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INJECTIVE BP_*BP-COMODULES AND LOCALIZATIONS OF BROWN-PETERSON HOMOLOGY

BY

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1. Introduction

BP is the Brown-Peterson spectrum for a fixed prime p; its homotopy is

$$BP_* \cong \mathbf{Z}_{(p)}[v_1, v_2, \ldots].$$

By convention, $v_0 = p$. $BP_*X = \pi_*(BP \wedge X)$ is a comodule over $BP_*BP \cong BP_*[t_1, t_2, \ldots]$. Let $\mathscr{BP}$ be the category of all BP_*BP-comodules and comodule maps. The only prime ideals of BP_* which are in $\mathscr{BP}$ are

$$I_0 = (0),\ I_1 = (p), \ldots, I_n = (p, v_1, \ldots, v_{n-1}), \ldots,$$

and

$$I_\infty = \bigcup_n I_n = (p, v_1, v_2, \ldots).$$

The Hurewicz homomorphism gives a right unit $\eta_R\colon BP_* \to BP_*BP$ and $\eta_R(v_n) \equiv v_n$ modulo $I_n BP_*BP$. (N.B. $\eta_R(v_1) = v_1 + pt_1 \neq v_1$.)

We say that a BP_*-module M is $\mathscr{BP}$-injective if $\mathrm{Ext}^i_{BP_*}(A, M) = 0$ for all $i > 0$ and all comodules A in $\mathscr{BP}$. We define the $\mathscr{BP}$-weak dimension of M, $\mathrm{w.dim}_{\mathscr{BP}}\, M$, to be less than $n + 1$ if $\mathrm{Tor}_j^{BP_*}(A, M) = 0$ for all $j > n$ and all comodules A in $\mathscr{BP}$. If M, itself, is a connected comodule in $\mathscr{BP}$, $\mathrm{w.dim}_{\mathscr{BP}}\, M$ is the same as the BP_*-projective dimension of M [8]. Our main algebraic result can be considered to be the dual of Landweber's exact functor theorem [8].

THEOREM 1.1. *For a BP_*-module M to be $\mathscr{BP}$-injective, it suffices that it satisfy two conditions:*

(i) *For each integer $n \geq 0$, $\mathrm{Hom}_{BP_*}(BP_*/I_n, M)$ is v_n-divisible.*
(ii) $\mathrm{w.dim}_{\mathscr{BP}}\, M < \infty$.

Miller, Ravenel, and Wilson [13] develop a "chromatic resolution" of

$$BP_*\colon 0 \to BP_* \to M^0 \to M^1 \to \cdots.$$

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It is defined by short exact sequences of $BP_* BP$-comodules

$$0 \to N^s \xrightarrow{f} M^s \to N^{s-1} \to 0$$

where $N^0 = BP_*$, $M^s = v_s^{-1}N^s$, and f is the localization homomorphism.

COROLLARY 1.2. *The Miller-Wilson chromatic resolution is a $\mathscr{BP}$-injective resolution of BP_* in that each M^s is $\mathscr{BP}$-injective.*

We prove this theorem and its corollary in Section 3. We employ the chromatic resolution to study spectra which are local for the direct sum homology theory $\oplus_{0 \leq n} v_n^{-1}BP_*(\ \ )$. Our discussion of localization with respect to BP-related periodic homology theories comes in the final section, Section 4. Before this discussion—and even before our study of $\mathscr{BP}$-injective modules—we can state and prove (in Section 2) our main localization result. For spectra X and Y, $[X; Y]_*$ denotes the group of stable homotopy classes of maps from X to Y. Let $M\mathbf{Z}_{(p)}$ be the $\mathbf{Z}_{(p)}$-Moore spectrum and $Y\mathbf{Z}_{(p)} = Y \wedge M\mathbf{Z}_{(p)}$. $Y\mathbf{Z}_{(p)}$ is the BP_*-localization of Y if Y is connective [1, Section III-6, III-14;2].

THEOREM 1.3. *Let Y be a connective spectrum such that the projective dimension of $BP_* Y$ over BP_* is finite. If X is a spectrum such that $v_n^{-1}BP_*X = 0$ for all $n \geq 0$, then $[X; Y\mathbf{Z}_{(p)}]_* = 0$.*

The mod p Eilenberg-MacLane spectrum $H\mathbf{F}_p$ has

$$\begin{aligned} v_n^{-1}BP_*H\mathbf{F}_p &\cong v_n^{-1}BP_*BP/(\eta_R(p), \eta_R(v_1), \ldots) \\ &\cong v_n^{-1}BP_*BP/(p, v_1, \ldots) = 0. \end{aligned}$$

If Y is a finite spectrum, $\text{w.dim}_{\mathscr{BP}}\ BP_* Y < \infty$ [4].

