. The existence of Λ is proved in [27].

LEMMA 3.13. *Under the hypothesis on $\pi_1(V)$ above, the $\{K_i\}$, $1 \le i \le m + 2$ (see 3.1 and the discussion following it) must be finitely generated Λ-torsion modules, i.e., $K_i \otimes_{\mathbf{Z}G_\theta} \Lambda = 0$.*

Remark. The condition on K_i is equivalent to the condition that there exist $t \in I$ such that $K_i \cdot (1 + t) = 0$.

Proof. The fact that the K_i are finitely generated follows from the fact that the algebraic mapping cone of the complementary map is a finitely generated projective complex and the fact that $\mathbf{Z}G_\theta$ is *noetherian.*

The remaining statements follow from the fact that the complementary map is a $\mathbf{Z}\pi_1(V)$-homology equivalence (see the remark following 1.2) and from Theorem 1 in [27]. ■

4. Homology realizations of a Poincaré imbedding

Before we can state and prove the main results of this section, we will need Theorem 2 of [26]:

THEOREM 4.1. *Under the assumptions of 3.1, a group G can be the fundamental group of the complement of a realization of θ if and only if the following hold.*

(1) *G is finitely presented;*

(2) *there exists a homomorphism $j\colon G \to G_\theta$, split by a homomorphism j_s such that:*

(a) *if $K = j^{-1}(C_\theta)$, then K is the normal closure within itself of $j_s(C_\theta)$, and*

(b) *$H_2(K; \mathbf{Z}) = 0$.* ■

Remark. Throughout the remainder of this section G and K will denote *any* groups satisfying the conditions above.

DEFINITION 4.2. Two realizations $f_0, f_1\colon M \to V$, will be said to be *concordant* if there exists an imbedding $F\colon M \times I \to V + I$ with $F(M \times \partial I)$, $V \times \partial I$ and $F(M \times I)$ intersecting $V \times \partial I$ transversally and with $F \mid M \times \{i\} = f_i$, $i = 0, 1$. ■

Remark. The proof of Theorem 2 in [26] shows that a realization of θ with complementary fundamental group any G satisfying the conditions 4.1 can be found in *any* concordance class.

Recall the involution $-: \mathbf{Z}\pi_1(V) \to \mathbf{Z}\pi_1(V)$ defined by

$$\overline{\sum_i n_i g_i} = \sum_i n_i w(g_i) g_i^{-1}$$

where $w: \pi_1(V) \to \mathbf{Z}_2 = \{\pm 1\}$ is the orientation character of V—by abuse of notation we will denote the induced conjugation operation on the Whitehead group (see [22, p. 373 and p. 378]) by $-: Wh(\pi_1(V)) \to Wh(\pi_1(V))$. With this in mind, we are in a position to state the main theorem of this section:

THEOREM 4.3. *Let $\theta = (E, \xi, h)$ be a Poincaré imbedding induced by an imbedding of compact manifolds $f: M^m \to V^{m+2}$ with $m \geq 3$, satisfying the conditions of* 3.1 *and, in addition:*

(A) *$\pi_1(V)$ is a finite extension of a polycyclic group;*

(B) *there exists a map $r: K(\pi_1(E), 1) \to E$ that induces an isomorphism of fundamental groups.*

If G is a group satisfying the conditions of 4.1 *and $\{A_j\}$, $1 \leq i \leq \mu$ (where $\mu = [(m+1)/2] - 1$) is a sequence of $\mathbf{Z}G_\theta$-modules, then there exists an imbedding $f': M \to V$ realizing θ and concordant to f with complementary map $c: E' \to E$ such that $c_*: \pi_1(E') \to \pi_1(E)$ is $j: G \to G_\theta$ and $H_i(E'; \mathbf{Z}G_\theta) = A_i$, $1 \leq i \leq \mu$ if $A_i = K_i \oplus H_i(E; \mathbf{Z}G_\theta)$ where the $\{K_i\}$ satisfy:*

