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ON THE ASSOCIATED GRADED RING OF AN IDEAL

BY

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Introduction

In this paper we study the graded ring of an ideal I in a Noetherian ring R. By definition this is the non-negatively graded algebra,

$$R/I \oplus I/I^2 \oplus \cdots \oplus I^n/I^{n+1} \oplus \cdots,$$

which we will denote by gr (I, R).

Section one deals the connection between gr (I, R) and $\mathscr{R}(I, R)$, the Rees algebra of I, defined to be the subalgebra $R[It]$ of the polynomial ring $R[t]$. We note that "Rees algebra" is also used to describe the ring $\mathscr{R}(I, R)[t^{-1}]$. However we will always refer to this ring as the extended Rees algebra. We will show that if R and $\mathscr{R}(I, R)$ are Cohen-Macaulay (respectively Gorenstein) then gr (I, R) is likewise Cohen-Macaulay (respectively Gorenstein.) (See Propositions 1.1 and 1.2.) We apply this to ideals of height two and projective dimension one, giving a sufficient condition (Proposition 1.3) which implies the graded algebra of such an ideal is Gorenstein.

If $I = p$ is prime then an important question is whether gr (p, R) is a domain. Let $p^{(n)} = p^n R_p \cap R$ be the nth symbolic power of p. If R_p is regular, then gr (p, R) is a domain if and only if $p^{(n)} = p^n$ for all $n \geq 1$. These questions have been examined by many researchers. (See [7], [9], [10], and [18]). If R is a commutative ring, by R^{red} we denote the reduced ring of R, namely R/N where N is the nilradical of R. In section two we find natural conditions which force gr $(p, R)^{\text{red}}$ to be a domain. These conditions are intimately connected with the concept of "analytic spread" first introduced by Northcott and Rees [16].

Let R be a local ring with maximal ideal m and let I be an ideal of R. Define $l(I)$ to be the Krull dimension of the ring

$$T = R/m \oplus I/mI \oplus I^2/mI^2 \oplus \cdots.$$

Observe that $T \approx \mathscr{R}(I, R)/m\mathscr{R}(I, R)$ and that T is also the homomorphic image of gr (I, R) by the ideal generated by the image of m in the 0th graded piece, R/I. We will always denote this ideal by $\bar{m}$. Since dim (gr (I, R)) = dim (R), the fact that T is a homomorphic image of gr (I, R) immediately shows $l(I)$ is at most dim (R).

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L. Burch [2] proved a fundamental inequality concerning the analytic spread:

$$l(I) \leq \dim(R) - \inf \operatorname{depth}(R/I^n). \tag{1}$$

Brodmann [1] has improved this recently by replacing "inf" by "lim inf".

Suppose $I = p$ is prime and $p^n = p^{(n)}$ for all n. Then

$$\operatorname{depth}(R/p^n) = \operatorname{depth}(R/p^{(n)}) \geq 1.$$

Since $(p_q)^{(n)} = (p^{(n)})_q$ for any prime q of R, we see that the equality $p^n = p^{(n)}$ forces depth $(R_q/p_q^n) \geq 1$. By Burch's inequality (1), this implies

$$l(I_q) < \dim(R_q), \text{ for } q \underset{\neq}{\supset} p. \tag{2}$$

In Section 2 we show the condition (2) is equivalent to gr $(p, R)^{\text{red}}$ being a domain, under certain weak conditions on R. (See Theorem 2.1.)

When (2) occurs, one may readily give a precise description of $p^{(n)}$. Recall if I is an ideal then the integral closure of I, denoted $\bar{I}$, is defined to be the set of all elements r of R which satisfy a monic polynomial $f(t) = t^n + t^{n-1}a_1 + \cdots + a_n$ whose coefficients a_j are in I^j. In Theorem 2.1 we show condition (2) is also equivalent to $p^{(n)} = p^n$ for all n.

It is simple to check that if we set $T = \mathscr{R}(I, R)[t^{-1}]$, then $T/Tt^{-1} \approx \operatorname{gr}(I, R)$. If the equivalent conditions of Theorem 2.1 hold then the nilradical of t^{-1} in T is a prime Q, and we will show t^{-1} generates Q generically (that is, in R_Q). Under these conditions we may apply a well-known lemma of Hironaka and are able to show.

PROPOSITION 2.3. *Suppose R is a universally catenarian Nagata domain and p is a prime such that R_p is regular. If* gr $(p, R)^{\text{red}}$ *is an integrally closed domain, then* gr (p, R) *is a domain.*

We apply these results to several examples. If p is a height two prime of projective dimension 1, then we give a simple criteria for gr (p, R) to be a domain (Proposition 2.4.)

We further illustrate how from these methods one can deduce properties of rings R which are isomorphic to gr (p, R) for some ideal p in R. These rings often arise in determinantal loci.

A ring R is said to satisfy R_k if R_q is a regular local ring for every prime q of height at most k.

PROPOSITION 2.1. *Suppose R is a commutative Noetherian universally catenarian ring and p a prime such that*

$$l(p_q) \leq \max\{\dim(R_p), \dim(R_q) - k\} \tag{\#}$$

Suppose either R or gr (p, R) *is Cohen-Macaulay. Then R/p satisfies R_{k-1} if and only if* gr (p, R) *satisfies R_{k-1}.*

From this result it is clear that many properties (normality, reducedness, etc.) can be determined in part by looking at the analytic spread.

Throughout the paper, 'ring' will always mean commutative with identity. When we refer to a theorem from another paper, we will always place an (A) after it. We use the notation and terminology of Matsumura's excellent book [13].