COROLLARY 1.4. (Margolis [11], Lin [10].) *If Y is a finite complex then $[H\mathbf{F}_p; Y]_* = 0$.* ■

Margolis and Lin each prove the stronger result that $[H\mathbf{F}_p; Y]_* = 0$ for any CW complex Y with finite skeleta. See Question 4.3.

Our work is motivated by Doug Ravenel's ideas on localization with respect to BP-related periodic homologies. We are grateful to Ravenel for making his typescript [14] available.

2. Proof of Theorem 1.3

Throughout this section, let $A \otimes B$ mean $A \otimes_{BP_*} B$. A BP_*-module B is $\mathscr{BP}$-flat ($\text{w.dim}_{\mathscr{BP}}\ B = 0$) if $\text{Tor}_j^{BP_*}(A, B) = 0$ for all $j > 0$ and all $BP_* BP$-comodules A.

LEMMA 2.1. *Let B be a $\mathscr{BP}$-flat BP_*-module. Suppose A is a BP_*-module such that $v_s^{-1}A = 0$ for all $s \leq n$. Then $\text{Ext}_{BP_*}^s(A, B) = 0$ for all $s \leq n$.*

Proof. We follow Miller-Ravenel-Wilson [13] and define BP_*BP-comodules $N^s, M^s\ s \geq 0$: $N^0 = BP^*$, $M^s = v_s^{-1}N^s$, and N^{s+1} is the cokernel of the localization homomorphism $N^s \to v_s^{-1}N^s = M^s$. The sequence

$$0 \to N^s \to M^s \to N^{s+1} \to 0 \tag{2.2}$$

is short exact and is in $\mathscr{BP}$. Since B is $\mathscr{BP}$-flat, (2.2) induces the following short exact sequence:

$$0 \to B \otimes N^s \to B \otimes M^s \to B \otimes N^{s+1} \to 0. \tag{2.3}$$

If $s \leq n$,

$$\begin{aligned}\mathrm{Ext}^*_{BP_*}(A, B \otimes M^s) &\cong \mathrm{Ext}^*_{BP_*}(A, v_s^{-1}(B \otimes N^s))\\ &\cong \mathrm{Ext}_{v_s^{-1}BP_*}(v_s^{-1}A, v_s^{-1}(B \otimes N^s))\\ &= 0 \quad (\text{since } v_s^{-1}A = 0).\end{aligned}$$

By the exactness of (2.3),

$$\begin{aligned}\mathrm{Ext}^s_{BP_*}(A, B) = \mathrm{Ext}^s_{BP_*}(A, B \otimes N^0) &\cong \mathrm{Hom}_{BP_*}(A, B \otimes N^s)\\ &\rightarrowtail \mathrm{Hom}_{BP_*}(A, B \otimes M^s) = 0 \quad \text{for } s \leq n.\end{aligned}$$ ■

COROLLARY 2.4. *Let B be a BP_*-module with* $\mathrm{w.dim}_{\mathscr{BP}}\, B < \infty$. *Suppose A is a BP_*-module such that $v_s^{-1}A = 0$ for all $s \geq 0$, then* $\mathrm{Ext}^*_{BP_*}(A, B) = 0$.

Proof. Induct over the $\mathscr{BP}$-weak dimension of B using Lemma 2.1 at the initial stage. ■

Let $\overline{BP} = BP/S$, the cofiber of the inclusion of the sphere spectrum S into BP. Let $\overline{BP}^s = \overline{BP} \wedge \cdots \wedge \overline{BP}$, s times. $BP_*\overline{BP}^s$ is a free BP_*-module and

$$BP_*(\overline{BP}^s \wedge Y) \cong BP_*(\overline{BP}^s) \otimes BP_* Y.$$

Hence the $\mathscr{BP}$-weak dimensions of $BP_*(\overline{BP}^s \wedge Y)$ and $BP_* Y$ are identical. So we have:

COROLLARY 2.5. *Let Y be a spectrum with* $\mathrm{w.dim}_{\mathscr{BP}}\, BP_* Y < \infty$. *Suppose X is a spectrum such that $v_n^{-1}BP_* X = 0$ for all $n \geq 0$. Then*

$$\mathrm{Ext}^*_{BP_*}(BP_* X, BP_*(\overline{BP}^s \wedge Y)) = 0 \quad \textit{for all } s \geq 0.$$ ■