(1) *$K_1 = H_1([K, K]; \mathbf{Z})$ (see* 3.1 *and* 3.5*);*

(2) *$K_2 = K_2' \oplus L$ where L is a submodule and there exists:*

(a) *a surjective homomorphism $\ell: L \to H_2([K, K]; \mathbf{Z})$*

(b) *a finitely generated right $\mathbf{Z}G_\theta$-module $\mathscr{H}$ and a short exact sequence*

$$0 \to F \to \mathscr{H} \to L \to 0$$

where F is a free $\mathbf{Z}G_\theta$-module and the composite $\ell \circ \jmath$ is s-equivalent to $\hbar(j_s): \mathscr{R}(j_s) \to H_2([K, K]; \mathbf{Z})$ (see 3.1, 3.6, 3.8, 3.9*);*

(3) *the $\{K_i\}$ are finitely generated $\mathbf{Z}G_\theta$-modules such that $K_i \otimes_{\mathbf{Z}G_\theta} \Lambda = 0$ (see* 3.13 *for a definition of Λ).*

(4) *the $\{K_i\}$ are all geometrically realizable (i.e. they are the single non-vanishing homology module of a connected space equipped with a free G_θ – action).*

Remarks. (1) Note that the condition on the map $\ell \circ \jmath$ in statement 2 above actually imposes conditions on the module $\mathscr{H}$ as well—it must be stably isomorphic to $\mathscr{R}(j_s)$.

(2) Note that conditions 1 and 3 are *necessary* as well as sufficient. Condition 4 and part of condition 2 could be eliminated if the question of the existence of equivariant Moore spaces (the Steenrod problem) could be resolved in the affirmative.

(3) The results of [1], [2], and [31] imply that condition 4 will be satisfied when each of the K_i satisfy any one of the following conditions:

(1) *K_i is of homological dimension ≤ 2;*
(2) *$K_i/pK_i = 0$ for all primes $p < 1 + (\dim_{ZG_\theta}(K_i))/2$;*
(3) *G_θ is a cyclic group.*

Note that these conditions are satisfied in all of the classical cases as well as the case in which G_θ is a *finite* group.

(4) Condition B will be satisfied whenever the corresponding condition for M is satisfied since E contains $S(\xi)$—an S^1-bundle over M. This happens, for instance, whenever M is simply-connected or whenever the fundamental group of M comes from factors that are aspherical (e.g. sufficiently large irreducible 3-manifolds).

Proof. Let $\mathscr{J}(j_s)$ be a relative Jacobian of a presentation of the pair $(G, j_s(G_\theta))$ such that the associated homomorphism $h(j_s)\colon \mathscr{R}(j_s) \to H_2([K, K]; \mathbf{Z})$ (see 3.6, 3.8, and 3.9) is s-equivalent to $t \circ s\colon \mathscr{H} \to H_2([K, K]; \mathbf{Z})$. It follows that there exists free modules F_1 and F_2 such that $\mathscr{H} \oplus F_1$ is isomorphic to $\mathscr{R}(j_s) \oplus F_2$. Without loss of generality, we can assume that $F_2 = 0$ since, if not, we can modify the presentation of $(G, j(G_\theta))$ by adjoining trivial relations to it to add a zero matrix to $\mathscr{J}(j_s)$ of suitable size. Thus we have a commutative diagram

$$\begin{array}{ccc}
\mathscr{H} \oplus F_1 & \xrightarrow{\imath} & \mathscr{R}(j_s) \\
{\scriptstyle t \circ s \oplus 0} \searrow & & \swarrow {\scriptstyle h(j_s)} \\
& H_2([K, K]; \mathbf{Z}). &
\end{array}$$