1. Relationship of $\mathscr{R}(I, R)$ and gr (I, R)

PROPOSITION 1.1 *Suppose R is Cohen-Macaulay, I is an ideal of height at least* 1, *and $\mathscr{R}(I, R)$ is Cohen-Macaulay. Then* gr (I, R) *is Cohen-Macaulay.*

Proof. Since gr (I, R) is graded, to show it is Cohen-Macaulay it is enough to show gr $(I, R)_Q$ is Cohen-Macaulay for every maximal ideal Q containing

$$I/I^2 \oplus \cdots \oplus I^n/I^{n+1} \oplus \cdots.$$

See [8] and [12].)

If r is an element of R, then by r^* we denote the leading form of r in gr (I, R). If J is an ideal of R, by J^* we denote the ideal in gr (I, R) generated by all the leading forms of elements in J. The maximal ideals in gr (I, R) containing I^* correspond to maximal ideals m of R which contain I. If we set $W = R - m$ then gr $(I, R)_Q$ is a localization of

$$\text{gr } (I, R)_{W^*} \approx \text{gr } (I_m, R_m).$$

Thus to show gr $(I, R)_Q$ is Cohen-Macaulay, it is enough to assume R is local with maximal ideal m, and that Q is the ideal of gr (I, R) generated by I^* and $\bar{m}$, where $\bar{m}$ is the ideal of gr (I, R) generated by the image of m/I in the 0th graded piece of gr (I, R). Let $P = (m, It)$ be the ideal in $\mathscr{R}(I, R)$ generated by m and It. Set $S = \mathscr{R}(I, R)$ and $T = \text{gr } (I, R)$. Then there are exact sequences,

$$0 \to It \to S_P \to R \to 0 \tag{1}$$

and

$$0 \to I \to S_P \to T_Q \to 0. \tag{2}$$

Let $d = \dim (R)$ so that depth $(R) = d$ and depth $(S_P) = d + 1$ by the assumption.

Recall in general if S is a local ring with maximal ideal J and $k = S/J$, then the depth of a finitely generated S-module M is characterized by the least non-vanishing $\text{Ext}_S^i (k, M)$ [13]. Let $K = S_P/P$ and apply $\text{Hom}_{S_P}(K, \ )$ to the exact sequence (1). By the assumptions, we obtain $\text{Ext}^i (K, It) = 0$ for $i < d + 1$. However, $It \approx I$ as S modules and so $\text{Ext}^i (K, I) = 0$ for $i < d + 1$. If we now apply $\text{Hom}_{S_P} (K, \ )$ to the exact sequence (2), we find $\text{Ext}^i (K, T_Q) = 0$ for $i < d$. Therefore depth T_Q is at least d; since it is also at most d, we see it is Cohen-Macaulay as required.

Recall a local ring is said to be Gorenstein if it has finite injective dimension.

If a local ring R is a homomorphic image of a local Gorenstein ring S, then R is Gorenstein if and only if R is Cohen-Macaulay and

$$\mathrm{Ext}_S^{\dim S - \dim R}(R, S) \approx R.$$

We say a non-local ring is Gorenstein if it is locally at every maximal ideal (equivalently every prime ideal).

PROPOSITION 1.2. *Suppose R is a Gorenstein ring and I is an ideal of height at least two. If $\mathscr{R}(I, R)$ is Gorenstein, then* gr (I, R) *is Gorenstein.*

Proof. Since gr (I, R) is graded, it is enough to show gr $(I, R)_Q$ is Gorenstein whenever Q is a maximal ideal containing I^* [12]. As in Proposition 1.1 we may reduce to R being a local ring with maximal ideal m, $P = (m, It)$ a maximal ideal of $S = \mathscr{R}(I, R)$ and set $T = \mathrm{gr}\,(I, R)$. By Proposition 1.1, T_Q is Cohen-Macaulay, and hence it is enough to show

$$\mathrm{Ext}^1_{S_P}(T_Q, S_P) \approx T_Q,$$

since $\dim(S_P) = \dim(T_Q) + 1$. It suffices to show $\mathrm{Ext}^1_S(T, S) \approx T$ as this will localize.

If x is an element of I which is a non-zero divisor, then as is well known,

$$\mathrm{Ext}^1_S(T, S) \approx \mathrm{Hom}_S(T, S/Sx) \approx (x\colon I)/Sx.$$

(Recall $T = S/SI$.) Therefore it is enough to show,

$$(x\colon I)/Sx \approx T = S/SI \tag{*}$$

for some such x in I.

LEMMA 1.1. *Suppose R and I are as above and x is an element of I such that*
(a) *x is a non-zero divisor,*
(b) *x is in $I - I^2$,*
(c) *x^* is a non-zero divisor in* gr (I, R).
Then $(x\colon SI) = (x, xt)$ and $(x\colon xt) = I$.

Proof. We will first show the former equality. Observe that xt is in $(x\colon SI)$ since $xt(I) = It(x)$ is an equation that holds in S. This also shows I is contained in $(x\colon xt)$. Thus it is enough to show $(x\colon SI)$ is contained in (x, xt). Since this ideal is homogeneous we need only show this for homogeneous elements contained in this ideal.

Suppose ct^n is in $(x\colon SI)$. Since the height of I is at least two and R is Cohen-Macaulay, we may choose a y in I such that x, y form an R-sequence. We have

$$y(ct^n) = x(dt^n),$$

and so

$$yc = xd.$$

Since x and y form an R-sequence, there is an element z in R such that

$$c = xz \quad \text{and} \quad d = yz.$$

Since ct^n is in S, c is in I^n; by assumptions (b) and (c) we may conclude z is in I^{n-1} and so if n is at least 1, zt^{n-1} is in S.

If n is at least one this shows $ct^n = (xt)(zt^{n-1})$ and so ct^n is in (x, xt). If $n = 0$, then $c = xz$ shows c is in (x, xt). Thus we have demonstrated the first equality of Lemma 1.1.