Geometric BP_*-resolutions exist [4]: for any spectrum X, there are cofibrations

$$\Sigma^{-1} X_{s+1} \to A_s \to X_s \xrightarrow{f_s} X_{s+1} \to \Sigma A_s$$

with (i) $X_0 = X$, (ii) $BP_*(f_s) \equiv 0$, and (iii) $BP_* A_s$ BP_*-free. The hypotheses, then, of Theorem III.13.6, p. 285 of [1] are satisfied. There is a universal coefficient spectral sequence

$$\mathrm{Ext}^*_{BP_*}(BP_* X, BP_*(\overline{BP}^s \wedge Y)) \Rightarrow [X; BP \wedge \overline{BP}^s \wedge Y]_*.$$

Thus for X and Y as in Corollary 2.5, $[X; BP \wedge \overline{BP}^s \wedge Y]_* = 0$ for all $s \geq 0$. We form an Adams resolution of Y by the cofibrations

$$\overline{BP}^{s+1} \wedge Y \xrightarrow{d_{s+1}} \overline{BP}^s \wedge Y \xrightarrow{e_s} BP \wedge \overline{BP}^s \wedge Y \to \overline{BP}^{s+1} \wedge Y.$$

The inclusion of the sphere spectrum into BP induces e_s. Since

$$[X; BP \wedge \overline{BP}^s \wedge Y]_* = 0,$$

$d_{s+1*}: [X; \overline{BP}^{s+1} \wedge Y] \to [X; \overline{BP}^s \wedge Y]$ is an isomorphism and

$$[X; Y]_* \cong \varprojlim_s [X; \overline{BP}^s \wedge Y]_*.$$

THEOREM 2.6. *Let Y be a p-local, connective spectrum such that the projective dimension of $BP_* Y$ over BP_* is finite. Then Y is $(\bigvee_{0 \leq n} v_n^{-1} BP)_*$-local: if X is any spectrum with $v_n^{-1} BP_* X = 0$ for all $n \geq 0$, then $[X; Y] = 0$.*

Proof. The hypotheses that Y be p-local and connective ensure that

$$\varprojlim_s [X; \overline{BP}^s \wedge Y]_* = 0.$$

See Theorem III.15.1, pp. 316 ff of [1]. ■

Remark 2.7. A reading of the proof of Theorem III.13.6 of [1] reveals that if

$$\mathrm{Ext}^j_{BP_*}(BP_* X, BP_*) = 0, \quad 0 \leq j \leq n,$$

then any map $g: X \to BP$ factors as a composite

$$g: X \xrightarrow{f_0} X_1 \to \cdots \to X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{g'} BP.$$

Since each $BP_*(f_i) \equiv 0$, g has Adams-Novikov (BP) filtration at least $n + 1$.

3. Injective $BP_* BP$-comodules

Recall that $\mathscr{BP}$ is the category of $BP_* BP$-comodules. Let $\mathscr{BP}_0$ be the subcategory of $\mathscr{BP}$ of finitely presented comodules. In this section, we study certain BP_*-module properties related to comodules in $\mathscr{BP}$ or $\mathscr{BP}_0$. Accordingly, we adopt the conventions that $\mathrm{Ext}^j(A, B)$ and $\mathrm{Tor}_j(A, B)$ mean $\mathrm{Ext}^j_{BP_*}(A, B)$ and $\mathrm{Tor}_j^{BP_*}(A, B)$, respectively, throughout this section.

Any such study of BP_*-module properties of $BP_* BP$-comodules properly begins with the Landweber filtration theorem [6], [7] which states that any comodule A in $\mathscr{BP}_0$ has a finite filtration whose associated subquotients are stably isomorphic to cyclic comodules of the form BP_*/I_n, $0 \leq n < \infty$. Here the I_n are the prime, $BP_* BP$-invariant ideals of BP_* defined by

$$I_0 = (0), I_1 = (p), \ldots, I_n = (p, v_1, \ldots, v_{n-1}).$$

We also define $I_\infty = \bigcup_n I_n = (p, v_1, v_2, \ldots)$. The effect of the filtration theorem extends to $\mathscr{BP}$ in that any comodule A of $\mathscr{BP}$ is a direct limit of comodules in $\mathscr{BP}_0$ [12, Lemma 2.11]. It is meet and right to list some homological properties of the cyclic comodules BP_*/I_n, $0 \leq n < \infty$. Throughout this list (3.1–3.6), M will stand for an arbitrary BP_*-module.