By hypothesis, we have an exact sequence $0 \to F \to \mathscr{H} \to L \to 0$ so that it follows that $F \oplus F_1 \subset \mathscr{H} \oplus F_1$ maps to 0 under $t \circ s \oplus 0$. Let $\hat{F}$ be the image of $F \oplus F_1$ under the isomorphism $\imath$. Then it is clear that the diagram

$$\begin{array}{ccccccccc}
0 & \to & \hat{F} & \to & \mathscr{R}(j_s) & \to & L & \to & 0 \\
& & & & {\scriptstyle h(j_s)} \searrow & & \downarrow {\scriptstyle s'} & & \\
& & & & & & H_2([K, K]; \mathbf{Z}) & &
\end{array}$$

commutes where the upper row is exact and s' is the composite of s with an automorphism of L. Now we are in a position to *realize* K_1 and L *geometrically*. Let $\langle x, y; r, s\rangle$ be a presentation of the pair $(G, j_s(G_\theta))$ whose relative Jacobian is the matrix $\mathscr{J}(j_s)$ in the dimension above. Here $\langle x; r\rangle$ is a presentation of G_θ and y and s represent the additional generators and relations, respectively, required to present G. We assume that E has a cell-decomposition whose 2-skeleton is a *cellular realization* of the presentation $\langle x; r\rangle$ of G_θ—see [14]. We will construct a complex E' by attaching cells to E representing $\{y\}$ and $\{s\}$. Let U be the union of E (off ∂E) with a wedge of circles that are in a 1-1 correspondence with the elements of $\{y\}$. Let u: $U \to E$ be an extension of the identity map of E to the attached 1-spheres that induces the composite homomorphism on fundamental groups:

$$G_\theta * F_y \underset{q}{\to} G \underset{j_s}{\to} G_\theta,$$

where f_y is the free group on the elements of $\{y\}$, and q is the projection $G_\theta * F_y \to G$ defined by the presentation of G. Define E_2 to be the result of attaching 2-cells to U corresponding to the relations $\{s\}$ and let p_2: $E_2 \to E$ be the (unique up to homotopy) extension of u to E_2. The universal coefficient spectral sequence for E_2 gives rise to the exact sequence

$$\tilde{\mathscr{R}}(j_s) \underset{ZG}{\otimes} ZG_\theta \underset{\varphi}{\to} \mathscr{R}(j_s) \xrightarrow[\not{h}(j_s)]{} H_2([K, K]; Z) \to 0$$

and since the Hurewicz homomorphism

$$\pi_2(E_2) \to \tilde{\mathscr{R}}(j_s) \oplus H_2(E; ZG_\theta)_{ZG_\theta} \otimes ZG = H_2(E_2; ZG)$$

is *surjective*, it follows that we can attach 3-cells to E_2 representing basis elements of $\hat{F}$ (strictly speaking, we are attaching cells representing basis elements of $\hat{F} \otimes_{ZG_\theta} ZG$).

Call the resulting complex E_3. It will clearly have the following properties:

(1) $\pi_1(E_3) = G$;
(2) $H_2(E_3; ZG_\theta) = L \oplus H_2(E; ZG_\theta)$;
(3) $H_i(E_3; ZG_\theta) = H_i(E; ZG_\theta)$, $i > 2$.

We have thus *geometrically realized* the module L.

In order to realize the higher dimensional homology modules, we use the results of [31] (specifically case 1 of Theorem 3). We get a CW-complex X with the properties:

(1) $\pi_1(X) = G_\theta$;
(2) $H_0(X) = Z$;
(3) $H_i(X; ZG_\theta) = K_i$, $i > 2$, $H_2(X; ZG_\theta) = K'_2$;
(4) *X contains an imbedded $K(G_\theta, 1)$.*

Remark. We may assume, without loss of generality, that X has a *finite number of cells in each dimension.* This follows from the results of [30] and the fact that the K_i and K'_2 are finitely generated and the ring ZG_θ is noetherian.

Let $\hat{E}$ be the result of forming the union of X with the mapping cylinder of the composite

$$K(G_\theta, 1) \xrightarrow{r} E \to E_3$$

along imbedded $K(G_\theta, 1)$'s (recall that the map $r\colon K(G_\theta, 1) \to E$ induces an isomorphism of fundamental groups and that its existence is guaranteed by condition C in the hypothesis). The complex $\hat{E}$ has the following properties.