We have noted I is contained in $(x\colon xt)$; to show the opposite conclusion assume ct^n is in $(x\colon xt)$. Then

$$(ct^n)(xt) = x(dt^{n+1}).$$

Since x is not a zero divisor, this shows $c = d$ and hence c is in I^{n+1}. This shows ct^n is in IS as required.

We now finish the proof of Proposition 1.2. We have shown it is enough to demonstrate (*). Since gr (I, R) is Cohen-Macaulay by Proposition 1.1, and

$$\text{height } (I) = \text{height } (I^*) \geq 2,$$

we may easily choose an x in $I - I^2$ satisfying conditions (a)–(c) of Lemma 1.1. Hence,

$$\begin{aligned}(x\colon I)/(x) &= (x, xt)/(x) \approx (xt)/((x) \cap (xt)) \\ &= (xt)/xt(x\colon xt) = (xt)/xt(SI) \approx S/SI = T.\end{aligned}$$

We now apply these results to ideals of height two and projective dimension one. We use the result of [11].

THEOREM 1(A). *Let R be a Cohen-Macaulay Noetherian domain and M an R-module having a finite free minimal resolution,*

$$0 \longrightarrow R^m \xrightarrow{A} R^n \longrightarrow M \longrightarrow 0, \quad A = (a_{ij}).$$

Let $I_t(A)$ denote the ideal in R generated by the $t \times t$ minors of A. Then the following are equivalent:

(1) *$S_R(M)$, the symmetric algebra of M, is a domain.*

(2) height $(I_t(A)) \geq m + 2 - t$ *for* $1 \leq t \leq m$.

(3) $v(M_p, R_p) \leq n - m +$ height $(p) - 1$ *for all non-zero primes p in R, where $v(M, R)$ is the minimal number of generators of M.*

If any (and hence all) of the above conditions hold, then $S_R(M)$ is a complete intersection in a polynomial ring $R[T_1, \ldots, T_n]$ over R. Hence $S_R(M)$ is Cohen-Macaulay and is Gorenstein if R is.

We recall some facts about $S_R(M)$. It is a non-negatively graded algebra whose nth piece, $S^n(M)$, equals $(M \otimes \cdots \otimes M)/N$ where N is the submodule generated by elements of the form,

$$\cdots \otimes m \otimes \cdots \otimes m' \otimes \cdots - \cdots \otimes m' \otimes \cdots \otimes m \otimes \cdots.$$

If M has a presentation,

$$R^m \xrightarrow{(a_{ij})} R^n \longrightarrow M \longrightarrow 0,$$

then M can be identified with the ring $R[T_1, \ldots, T_n]/J$ where J is the ideal generated by the linear forms, $\sum_{j=1}^n a_{ij} T_j$.

If $M = I$ is an ideal, there is always an onto homomorphism from $S_R(I)$ to $\mathscr{R}(I, R)$. In [14] it was shown this is an isomorphism if and only if $S_R(I)$ is torsion free over R. Thus, if $S_R(I)$ is a domain, it is necessarily isomorphic to $\mathscr{R}(I, R)$.

PROPOSITION 1.3. *Let R be a Cohen-Macaulay domain and I an ideal of height two having a minimal free resolution,*

$$0 \longrightarrow R^n \xrightarrow{A} R^{n+1} \longrightarrow I \longrightarrow 0, \quad A = (a_{ij}).$$

If either

$$\text{height } (I_t(A)) \geq n + 2 - t \quad \textit{for } 1 \leq t \leq n,$$

or equivalently if

$$v(I_p, R_p) \leq \text{height } (p)$$

for every non-zero prime p in R, then gr (I, R) *is Cohen-Macaulay. If moreover R is Gorenstein, then* gr (I, R) *is also.*

Proof. By Theorem 1(A), if the two equivalent conditions hold, then $S_R(I)$ is a domain, and the comments above show it is then isomorphic to the Rees algebra. The conclusion follows immediately from the second statement of Theorem 1(A) and from Propositions 1.1 and 1.2.

Example 1.1. Let p be the prime in $R = k[X, Y, Z]$ defined to be the kernel of the homomorphism from R onto $k[t^{n_1}, t^{n_2}, t^{n_3}]$ where the n_i are positive integers. Herzog [5] has shown p is generated by at most three elements. Since height $(p) = 2$ and R/p is Cohen-Macaulay, p has projective dimension one and since R is a polynomial ring, has a free resolution of length 1. As R_p is regular, $v(p_p, R_p) = 2 = \dim (R_p)$ while if q contains p properly, $v(p_q, R_q) \leq v(p, R) = 3 = \dim (R_q)$. Thus Proposition 1.3 allows us to conclude that gr (p, R) is Gorenstein.

Example 1.2. Let p be the prime in $R = k[X, Y, Z, U]$ defining the twisted cubic, i.e. $R/p \approx k[t^3, t^2v, tv^2, v^3]$. It is known that p is defined by the 2×2 minors of the matrix

$$A = \begin{pmatrix} X & Y & Z \\ Y & Z & U \end{pmatrix}$$

and has a resolution,

$$0 \longrightarrow R^2 \xrightarrow{A} R^3 \longrightarrow p \longrightarrow 0.$$

The same reasoning as above shows gr (p, R) is Gorenstein.

Example 1.3. The "generic" ideal of projective dimension one is given by the ideal defined by the exact sequence

$$0 \longrightarrow R^n \xrightarrow{X} R^{n+1} \longrightarrow I \longrightarrow 0,$$

where $X = (x_{ij})$ is a generic n by $n + 1$ matrix over a field k, and $R = k[x_{ij}]$. We may assume I is generated by the n by n minors of X. Since

$$\text{height } (I_t(X)) = (n - t + 1)(n + 2 - t),$$

we see the condition (b) of Theorem 1(A) is certainly verified. Hence gr (I, R) is a Gorenstein ring. This appears in several papers.