(3.1) For $0 \leq s < \infty$, there are exact sequences in $\mathscr{BP}_0$:

$$0 \to BP_*/I_s \xrightarrow{v_s} BP_*/I_s \to BP^*/I_{s+1} \to 0.$$

(3.2) As a BP_*-module, the projective dimension of BP_*/I_s is s. Hence

$$\mathrm{Ext}^t\,(BP_*/I_s, M) = 0 = \mathrm{Tor}_t\,(BP_*/I_s, M) \quad \text{for } t > s.$$

(3.3) The sequence (3.1) induces the exact sequence

$$\begin{aligned} &\mathrm{Ext}^t\,(BP_*/I_s, M) \xrightarrow{v_s{}^\#} \mathrm{Ext}^t\,(BP_*/I_s, M) \\ &\qquad \to \mathrm{Ext}^{t+1}\,(BP_*/I_{s+1}, M) \to \mathrm{Ext}^{t+1}\,(BP_*/I_s, M) \cdots. \end{aligned}$$

(3.4) By an induction using the sequence 3.3 (with $s = t$), we may identity $M/I_nM \cong \mathrm{Ext}^n\,(BP_*/I_n, M)$.

(3.5) Define ${}_nM = \{x \in M\colon I_nx = 0\}$. So $M = {}_0M \supset {}_1M \supset {}_2M \supset \cdots$ gives a decreasing filtration of M by BP_*-submodules. We may identify ${}_nM \cong \mathrm{Ext}^0\,(BP_*/I_n, M)$.

(3.6) There is a Koszul duality isomorphism

$$\mathrm{Ext}^s\,(BP_*/I_n, M) \cong \mathrm{Tor}_{n-s}\,(BP_*/I_n, M).$$

See pages 150–153 and 159 (Exercise 7) of [3].

A BP_*-module M has $\mathscr{BP}_0$-injective dimension $\leq n$ if $\mathrm{Ext}^j\,(A, M) = 0$ for all $j > n$ and all comodules A in $\mathscr{BP}_0$; we write $\mathrm{inj\ dim}_{\mathscr{BP}_0} M \leq n$. If $\mathrm{inj\ dim}_{\mathscr{BP}_0} M = 0$, we say M is $\mathscr{BP}_0$-injective. $\mathscr{BP}$-injectivity is defined similarly. Dually, M has $\mathscr{BP}$-weak dimension $\leq n$ ($\mathrm{w.dim}_{\mathscr{BP}} M \leq n$) provided that $\mathrm{Tor}_j\,(A, M) = 0$ for all $j > n$ and all comodules A in $\mathscr{BP}$.

LEMMA 3.7. *Let M be a BP_*-module. Then* $\mathrm{Inj\ dim}_{\mathscr{BP}_0} M \leq n$ *if and only if for each $s \geq 0$,* $\mathrm{Ext}^n\,(BP_*/I_s, M)$ *is v_s-divisible.*

Proof. By (3.2), $\mathrm{Ext}^{n+1}\,(BP_*/I_s, M) = 0$ for $0 \leq s \leq n$. Use (3.3) $(t = n)$ to begin an induction on $s \geq n$ to prove that if $\mathrm{Ext}^n\,(BP_*/I_s, M)$ is v_s-divisible then

$$\mathrm{Ext}^{n+1}\,(BP_*/I_{s+1}, M) = 0.$$

If $\mathrm{Ext}^{n+1}\,(BP_*/I_s, M) = 0$ for all $s \geq 0$, then (3.3) shows how to prove that

$$\mathrm{Ext}^t\,(BP_*/I_s, M) = 0 \quad \text{for all } t \geq n+1,\ s \geq 0.$$

By the Landweber filtration theorem, this implies $\mathrm{inj\ dim}_{\mathscr{BP}_0} M \leq n$. The converse should now be obvious. ∎

COROLLARY 3.8. *Let M be a BP_*-module. M is $\mathscr{BP}_0$-injective if and only if for each $s \geq 0$, ${}_sM$ is v_s-divisible.*

Proof. (3.5) and (3.7). ■

COROLLARY 3.9. *Let M be a BP_*-module with* inj $\dim_{\mathscr{BP}_0} M \leq n$. *Then M/I_nM is v_n-divisible.*

Proof. (3.4) and (3.7). ■

PROPOSITION 3.10. *Let M be a connected BP_*-module. If* inj $\dim_{\mathscr{BP}_0} M \leq n$, *then M is a $\mathbf{Q}$ vector space and M is $\mathscr{BP}_0$-injective. (It will follow from Theorem* 3.14 *that M is $\mathscr{BP}$-injective.)*

Proof. By (3.9), M/I_nM is v_n-divisible. For $n > 0$, the connectivity of M allows this only if $M/I_nM = 0$. By a downward induction using (3.3, 3.4) ($t = s$), we see that M/I_kM is v_k-divisible (and hence 0) for $k = n, n-1, \ldots, 1$. Thus $M = M/I_0M$ is p-divisible. When $t = s - 1$, (3.3) has the form

$$\mathrm{Ext}^{s-1}(BP_*/I_s, M) \xrightarrow{v_s\#} \mathrm{Ext}^{s-1}(BP_*/I_s, M) \to \mathrm{Ext}^s(BP_*/I_{s+1}, M) \to 0.$$