(1) $\pi_1(\hat{E}) = G$;
(2) *$\hat{E}$ contains a subcomplex E such that:*
(a) *the inclusion of E in $\hat{E}$ induces $j_s\colon G_\theta \to G$ on fundamental groups;*
(b) *$H_j(\hat{E}, E; \mathbf{Z}G_\theta) = K_j$ for all $1 \le j \le \mu$;*
(c) *$H_j(\hat{E}, E; \mathbf{Z}G_\theta) = 0$ for all $\mu < j \le [(m+1)/2]$;*
(d) *$H_j(\hat{E}, E; \mathbf{Z}\pi_1(V)) = 0$ for all $j > 0$.*

Claim. We may assume, without loss of generality, that $(\hat{E}, E)$ is actually simply acyclic with local coefficients in $\mathbf{Z}\pi_1(V)$.

Suppose the Whitehead torsion of the inclusion of E in $\hat{E}$ is represented by an invertible $n \times n$ matrix with entries in $\mathbf{Z}\pi_1(V)$. This lifts to a matrix A with entries in $\mathbf{Z}G_\theta$. Let U be the one point union of a $K(G_\theta, 1)$ with n l-spheres where $l = 2[(m+1)/2] + 1$. Then $\pi_l(U) = F$, where F is a free $\mathbf{Z}G_\theta$-module of rank n and with canonical basis elements represented by the l-spheres above. Now attach n $(l+1)$-cells via maps representing the images of the canonical basis elements under A. Call the result U'. The union of U' with $\hat{E}$ along suitably imbedded $K(G_\theta, 1)$'s will clearly have the required properties.

We are now ready to construct the imbedding $f'\colon M \to V$. An argument exactly like that used in the proof of Lemma 4.3 shows that the simple $\mathbf{Z}\pi_1(V)$-homology equivalence $i\colon E \to \hat{E}$ (i.e. the inclusion) is $\mathbf{Z}\pi_1(V)$-homology s-cobordant rel $S(\xi)$ to a simple homology equivalence that is μ-connected. Let the cobordism be $F\colon (W; E, E') \to \hat{E}$. It follows, by the s-cobordism theorem, that there exists a homeomorphism

$$H\colon (W; E, E') \bigcup_{S(\xi)\times I} T(\xi) \times I \to V \times I$$

that is essentially the *identity map* on $E \bigcup_{S(\xi)\times\{0\}} T(\xi) \times \{0\}$. Define the imbedding $f'\colon M \to V$ to be the composite

$$M \xrightarrow{z} T(\xi) \to E' \bigcup_{S(\xi)\times\{1\}} T(\xi) \times \{1\} \xrightarrow{H|E' \cup T(\xi)} V \times \{1\}$$

The *complementary map* of this realization is defined to be the composite

$$H(E') \xrightarrow{H^{-1}} E' \xrightarrow{F|E'} \hat{E} \xrightarrow{\beta} E$$

where β is the composite

$$E_r \bigcup_{K(G_\theta, 1)} X \xrightarrow{1 \cup \gamma} E_r \bigcup_{K(G_\theta, 1)} K(G_\theta, 1) = E_r \xrightarrow{\delta} E$$

where E_r is the mapping cylinder of $r\colon K(G_\theta, 1) \to E$, γ is essentially the characteristic map of X to $K(G_\theta, 1)$ (if necessary r in the middle portion of the diagram above is replaced by its composite with an auto-homotopy equivalence of $K(G_\theta, 1)$), and δ is just the standard deformation retraction of E_r onto E. ∎

COROLLARY 4.4. *The conclusions of Theorem 4.3 if we replace the condition that the Poincaré imbedding θ be induced by an actual imbedding of compact manifolds by the condition that there exists a map $K(\pi_1(M), 1) \to M$ inducing an isomorphism of fundamental groups and that θ possess a realization.*

Proof. Lemma 3.2 implies that the imbedding f that is a realization of θ is concordant to a realization $f'M \to V$ whose complementary map is $[(m+1)/2]$-connected. If $\theta_{f'}$ is the Poincaré imbedding induced by f' then $\theta_{f'}$ satisfies all of the hypothesis of Theorem 4.3 and any realization of $\theta_{f'}$ is also a realization of θ. ∎