2. The main theorem

In this section we wish to prove the theorem discussed in the introduction. Before we begin the proof, we will discuss some facts concerning the analytic spread. Most of these results can be found in either [2] or [16]. Throughout the following remarks, R will denote a local Noetherian ring with maximal ideal m. Recall the analytic spread, $l(I)$, of an ideal I is defined to be the dimension of the ring

$$T = R/m \oplus I/mI \oplus \cdots \oplus I^n/mI^n \oplus \cdots.$$

(1) We always have $l(I) \leq \dim(R)$. This was proved in the introduction.

(2) In addition, $l(I) \leq v(I, R)$. If I has n generators, then T is a homomorphic image of $R/m[T_1, \ldots, T_n]$ which shows the inequality.

(3) The analytic spread can be characterized as the degree plus one of the polynomial $p(n)$ which gives $\dim(I^n/mI^n)$ for large n. This follows immediately from the theory of the Hilbert-Samuel polynomial. In particular, $l(I) = l(I^n)$ for every $n \geq 1$.

If I and J are two ideals of R, I is said to be a *reduction* of J if I is contained in J and if $IJ^n = J^{n+1}$ for all large n.

(4) If J is an ideal in R, then there exist reductions I of J minimal with respect to inclusion among the set of all reductions of J. Any such minimal reduction is minimally generated by $l(J)$ elements. Here we assume R/m is infinite.

Recall elements $a_1, \ldots, a_n$ in R are said to be *analytically independent* if whenever $F(X_1, \ldots, X_n)$ is a homogeneous polynomial with coefficients in R and

$$F(a_1, \ldots, a_n) = 0,$$

then all the coefficients of F lie in m. Any system of parameters or any R-sequence is analytically independent.

If I is a minimal reduction of J, then any minimal generating set of I is analytically independent.

(5) Given an ideal J, there exists a unique ideal $\bar{J}$ containing J and maximal under the condition that any reduction of J is a reduction of $\bar{J}$. In particular, $l(J) = l(\bar{J})$. This ideal $\bar{J}$ may be described as the integral closure of J. (See the introduction; the integral closure of an ideal is the set of all elements which are integral over it.)

(6) If I contained in J is an ideal generated by $l(J)$ elements which are analytically independent, then I is a minimal reduction of J.

(7) If q is a prime ideal containing I, then $l(I_q) \leq l(I)$. This follows from the comments above. If $k = l(I)$, and $J = (x_1, \ldots, x_k)$ is a minimal reduction of I, then I is integral over J and hence I_q is integral over J_q. Therefore $l(I_q) = l(J_q) \leq v(J_q, R_q) \leq k$.

(8) Let R be a domain and I an ideal. Then the integral closure of $\mathscr{R}(I, R)$ in $R[t]$ is $R + \bar{I}t + \overline{I^2}t^2 + \cdots$ (see [19]). If $\bar{R}$ denotes the integral closure of R, then the ring

$$T = \bar{R} + \bar{I}t + \overline{I^2}t^2 + \cdots$$

is integrally closed.

THEOREM 2.1. *Let R be a universally catenarian Nagata domain and let p be a prime ideal such that* gr (p_p, R_p) *is a domain. (In particular this holds if R_p is regular.) Then the following are equivalent:*

(1) $l(p_q) < \dim (R_q)$ *for all primes q strictly containing p.*

(2) $\overline{p^n} = p^{(n)}$ *for all $n \geq 1$.*

(3) gr $(p, R)^{\mathrm{red}}$ *is a domain.*

Proof. We will first show (1) implies (2). Let S be the Rees algebra of p, and let T be as in Remark (8) above. Then T is an integral extension of S with the same quotient field. Since R is Nagata, so is S and hence T is a finite S-module, in particular a Noetherian ring. By 34.8 of [15], if m is an ideal of R,

$$\text{height } (mT) = \text{height } (mT \cap S),$$

and since $mT \cap S$ contains mS we have

$$\text{(a)} \qquad \text{height } (mT) \geq \text{height } (mS).$$

Suppose (1) holds but (2) does not. Let m be a minimal prime of R containing p such that $(\overline{p^n})_m \neq (p^{(n)})_m$. Since $(\overline{p^n})_m = \overline{(p_m)^n}$ and $(p_m)^{(n)} = (p^{(n)})_m$, we may assume R is local with maximal ideal m. As gr (p_p, R_p) is a domain, $\overline{p_p^n} = p^n R_p$ and this shows $\overline{p^n}$ is contained in $p^{(n)}$. Hence it is enough to show the opposite conclusion.

R is universally catenarian and hence so is S. In particular,

$$\text{height } (mS) + \dim (S/mS) = \dim (S) = \dim (R) + 1.$$

However, $\dim (S/mS) = l(p)$. The prime m strictly contains p since gr (p_p, R_p) a domain forces the equality of $\overline{p_p^n}$ and $p_p^{(n)}$. Hence, (1) shows $l(p) < \dim (R)$. We conclude that height $(mS) \geq 2$.

Suppose s is in $p^{(n)}$ but not in $\overline{p^n}$. By choice of m there is a N such that $m^N(s)$ is contained in $\overline{p^n}$. Therefore, the element st^n of the quotient field of T has the property that $m^N(st^n)$ is contained in T. For if ct^k is in $m^N T$, then c is in $m^N\overline{p^k}$ and so $st^n ct^k = (sc)t^{n+k}$ is in $(sm^N)\overline{p^k}$ which is contained in $\overline{p^{n+k}}$. We have shown that $st^n(m^N T)$ is contained in T. By (a) above,

$$\text{height } (m^N T) \geq \text{height } (m^N S) \geq 2.$$

Since T is integrally closed by Remark (8), we see st^n must lie in T. This shows s is in $\overline{p^n}$, and this contradiction completes the proof of this implication.