Each $\mathrm{Ext}^{s-1}(BP_*/I_s, M)$ is dominated by the connected module ${}_1M = \mathrm{Ext}^0(BP_*/I_1, M)$. By a second downward induction,

$$\mathrm{Ext}^{s-1}(BP_*/I_s, M)$$

is v_s-divisible (and hence 0) for $s = n+1, n, \ldots, 1$. Hence ${}_1M = \{x \in M: px = 0\} = 0$; M is a $\mathbf{Q}$-vector space. Since ${}_sM \subset {}_1M = 0, s > 0$, M is $\mathscr{BP}_0$-injective by Corollary 3.8. ■

COROLLARY 3.11. *Let $M \neq 0$ be a BP_*-module with* inj $\dim_{\mathscr{BP}_0} M \leq n$. *Then there is no integer t such that $v_s^tM = 0$ for $s = 0, 1, \ldots, n$.*

Proof. M/I_nM is v_n-divisible (3.9). If $v_n^tM = 0$, then $v_n^t(M/I_nM) = 0$ meaning that $M/I_nM = 0$. As in the proof of (3.10), this begins a downward induction concluding that $M/I_0M = M$ is p-divisible. Since $p^tM = 0$, we reach the contradiction that $M = 0$. ■

PROPOSITION 3.12. *Let M be a $\mathscr{BP}_0$-injective BP_0-injective BP_*-module. Then* w.dim.$_{\mathscr{BP}}$ *$M \leq n$ if and only if ${}_{n+1}M = 0$.*

Proof. By the Landweber filtration theorem as extended in [12], w.dim$_{\mathscr{BP}}$ $M \leq n$ provided that $\mathrm{Tor}_j(BP_*/I_s, M) = 0$ for $j > n$ and all s. By Koszul duality (3.6), $\mathrm{Tor}_j(BP_*/I_s, M) \cong \mathrm{Ext}^{s-j}(BP_*/I_s, M)$. Since M is $\mathscr{BP}_0$-injective, this latter group is 0 for $j \neq s$. So the obstructions to w.dim$_{\mathscr{BP}}$ $M \leq n$ are precisely the modules $\mathrm{Ext}^0(BP_*/I_s, M) = {}_sM \subset {}_{n+1}M$, $s \geq n+1$. ■

The radical, $\sqrt{J}$, of an ideal J in the ring BP_* is the ideal

$$\sqrt{J} = \{x \in BP_*: x^s \in J, \text{ some } s > 0\}.$$

If J is BP_*BP-invariant, Landweber [9] has proved that $\sqrt{J} = I_n$ for some $n = 0, 1, 2, \ldots, \infty$.

LEMMA 3.13. *For a BP_*-module M to be $\mathcal{BP}$-injective, it is necessary and sufficient for M to:*

(i) *be $\mathcal{BP}_0$-injective;*
(ii) *have* $\operatorname{Ext}^1(BP_*/J, M) = 0$ *for any BP_*BP-invariant ideal J in BP_* with $\sqrt{J} = I_\infty$.*

Proof. Necessity is obvious. To prove sufficiency, we must show that given any inclusion of comodules $i: A \to C$ in $\mathcal{BP}$, $i^\#: \operatorname{Hom}(C, M) \to \operatorname{Hom}(A, M)$ is onto. Fix $f \in \operatorname{Hom}(A, M)$. Let $\mathcal{C}$ be the class of extensions of f of the following form: an element of $\mathcal{C}$ is a BP_*-homomorphism $g: B \to M$ where $A \subset B \subset C$ as sub-comodules in $\mathcal{BP}$ and $g \mid A = f$. Partially order $\mathcal{C}$ by domain inclusion: given $g_i: B_i \to M$, $i = 1, 2$, $g_1 \leq g_2$ if and only if $B_1 \subset B_2$ as comodules in $\mathcal{BP}$ and $g_2 \mid B_1 = g_1$. By a classical Zorn's lemma argument, $\mathcal{C}$ has a maximal element $g': B' \to M$. Suppose $B' \neq C$. We can then choose $0 \neq c + B' \in C/B'$ which is primitive. By Theorems 1 and 2 of [9], we may assume that

$$J = \{\lambda \in BP_*: \lambda c \in B'\}$$

either is I_t, $t < \infty$, or has $\sqrt{J} = I_\infty$. Let $B'' = \{b + \lambda c: b \in B', \lambda \in BP_*\}$. Then $A \subset B' \subset B'' \subset C$ as comodules in $\mathcal{BP}$. So, the sequence