We will try to sharpen these results somewhat using the χ-invariant described in Section 2. Note that, so far, we have been able to construct prescribed homology modules only up to two dimensions below the middle dimension. Furthermore, the homology module one dimension below the middle dimension was generally not mapped isomorphically by the complementary map—there was a kernel that could not be prescribed or made to vanish. This kernel measured the error that resulted from approximating a possibly infinite dimensional chain complex by one bounded by the middle dimension of the Poincaré imbedding.

We will use the following notation: If A is a right $\mathbf{Z}G_\theta$-module $e^i(A)$ will denote $\overline{\mathrm{Ext}^i_{\mathbf{Z}G_\theta}(A, \mathbf{Z}G_\theta)}$—this is similar to the notation of Levine in [20] except that we take the conjugate of the Ext.

With this in mind, our main result is the following.

THEOREM 4.5. *Let $\theta = (E, \xi, h)$ be a Poincaré imbedding induced by an imbedding of compact manifolds $f: M^m \to V^{m+2}$ with $m \geq 3$, satisfying the conditions in* 3.1 *and, additionally:*

(A) *$\pi_1(V)$ is a finite extension of a polycyclic group;*

(B) *there exists a map $r: K(G_\theta, 1) \to E$ inducing an isomorphism of fundamental groups—see Remark* 3 *following* 4.3.

Suppose G is a group satisfying the conditions of 4.1 (*also see* 3.3 *and* 3.4) *and $\{K_i\}$, $1 \leq i \leq [(m+1)/2]$, is a sequence of right $\mathbf{Z}G_\theta$-modules such that:*

(1) *$K_1 = H_1([K, K]; \mathbf{Z})$ (see* 3.1 *and* 3.5);

(2) *There exists a surjective homomorphism $\ell: K_2 \to H_2([K, K]; \mathbf{Z})$, a finitely generated right $\mathbf{Z}G_\theta$-module $\mathscr{H}$, and a homomorphism $\jmath: \mathscr{H} \to K_2$, such that the map $\ell \circ \jmath$ is s-equivalent (see* 3.1. *and* 3.8) *to $\wp(j_s): \mathscr{R}(j_s) \to H_2([K, K]; \mathbf{Z})$ (see* 3.6 *and* 3.9);

(3) *The* $\{K_i\}$ *are finitely generated* $\mathbf{Z}G_\theta$, *modules such that* $K_i \otimes_{\mathbf{Z}G_\theta} \Lambda = 0$ (*see* 3.11 *for a definition of* Λ);

(4) $$\dim_{\mathbf{Z}G_\theta}(K_i) \leq \begin{cases} [(m+1)/2] - i + 1 & \text{if } i > 2; \\ [(m+1)/2] - i & \text{if } i \leq 2: \end{cases}$$

(5) $\sum(-1)^i\chi(K_i) = \Psi$ (*see Section* 2 *for a definition of* $\chi(K_i)$), *where* Ψ *is in the kernel of the boundary homomorphism* $K_0'(\mathbf{Z}G_\theta \to \mathbf{Z}\pi_1(V)) \to K_0(\mathbf{Z}G_\theta)$ (*see* [1, *p.* 447]);

(6) *The* $\{K_i\}$ *are geometrically realizable—see the remarks following* 4.3.