Now assume (2) and we will show (3). Let N be the nilradical of gr (p, R). Suppose a^* and b^* are homogeneous elements of gr (p, R) such that a^*b^* is in N. It is enough to show either a^* or b^* is in N. Let a and b be liftings of a^* and b^* respectively to R. We may assume a^* is a zero divisor since otherwise b^* will be nilpotent. If $W = R - p$, then by assumption gr $(p_p, R_p) = \text{gr } (p, R)_{W^*}$ is a domain, and consequently either a^* or $(0: a^*)$ goes to zero in this localization. Therefore either a^* or b^* goes to zero and we may assume without loss of generality that there is an s^* in W^* such that $s^*a^* = 0$. Since s is not in p, this implies a is in $p^{(n)}$ for some n with a not in p^n.

By assumption, $p^{(n)} = \overline{p^n}$ and so

$$\text{(b)} \qquad a^M + a^{M-1}b_1 + \cdots + b_M = 0$$

with b_i in p^{ni}. Suppose a is in p^k but not p^{k+1}. Then (b) shows that a^M is in $p^{k(m-1)+n}$. This means a^* is nilpotent.

Finally, assume (3); we will demonstrate (1). If m is a prime containing p and $W = R - m$, then gr $(p_m, R_m) \approx \text{gr } (p, R)_{W^*}$ and since condition (3) remains true under localization, we see it is enough to assume R is local with maximal ideal m and to show that (3) forces $l(p) < \dim (R)$.

Let $\bar{m}$ be the ideal generated by the image of m in the 0th graded piece of gr (p, R). Then $l(p) = \dim (\text{gr } (p, R)/\bar{m})$ and so since by Remark (1), $l(p) \leq \dim (R)$, if $l(p) = \dim (R)$ then height $(\bar{m}) = 0$. By assumption, gr $(p, R)^{\text{red}}$ is a domain so there is a unique minimal prime, which is nilpotent. Thus $\bar{m}$ would be nilpotent; this is obviously not the case and this contradiction finishes the proof of the theorem.

There are other conditions under which (1) and (2) are equivalent which are useful in some examples. We note one such case here.

THEOREM 2.2. *Suppose R is a universally catenarian Noetherian ring and p a prime such that* gr (p_p, R_p) *is a domain. If* gr (p, R) *is equidimensional, that is to say, if* dim (gr $(p, R)/Q)$ = dim (gr (p, R)) *for every minimal prime Q, then the following are equivalent*:

(1) $l(p_q) < \dim R_q$ *for every prime q strictly containing p;*

(2) gr $(p, R)^{\text{red}}$ *is a domain.*

Proof. We use the following easy observation.

LEMMA 2.1. *Suppose R is a Noetherian ring and W a multiplicatively closed subset of R such that R_W is a domain, and such that if w and w' are two elements of W, then $ww' \neq 0$. Then there is a unique maximal proper ideal of the form* $(0: w)$ *and it is prime.*

Proof. This is in any elementary textbook.

Now suppose (1) holds. Let $W = R - p$ and let Q be the prime in gr (p, R) determined by the set W^* and guaranteed by Lemma 2.1. Let q be a prime in R containing p which is minimal over the R/p torsion of gr (p, R). If there is none, then gr (p, R) is a domain and there is nothing to prove. Since R/p is a domain, the set of torsion elements is a prime, necessarily equal to q. Thus if s is not in q, s^* is not a zero-divisor in gr (p, R). Hence gr (p, R) imbeds in gr $(p, R)_{W^*}$ and consequently gr $(p, R)^{\text{red}}$ imbeds in gr $(p_q, R_q)^{\text{red}}$. Thus it is enough to show this latter ring is a domain. We may thus assume R is local with maximal ideal m which is minimal over the R/p torsion in gr (p, R). By (1), $l(p) < \dim(R)$ and since R (and hence gr (p, R)) is universally caternarian and equidimensional we have

$$\text{height}(\bar{m}) + l(p) = \dim(R).$$

It follows that height $(\bar{m})$ is at least one. However by choice of m, it follows that $\bar{m}^k Q = 0$ for some k. Since Q is prime and the height of $\bar{m}$ is at least one it follows that Q is the unique minimal prime; therefore gr $(p, R)^{\text{red}}$ = gr $(p, R)/Q$ is a domain.

Before we further discuss the relationship of the analytic spread to properties of the graded ring, we need the following simple lemma.

LEMMA 2.2. *Let R be a Noetherian local ring and p a prime such R_p is regular. Suppose either R or* gr (p, R) *is Cohen-Macauly. If $l(p) = \dim(R_p)$ then p is generated by an R-sequence.*

Proof. The case where R is Cohen-Macaulay is due to Cowsik-Nori [3]. However, we repeat the argument here.

Assume R is Cohen-Macaulay. Set n = height (p) and let I be a minimal reduction of p. Since I is generated by n elements (Remark (4)), height $(I) = n$, and R is Cohen-Macaulay, I is generated by an R-sequence. Thus, I_p is generated by a system of parameters in R_p; since p_p is integral over I_p, I_p is a reduction of p_p. However, since R_p is regular, Remark (6) shows p_p is a minimal reduction of itself. Thus $I_p = p_p$. Remark (3) shows p is the only minimal prime over I since p^N is contained in I for some n. Since R is Cohen-Macaulay, I is unmixed. The fact that $I_p = p_p$ shows the primary component of p in I is p itself and together these remarks show $I = p$.

Now suppose gr (p, R) is Cohen-Macaulay. We induct on dim (R_p) to show that p is generated by a R-sequence. If the dimension is 0, we must show $p = 0$. Let m be a minimal prime containing p such that $p_m \neq 0$. This ideal must strictly contain p since R_p is regular (and so is a field). Localize at m. By Remark (7), $l(p_m)$ is at most $l(p)$ and is always at least dim (R_p). Hence we still have equality after localizing at m. In addition gr (p_m, R_m) is a localization of gr (p, R) and so is still Cohen-Macaulay. Thus we may assume R is local with maximal ideal m, and that $p_q = 0$ for every prime in R containing p and not equal to m.