$$\operatorname{Hom}(B'', M) \to \operatorname{Hom}(B', M) \to \operatorname{Ext}^1(B''/B', M)$$

is exact. $\operatorname{Ext}^1(B''/B', M) = 0$ since B''/B' is stably isomorphic to BP_*/J which either is in $\mathcal{BP}_0$ or has $\sqrt{J} = I_\infty$. Thus the homomorphism g' extends to $g'': B'' \to M$, $B' \subset_{\neq} B''$. This contradicts the maximality of g' in $\mathcal{C}$. Thus $B' = C$, and f extends to $g: C \to M$ as required. ∎

THEOREM 3.14. *Let M be a BP_*-module so that*

(i) *for each $n \geq 0$, ${}_nM$ is v_n-divisible, and*
(ii) $\text{w.dim}_{\mathcal{BP}} M < \infty$.

Then M is $\mathcal{BP}$-injective.

Proof. Corollaries 2.4 and 3.8, Lemma 3.13. ∎

The Landweber filtration theorem (as extended) leads to proofs of the following dual statements.

LEMMA 3.15. *Let M be a BP_*-module.*

(i) $\text{w.dim}_{\mathscr{BP}}\, v_n^{-1}M \leq \{n, \text{w.dim}_{\mathscr{BP}}\, M\}$.
(ii) $\text{inj dim}_{\mathscr{BP}_0}\, v_n^{-1}M \leq \{n, \text{inj dim}_{\mathscr{BP}_0}\, M\}$. ■

LEMMA 3.16. *Let M be a BP_*-module. Then $v_s^{-1}({}_tM) = {}_t(v_s^{-1}M)$.*

Proof. A proof follows from a five-lemma induction using the exact sequence (3.3) ($t = 0$) and (3.5), $0 \to {}_{t+1}M \to {}_tM \to {}_tM$, and the fact that $v_s^{-1}(\;\;)$ is an exact functor. The induction begins with the observation that ${}_0M = M$. ■

Recall the short exact sequences (2.2) which describe the Miller-Ravenel-Wilson "chromatic resolution" of BP_*: $0 \to N^s \to M^s \to N^{s+1} \to 0$, $N^0 = BP^*$, $M^s = v_s^{-1}N^s$.

COROLLARY 3.17. *The comodules M^s in the Miller-Ravenel-Wilson chromatic resolution* (2.2) *are $\mathscr{BP}$-injective. The chromatic resolution is a $\mathscr{BP}$-injective resolution of BP_*.*

Proof. The corollary follows from Theorem 3.14 once one applies Proposition 3.12 and Lemma 3.16 to (2.2) to show that $\text{w.dim}_{\mathscr{BP}}\, M^s = s$ and that ${}_n(M^s)$ is v_n-divisible. ■

For a BP_*-module M, let ${}_{(n)}M = \{x \in M\colon I_n^s x = 0, \text{ some } s > 0\}$. Note that ${}_nM \subset {}_{(n)}M$ and that ${}_nM = 0$ if and only if ${}_{(n)}M = 0$. In the special case that M is a comodule in $\mathscr{BP}$, ${}_{(n+1)}M$ can be characterized as the kernel of the localization $M \to v_n^{-1}M$ [5, Theorem 0.1]. Observe that the proof of Proposition 3.12 actually shows that if $\text{w.dim}_{\mathscr{BP}}\, M \leq n$, then ${}_{n+1}M = {}_{(n+1)}M = 0$. The converse holds if M is $\mathscr{BP}_0$-injective.

LEMMA 3.18. *Let M be a BP_*-module which is $\mathscr{BP}_0$-injective. Then for each $n \geq 0$, ${}_{(n)}M$ is v_n-divisible.*

Proof. It will suffice to display ${}_{(n)}M$ as a direct limit of v_n-divisible modules. Consider the collection of modules $\text{Ext}^0\,(BP_*/J, M)$ where J is any finitely-presented, BP_*BP-invariant ideal of BP_* such that $\sqrt{J} = I_n$. Inclusion $J \subset J'$ of two such ideals induces $BP_*/J \to BP_*/J'$ which induces, in turn, $\text{Ext}^0\,(BP_*/J', M) \to \text{Ext}^0\,(BP_*/J, M)$. This forms a direct system whose limit is ${}_{(n)}M$. For each such ideal J, there is some high power v_n^s of v_n such that $K = J + (v_n^s)$ is a BP_*-ideal belonging to $\mathscr{BP}_0$. Since M is $\mathscr{BP}_0$-injective, $\text{Ext}^1\,(BP_*/K, M) = 0$. Hence, v_n^s-multiplication induces the exact sequence

$$\text{Ext}^0\,(BP_*/J, M) \xrightarrow{v_n{}^{s\#}} \text{Ext}^0\,(BP_*/J, M) \to \text{Ext}^1\,(BP_*/K, M) = 0.$$