Then there exists a realization $f'\colon M \to V$, *where* V' *is a manifold h-cobordant to* V, *of* θ *with complementary map* $c\colon E' \to E$ *such that:*

(i) $\pi_1(E') = G$;
(ii) $H_i(E'; \mathbf{Z}G_\theta) = H_i(E; \mathbf{Z}G_\theta) \oplus K_i$, *for* $i < [(m+1)/2]$;
(iii) $H_k(E'; \mathbf{Z}G_\theta) = H_k(E; \mathbf{Z}G_\theta) \oplus K_k \oplus_1^{k-2} e^i(K_{k-i+1}))$,

where $k = [(m+1)/2]$, *and the Whitehead torsion of the homotopy equivalence* $V' \to V$ *is* $\rho + (-1)^{m+1}\bar{\rho}$, *where* ρ *is any element of* $K_1(\mathbf{Z}\pi_1(V))$ *that maps to* Ψ *under the connecting homomorphism* $K_1(\mathbf{Z}\pi_1(V)) \to K_0'(\mathbf{Z}G_\theta \to \mathbf{Z}\pi_1(V))$ (*see* [4, *p.* 447]).

Furthermore, if ψ *is zero, an imbedding* f' *with the properties described above can be constructed that is a normal realization of* θ *that is concordant to* f.

Remark. As in 4.3, the map induced in homology by the complementary map is projection of the right-hand sides of the expressions in (i) and (ii) onto the first factor.

Proof. Since the proof is very similar to that of 4.3, we will only indicate the differences—our notation will be the same as that of the proof of 4.3.

The restrictions on the homological dimensions of the $\{K_i\}$ imply that the complex $\hat{X}$ will be *homotopy-equivalent* to a complex that has no cells of dimension larger than $[(m+1)/2]$.

Theorem 2.9 shows that $\chi(\hat{E}, E) = \tilde{\psi}'$ is an element of $K_0'(\mathbf{Z}G \to \mathbf{Z}\pi_1(V))$ that maps to ψ in $K_0'(\mathbf{Z}G_\theta \to \mathbf{Z}\pi_1(V))$ under the change of rings $\mathbf{Z}G \to \mathbf{Z}G_\theta$.

The exact sequence of a triple in algebraic K-theory (see p. 448 of [4]) and statement 7 of 2.6 imply that there exists a finitely generated projective right $\mathbf{Z}G$-chain complex P_* with non-vanishing chain-module in dimensions 2 and 3 only such that $\tilde{\psi}' + \chi(P_*) = \tilde{\psi}$ is contained in $\ker K_0'(\mathbf{Z}G \to \mathbf{Z}\pi_1(V)) \to K_0(\mathbf{Z}G)$ and $P_* \otimes_{\mathbf{Z}G} \mathbf{Z}G_\theta$ is *acyclic*.

We may clearly attach 2- and 3-cells to $\hat{E}$ forming $\hat{E}'$ such that the cellular chain complex of $(\hat{E}', \hat{E})$ is P_* (if necessary, perform the Eilenberg trick to replace P_* by an infinitely generated *free* chain complex).

The theory of the Wall finiteness obstruction now implies that the pair $(\hat{E}', E)$ is homotopy equivalent to a *finite* relative CW-complex which, by abuse of notation, we will denote by $(\hat{E}, E)$.

In the last stage of the proof we use an argument similar to that in the proof of Theorem 1.2 of [29] rather than that of Lemma 4.3 in [6] (which was done in the proof of Theorem 4.3 of the present paper). In other words we consider the surgery problem $i \times p \colon E \times I \to \hat{E}$ where i is the inclusion $E \to \hat{E}$ and p is the projection $p \colon I \to \{0\}$, and we attach handles to $E \times \{1\}$ corresponding to the cells that were attached to E to form $\hat{E}$ (since i is an integral homology equivalence it can clearly be *framed*). The resulting cobordism will clearly be a $\mathbf{Z}\pi_1(V)$-homology h-cobordism rel $S(\xi)$ and its upper end will clearly map to $\hat{E}$ via a homology equivalence. The statement about the Whitehead torsion of the homotopy equivalence $V' \to V$ follows from statement 5 of 2.6, and statement (iii) about $H_k(E'; \mathbf{Z}G_\theta)$ is a direct consequence of a description of the operation of attaching handles on the *chain level.*