Since gr (p, R) is Cohen-Macaulay,

$$\text{height }(\bar{m}) + \dim (\text{gr }(p, R)/\bar{m}) = \dim (R).$$

Since by assumption $l(p) = 0$, the height of $\bar{m}$ is equal to the dimension of R which is at least one since m is not p. Choose an x in m but not p such that x^* is in no minimal prime of gr (p, R). Since gr (p, R) is Cohen-Macaulay, x^* is not a zero-divisor.

Our choice of q shows $p_x = 0$ and so $x^k p = 0$ for some k. If p is non-zero, choose an r in $p^n - p^{n+1}$. Since $x^k r = 0$, $(x^*)^k r^* = 0$ and since x^* is not a zero-divisor, $r^* = 0$. This contradiction shows $p = 0$.

If height (p) is at least one, let m be the maximal ideal of R and choose a y in p which is in no minimal prime of R and such that y^* is in no minimal prime of gr (p, R) nor in any minimal prime of $\bar{m}$ whose dimension is equal to dim (gr $(p, R)/\bar{m}$) which is at least one.

Let $R' = R/Ry$ and $p' = p/Ry$. Since y^* is a non-zero divisor, it is easy to check that gr $(p, R)/(y^*) \approx$ gr (p', R'), and so this latter graded ring is still Cohen-Macaulay. Further, since

$$\text{height }((\bar{m}, y^*)/(y^*)) = \text{height }(\bar{m}),$$

we see that $l(p) = l(p') + 1$, and so $l(p') = \dim R'_{p'} = \dim R_p - 1$ since y is in no minimal prime of R. The induction shows p' is generated by an R-sequence. To finish the proof, it is enough to show y is not a zero-divisor in R. However, the fact that y^* is not a zero-divisor shows that $(p^n : y)$ is contained in p^{n-k} if y is in p^k but not in p^{k+1}. Since R is local Noetherian, $\bigcap p^n = 0$. Thus,

$$(0 : y) \subseteq (p^n : y) \subseteq \bigcap p^{n-k} = 0.$$

PROPOSITION 2.1. *Let R be a homomorphic image of a regular domain and let p be a prime such that R_p is regular. Assume either R is a Cohen-Macaulay Nagata domain or that* gr (p, R) *is Cohen-Macaulay. Also assume*

$$l(p_q) \leq \max \{\dim R_p, \dim R_q - k - 1\} \qquad (\#)$$

for all primes q containing p. Then gr (p, R) *satisfies condition R_k if and only if R/p satisfies R_k.*

Proof. First suppose R/p satisfies R_k and let Q be a prime in gr (p, R) of height at most k. Set $\bar{m} = Q \cap R/p$ and let m be the lifting of $\bar{m}$ to R. As above, gr $(p, R)_Q$ is a localization of gr (p_m, R_m) and therefore since all the assumptions remain true under localization, we may assume R is local with maximal ideal m, and Q is a prime of gr (p, R) of height at most k such that Q contains $\bar{m}$. This shows that height $(\bar{m}) \leq k$.

We claim gr (p, R) is equidimensional. If gr (p, R) is Cohen-Macaulay, then since it is the homomorphic image of a regular domain, it must be equidimensional. If R is Cohen-Macaulay and a domain, then since it is also Nagata, the conditions of Theorem 2.1 show that gr $(p, R)^{\text{red}}$ is a domain and consequently gr (p, R) is equidimensional. Observe that unless $k \geq 0$ there is nothing to prove and so

$$l(p_q) < \dim (R_q)$$

for all primes q strictly containing p.

Since gr (p, R) is equidimensional and universally catenarian,

$$\text{height } (\bar{m}) + \dim (\text{gr } (p, R)/\bar{m}) = \dim R.$$

Hence,

$$\text{(c)} \qquad \text{height } (\bar{m}) = \dim R - l(p).$$

We claim $\dim R_p$ is strictly greater than $\dim (R) - k - 1$. If not,

$$\text{height } (\bar{m}) = \dim (R) - l(p) \geq \dim (R) - (\dim (R) - k - 1)$$

by (#), and so height $(\bar{m}) \geq k + 1$ which is a contradiction. Hence $\dim (R_p)$ is greater than $\dim (R) - k - 1$ and so (#) shows that $l(p) = \dim (R_p)$. Now Lemma 2.2 shows p is generated by a R-sequence and consequently,

$$\text{gr } (p, R) \approx R/p[T_1, \ldots, T_n] \quad \text{where } n = \text{height } (p).$$

In particular if R/p satisfies R_k so does gr (p, R).

Conversely suppose gr (p, R) satisfies R_k and let m' be a prime of R/p of height at most k. Let Q be a minimal prime of gr (p, R) containing the image of m' in the 0th graded piece and such that

$$\dim (\text{gr } (p, R)/Q) = \dim (\text{gr } (p, R)/\bar{m}).$$

Localize at m, the pullback of $\bar{m}$ to R. As above, height $(\bar{m}) = \dim (R) - l(p)$. Since height $(\bar{m})$ is at most the height of m', we see that $k \geq \dim (R) - l(p)$. Hence $l(p) \geq \dim (R) - k$. Combining this with (#) yields $l(p) = \dim (R)$. The proof may now be completed as above using Lemma 2.2.

COROLLARY 2.1. *Let R be a Noetherian ring which is a homomorphic image of* *a regular domain and let p be a prime such that R_p is regular. Assume* gr (p, R) *is Cohen-Macaulay. Then*:

(1) gr (p, R) *is a domain if and only if*

$$(*) \qquad l(p_q) \leq \max\{\dim(R_p), \dim(R_q) - 1\}$$

holds for all primes q containing p;

(2) *Suppose further that R/p is integrally closed. Then* gr (p, R) *is integrally closed if*

$$(**) \qquad l(p_q) \leq \max\{\dim(R_p), \dim(R_q) - 2\}$$

holds for all primes q containing p.