Thus the $\text{Ext}^0\,(BP_*/J, M)$ are v_n-divisible as required. ■

COROLLARY 3.19. *For any $\mathscr{BP}_0$-injective BP_*-module M, there is a short exact sequence*

$$0 \to {}_{(n+1)}M \to {}_{(n)}M \to v_n^{-1}{}_{(n)}M \to 0. \qquad \blacksquare$$

This sequence leads to the following generalization of Theorem 3.14.

PROPOSITION 3.20. *Let M be a $\mathscr{BP}_0$-injective BP_*-module such that both*

$$\varprojlim_n {}_{(n)}M = \bigcap_n {}_{(n)}M = 0 \quad \text{and} \quad \varprojlim_n{}^1 {}_{(n)}M = 0.$$

Then M is $\mathscr{BP}$-injective.

Proof. Let D be a BP_*-module like BP_*/J in Lemma 3.13 such that $v_n^{-1}D = 0$ for all $n \geq 0$. By Corollary 3.19,

$$\operatorname{Ext}^j(D, {}_{(n+1)}M) \cong \operatorname{Ext}^j(D, {}_{(n)}M) \cong \varprojlim_n \operatorname{Ext}^j(D, {}_{(n)}M).$$

Recall that a theorem of Roos [16] gives two spectral sequences

$$E_2^{i,j} = \varprojlim_n{}^i \operatorname{Ext}^j(D, {}_{(n)}M) \quad \text{and} \quad \bar{E}_2^{i,j} = \operatorname{Ext}^i\left(D, \varprojlim_n{}^j {}_{(n)}M\right)$$

which converge to the same module. By our hypotheses on M, $\bar{E}_2^{i,0} = 0 = \bar{E}_2^{i,1}$ for all i. $\bar{E}_2^{i,j} = 0$ for $j > 1$ since the inverse system $\{{}_{(n)}M\}$ is indexed by the natural numbers. Thus

$$\begin{aligned} 0 = E_2^{0,j} &= \varprojlim_n \operatorname{Ext}^j(D, {}_{(n)}M) \cong \operatorname{Ext}^j(D, {}_{(0)}M) \\ &= \operatorname{Ext}^j(D, M) \quad \text{for all } j. \end{aligned}$$

Apply Lemma 3.13. $\blacksquare$

4. Localization and periodic spectra related to BP

Fix a spectrum E. A second spectrum X is E_*-acyclic if $E_*X = 0$. A spectrum Y is E_*-local if $[X; Y] = 0$ for each E_*-acyclic spectrum X. By Bousfield [2], there is a natural map $\eta\colon X \to X_E$ with X_E being E_*-local and $E_*(\eta)$ an isomorphism. We call $\eta\colon X \to X_E$ the E_*-localization of X.

The Brown-Peterson spectrum BP has been a center of our attention. Algebraically, we can localize the coefficient ring $BP_* = \pi_*BP$ to form the ring $v_n^{-1}BP_* \cong \mathbf{Z}_{(p)}[v_n^{-1}, v_1, v_2, \ldots]$. There are maps $\sum^{2p^n-2} BP \to BP$ inducing v_n-multiplication in homotopy. A mapping telescope using these maps realizes $v_n^{-1}BP_*$ as the homotopy of a spectrum which we call $v_n^{-1}BP$. Localization with respect to $v_n^{-1}BP_*$ seriously alters spectra. In particular, the $v_0^{-1}BP_*$-localization of BP is $v_0^{-1}BP$. If we take all the spectra $v_n^{-1}BP$ together, nice spectra are unchanged. The direct sum homology theory $\oplus_{0 \leq n} v_n^{-1}BP_*(\ )$ is represented by the wedge spectrum $W = \bigvee_{0 \leq n} v_n^{-1}BP$. Theorem 2.6 tells us that if Y is connective and p-local and if $BP_* Y$ has finite BP_*-projective dimension, then Y is W_*-local. Hence all finite complexes, BP, each $v_n^{-1}BP$, and $W = \bigvee_{0 \leq n} v_n^{-1}BP$, itself, are W_*-local.