We will conclude this paper with an application of the preceding theorem to knotted lens spaces. Throughout this discussion L_1^{2k-1} and L_2^{2k+1} will denote homotopy lens spaces of index n (i.e., quotients of spheres by free $\mathbf{Z}_n$-actions), where n is an odd integer and such that there exists a locally-flat imbedding of L_1 in L_2—Theorem 9.5 in [6] gives *necessary* and *sufficient* conditions for this to happen. First recall Corollary 4 in [26] which characterizes complementary fundamental groups of knotted lens spaces:

PROPOSITION 4.6. *A group G can be the fundamental group of the complement of a locally-flat imbedding of L_1 in L_2 if and only if:*

(1) *G is finitely presented;*

(2) *G is the normal closure of an element x such that $G/(x^n)^G = \mathbf{Z}_n$, where $(x^n)^G$ is the normal closure of x^n;*

(3) *$H_1((x^n)^G; \mathbf{Z}) = \mathbf{Z}$ and $H_2((x^n)^G; \mathbf{Z}) = 0$.* ∎

Our result on the higher dimensional homology is as follows.

THEOREM 4.7. *Suppose, in addition to the assumption that there exists an imbedding of L_1 in L_2, that L_2 is h-cobordant to a suspension of L_1. If G is a group satisfying the conditions in Proposition 4.6 and $\{A_i\}$, $1 \le i \le k$, are a sequence of $\mathbf{Z}[\mathbf{Z}]$-modules satisfying the conditions:*

(1) *$K_1 = H_1([x^n)^G, (x^n)^G]; \mathbf{Z})$;*

(2) *There exists a surjective homomorphism from the $\mathbf{Z}$-torsion free summand of K_2 to $H_2([x^n)^G, (x^n)^G]; \mathbf{Z})$;*

(3) *The K_i are finitely generated and $K_i \otimes_{\mathbf{Z}[\mathbf{Z}]} \Lambda = 0$—see 3.12 and 3.13;*

(4) *K_k is $\mathbf{Z}$-torsion free;*

(5) *$\prod_{i=1}^k \{P_{f(K_i)}(\tau)P_{f(K_i)}(\tau^{-1})\}^{(-1)^i} = \Delta(L_1)\Delta(L_2)^{-1}(\tau^d - 1)$, up to multiples by nth roots of unity, where τ is a primitive nth root of unity, $P_{f(K_i)}$ is defined in 2.11, and Δ and d are defined in the discussion preceding 1.9.*

Then there exists a locally-flat imbedding of L_1 in L_2 with complement E such that:

(i) $\pi_1(E) = G$;
(ii) $H_i(E; \mathbf{Z}[\mathbf{Z}]) = K_i$, *for* $1 \leq i < k$;
(iii) $H_k(E; \mathbf{Z}[\mathbf{Z}]) = K_k \oplus e^1(K_k) \oplus e^2(K_{k-1})$—*see the discussion preceding* 4.5.

Remark. (1) If we identify $\mathbf{Z}[\mathbf{Z}]$ with $\mathbf{Z}[t, t^{-1}]$ the condition that $K_i \otimes_{\mathbf{Z}[\mathbf{Z}]} \Lambda = 0$ is *equivalent* to the condition that K_i be *annihilated* by a Laurent polynomial, $p(t^n)$, such that $p(1) = \pm 1$—see Corollary 3 in [27].

(2) The requirement that L_2 be h-cobordant to a suspension of L_1 results from our working below the middle dimension. All of the imbeddings we construct realizing homology modules are *cobordant* to a *standard imbedding.* In general, however, not only will no *unknotted* imbeddings of L_1 in L_2 exist (see the discussion following 2.12)—there may not even exist any imbeddings *cobordant* to a standard imbedding.

A later paper in this series will prove a more general result that takes the cobordism theory into account (as well as its interaction with the middle-dimensional homology) and the requirement that L_2 be h-cobordant to a suspension of L_1 will be eliminated. ∎

Proof. This theorem is an immediate consequence of Theorem 4.3, Lemma 3.11, 2.12, and the fact that $\mathbf{Z}[\mathbf{Z}]$ has global homological dimension 2—here, θ is the Poincaré imbedding defined by the *standard* inclusion of L_1 into its suspension with invariant d—see 1.9.

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