Proof. Recall that a Cohen-Macaulay ring R is reduced if and only if R satisfies R_0. [13] Since R/p obviously satisfies R_0, the assumptions together with Proposition 2.1 show that gr (p, R) is reduced, and in addition show gr (p, R) is equidimensional. (See the proof of Proposition 2.1.)

Assume (*) holds. Then Theorem 2.2 shows that gr $(p, R)^{\text{red}}$ is a domain. Since gr (p, R) is reduced, this shows it is a domain. Conversely, if gr (p, R) is a domain, then the discussion of the introduction shows that (*) must hold.

We now show (2). Assume that (**) holds. Then Proposition 2.1 shows that gr (p, R) satisfies R_1. Since gr (p, R) is also Cohen-Macaulay, this implies gr (p, R) is normal [13]. As gr (p, R) is a domain by above, it is integrally closed.

PROPOSITION 2.2. *Suppose R is a universally catenarian Nagata domain and p a prime such that R_p is regular. If* gr $(p, R)^{\text{red}}$ *is an integrally closed domain, then* gr (p, R) *is a domain (and hence is integrally closed).*

Proof. Let $S = \mathscr{R}(p, R)[t^{-1}]$ be the extended Rees algebra. There is an isomorphism, $S/St^{-1} \approx \text{gr}(p, R)$ and so the assumptions show there is a unique minimal prime Q in S containing t^{-1}.

We claim QS_Q is generated by t^{-1}. To show this, it is enough to show that gr $(p, R)_q$ is a domain where q is the image of Q in gr (p, R). However, since q is nilpotent, $q \cap R/p = (0)$ and so gr $(p, R)_q$ is a localization of gr $(p, R)_{W^*}$ where $W = R - p$. Since R_p is regular, this is a domain.

We may now apply the following lemma of Hironaka:

LEMMA 2.3(A) (36.10 of [15]). *Let R be a universally catenarian Nagata domain and suppose a is a non-zero element such that*

(1) *Ra has only one minimal prime divisor p,*

(2) *$aR_p = pR_p$,*

(3) *R/p is an integrally closed domain.*

Then $p = (a)$ and R is integrally closed.

These conditions are satisfied with $a = t^{-1}$ and $p = Q$. We may conclude $(t^{-1}) = Q$ and so gr $(p, R) = \text{gr}(p, R)^{\text{red}}$ is an integrally closed domain.

PROPOSITION 2.3. *Let R be a Cohen-Macaulay Noetherian domain which is an image of a regular domain and let p be a height two prime with resolution*

$$0 \longrightarrow R^n \xrightarrow{A} R^{n+1} \longrightarrow p \longrightarrow 0.$$

If height $(I_t(A)) \geq n + 3 - t$ *for* $1 \leq t \leq n - 1$, *then* gr (p, R) *is a domain. If moreover* height $(I_t(A)) \geq n + 4 - t$ *for* $1 \leq t \leq n - 1$, *and* R/p *is integrally closed, then* gr (p, R) *is integrally closed.*

Proof. Proposition 1.3 shows that under either of these two conditions, gr (p, R) is Cohen-Macaulay. In addition under these conditions, $S_R(p) \approx \mathscr{R}(p, R)$ and so the analytic spread of p at any prime q is just the dim $(S_{R_q}(p_q)/q)$ which is easily seen to be $v(p_q, R_q)$. By Corollary 2.1, to show gr (p, R) is a domain it is enough to show,

$$\text{(d)} \qquad l(p_q) \leq \max\{\dim(R_p), \dim(R_q) - 1\}$$

for every prime q which contains p. From the remarks above we may replace $l(p_q)$ with $v(p_q, R_q)$. It is easily seen that $v(p_q, R_q) \leq n - t + 1$ if and only if $I_t(A) \nsubseteq q$.

Suppose at some prime q condition (d) does not hold. Then

$$v(p_q, R_q) > \max\{\dim(R_p), \dim(R_q) - 1\}$$

and in particular, $v(p_q, R_q)$ is greater than dim $(R_q) - 1$. Set

$$v(p_q, R_q) = n - t + 1.$$

Then $I_{t+1}(A)$ is contained in q else $v(p_q, R_q) \leq n + 1 - (t + 1)$. As we may assume q contains p, the minimal number of generators of p_q is at least 2 and so t is at most $n - 1$. Thus the assumption shows

$$\text{height}\ (I_{t+1}(A)) \geq n + 3 - (t + 1) = n + 2 - t.$$

Consequently, the height of q is at least this number. Thus the inequality

$$\begin{aligned} n + 1 - t = l(p_q) &> \max\{\dim(R_p), \dim(R_q) - 1\} \\ &\geq \max\{\dim(R_p), n + 1 - t\} \end{aligned}$$

gives a contradiction and so (d) has been verified. Therefore gr (p, R) is a domain.

To show gr (p, R) is integrally closed, it suffices by Corollary 2.1 to show

$$\text{(e)} \qquad l(p_q) \leq \max\{\dim(R_p), \dim(R_q) - 2\}$$

for all primes q containing p.

As above the analytic spread of p at q is equal to the minimal number of generators of p_q. Assume q is a prime containing p such that (e) does not hold. If q has height less than or equal to one over p, then since height $(I_{n-1}(A))$ is at least 4, we see that q cannot contain this ideal and so $v(p_q, R_q) = 2$. In this case,

(e) is satisfied. Hence we may assume that the maximum on the left hand side of (e) is given by $\dim(R_q) - 2$. Set $v(p_q, R_q) = n + 1 - t$. As above, this shows $I_{t+1}(A)$ is contained in q and hence

$$\dim(R_q) \geq n + 4 - (t + 1) = n + 3 - t.$$

From (e) we now obtain,

$$\begin{aligned} n + 1 - t = l(p_q) &> \max\{\dim(R_p), \dim(R_q) - 2\} \\ &\geq \max\{\dim(R_p), n + 1 - t\}. \end{aligned}$$

This contradiction finishes the proof.