In addition to the homology theories $v_n^{-1}BP_*(\ \ )$, there are two other important families of periodic homology theories associated to BP:

(i) $E(n)_*(\ \ )$ with coefficients $E(n)_* \cong \mathbf{Z}_{(p)}[v_1, \ldots, v_n, v_n^{-1}]$, represented by the spectrum $E(n)$;

(ii) the Morava K-theories $K(n)_* \cong \mathbf{F}_p[v_n, v_n^{-1}]$, represented by the spectrum $K(n)$.

Remark 4.1. The meaning of W_*-acyclic spectrum is the same regardless of whether W stands for $\bigvee_{0 \leq n} v_n^{-1}BP$, $\bigvee_{0 \leq n} E(n)$, or $\bigvee_{0 \leq n} K(n)$.

Proof sketch. Recall there are homology theories $P(n)_*(\ \ )$ with coefficients $P(n)_* \cong BP_*/I_n$. From [9, Corollary 4.12], we know that $v_n^{-1}P(n)_* X = 0$ if and only if $K(n)_* X = 0$. Assume that $K(m)_* X = 0$ for all m, $0 \leq m \leq n$. We want to show that $v_n^{-1}P(m)_* X = 0$ for all m, $0 \leq m \leq n$ by a downward induction on m starting at $m = n$. With the exact sequence

$$\cdots \to v_n^{-1}P(m)_* X \xrightarrow{v_m} v_n^{-1}P(m)_* X \to v_n^{-1}P(m+1)_* X \to \cdots,$$

the inductive hypotheses $v_n^{-1}P(m+1)_* X = 0$ implies that $v_n^{-1}P(m)_* X \cong v_n^{-1}v_m^{-1}P(m)_* X$ which is 0. Consequently, we obtain $v_n^{-1}BP_* X = v_n^{-1}P(0)_* X = 0$ which is equivalent to $E(n)_* X = 0$. ∎

References [5] and [9] also show that $v_n^{-1}BP_* X = 0$ (or $\mathrm{E}(n)_* X = 0$) implies that $v_j^{-1}BP_* X = 0$ (or $E(j)_* X = 0$) for all $j \leq n$. Thus Lemma 2.1 and Remark 2.7 imply:

PROPOSITION 4.2. *If $E(n)_* X = 0$, then every element of BP^*X has Adams-Novikov (BP) filtration at least $n + 1$: any $g \colon X \to BP$ factors as*

$$g = g' \circ f_n \circ \cdots \circ f_1 \circ f_0$$

with $BP_*(f_i) \equiv 0$, $0 \leq i \leq n$.

Recall that our corollary is not optimal: $[H\mathbf{F}_p;\ Y]_* = 0$ for spectra Y (e.g. CW complexes with finite skeleta) which we do not know to be W_*-local. An answer to the following question of Ravenel seems to require a deep understanding of the unstable properties of BP.

Question 4.3 [14; 4.13]. If $v_n^{-1}BP_* X = 0$ for every $n \geq 0$ and if Y is a CW complex, must $[X;\ Y\mathbf{Z}_{(p)}]_* = 0$?

An algebraic analog of the question leads to the following simple examples of $BP_* BP$-comodules:

$$A = \bigoplus_{0 \leq n} BP_*/I_n \quad \text{and} \quad B = \bigoplus_{0 \leq n} \sum^n BP_*/I_n = \prod_{0 \leq n} \sum^n BP_*/I_n.$$

The representation of

$$\mathbf{Z}/p = BP_*/I_\infty = \lim_n BP_*/I_n$$

yields a non-zero element of $\mathrm{Ext}^1_{BP_*}(BP_*/I_\infty, A) \neq 0$. Let D be any BP_*-module with $v_n^{-1}D = 0$ for all $n \geq 0$ (e.g. $D = BP_*/I_\infty$). Then

$$\mathrm{Ext}^*_{BP_*}(D, B) \cong \prod \mathrm{Ext}^*_{BP_*}(D, \textstyle\sum^n BP_*/I_n) = 0 \quad \text{(Corollary 2.4).}$$

By Ravenel and Wilson's solution of the Conner-Floyd conjecture [15], B is a subcomodule of $BP_* K$ where K is the CW complex $\bigvee_{0 \leq n} K(\mathbf{Z}/p, n)$ (p odd). Our intuition is that A cannot be so represented as a subcomodule of the BP homology of an (unstable) CW complex. We are led to seek constraints on the annihilator ideals of elements in $BP_* X$ when X is a complex.

Question 4.4. Let X be a CW complex and let $0 \neq y \in BP_n X$. Let $J = \{\lambda \in BP_*: \lambda y = 0\}$. By [9], $\sqrt{J} = I_m$, $0 \leq m \leq \infty$. Must m always be finite? Better still, must $\sqrt{J} = I_m$, $0 \leq m \leq n$?

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