Example 2.1. Consider the example 1.2 of the prime p defining

$$k[t^3, t^2v, tv^2, v^3].$$

Since height $I_2(A) = 2$ and height $I_1(A) = 4$ we see that gr (p, R) is a domain. However, we cannot conclude that it is integrally closed.

We now consider a generalization of a technique originated by Hochster [6] and Samuel [19]. Both of these authors considered rings of the form

$$R[T_1, \ldots, T_n]/(a_1 T_1 + \cdots + a_n T_n).$$

In [6] Hochster showed:

PROPOSITION 2.4(A). *Let R be a noetherian ring and*

$$S = R[T_1, \ldots, T_n]/(f) \quad \textit{where} \quad f = \sum_1^n a_i T_i.$$

Set $I = (a_1, \ldots, a_n)$.

(1) *If* grade $(I) \geq 2$ *and R is a domain, then S is a domain.*

(2) *If* grade $(I) \geq 3$ *and R is integrally closed, then S is integrally closed.*

(3) *If* grade $(I) \geq 3$ *and R is a UFD, then S is a UFD.*

One may perceive these results as phenomena connected with graded algebras of ideals. We replace "grade" by "height" in the above proposition and reach somewhat similar conclusions from which one can easily deduce Proposition 2.4.A. The result relies upon the fact one can find a prime J in S such that $S \approx$ gr (J, S).

PROPOSITION 2.5. *Let R, S, and I be as above, and assume R is universally catenarian.*

(1) *If* height $(I) \geq 2$ *and R is a domain, then S^{red} is a domain.*

(2) *If* height $(I) > k + 1$ *then S^{red} satisfies R_k if and only if R satisfies R_k.*

Proof. Let t_i denote the image of T_i in S and set $J = (t_1, \ldots, t_n)$. Let

$$D = R[T_1, \ldots, T_n]$$

and set $J' = (T_1, \ldots, T_n)$. It is clear that $D \approx \mathrm{gr}\,(J', D)$. Since $f = f^*$ is not a zero-divisor on D, we see that

$$S = D/Df \approx \mathrm{gr}(J', D)/(f) \approx \mathrm{gr}\,(J'/(f), D/(f)) = \mathrm{gr}\,(J, S).$$

Now suppose R is a domain. We claim S is equidimensional. Every minimal prime of S corresponds to a prime in T minimal over f. Since R is a domain so is T and hence every such prime has height one. However R and thus T are universally catenarian and so the dimension of each of these primes is the same. Thus S is equidimensional. This shows $\mathrm{gr}\,(J, S)$ is equidimensional since it is isomorphic to S. If we are able to show that $l(J_q) < \dim\,(S_q)$ for every prime q strictly containing J, then Theorem 2.2 shows that $\mathrm{gr}\,(J, S)^{\mathrm{red}} = S^{\mathrm{red}}$ is a domain.

By assumption, height $(J + I) \geq$ height $(J) + 2 = n + 1$. Hence if q contains this ideal, then the above inequality holds as $l(J_q) \leq l(J) \leq v(J, S) = n$. Thus we may assume q does not contain I. In this case some a_i is not in q and then S_q becomes the localization of a polynomial ring in $n - 1$ variables over R, and J becomes generated by $n - 1$ elements. This shows (1).

We now demonstrate (2). Suppose R satisfies R_k, and let Q be a prime in S of height at most k. Since height (I) is at least $k + 2$, the height of IS is at least $k + 1$ and so Q cannot contain I. As above this shows S_Q is a localization of a polynomial ring over R and hence S_Q satisfies R_k; in particular it is regular. In this case, $S_Q = S_Q^{\mathrm{red}}$ which shows S^{red} satisfies R_k.

Conversely assume S^{red} satisfies R_k and let q be a prime in R of height at most k. Since the height of I is greater than this, q cannot contain I and so f is not in q expanded to T. If Q is a minimal prime containing (q, f) and having the same dimension, then the image of Q in S can have height at most k also. By abuse of notation we call this ideal Q. As above $S_Q = S_Q^{\mathrm{red}}$ is a localization of a polynomial ring over R_q; since S_Q is regular, R_q must be regular. (Note that $S_Q^{\mathrm{red}} = S_Q$ since S_Q^{red} is a domain because it is regular.)

Finally we wish to illustrate how the representation of a graded algebra as the graded algebra of some ideal can allow one to deduce some of its properties.

Example 2.2. Let k be a filed and $X = (x_{ij})$ a generic $r \times s$ matrix over k with $r \leq s$. Let $p = I_t(X)$ and set $R = k[x_{ij}]/p$. Set J equal to the ideal of R generated by the images of $x_{11}, \ldots, x_{r1}$. Let X' be the matrix X with the first column deleted.

In [4], Eagon and Hochster showed that R is a Cohen-Macaulay integrally closed domain. We wish to illustrate how the normality of this variety can be deduced from the knowledge that it is Cohen-Macaulay and a domain.

As R/J sits in R in a natural way it is easy to see there is an onto map from R to $\mathrm{gr}\,(J, R)$. Since these rings have the same dimension and we know R is a domain, these rings must be isomorphic. By induction on the size of the matrix, we may assume we know R/J is integrally closed. Now Corollary 2.1 shows $\mathrm{gr}\,(J, R)$ and hence R will be integrally closed if,

(f) $$l(J_q) \leq \max\,\{\dim\,(R_j), \dim\,(R_q) - 2\}.$$

The height of J is $t - 1$, and J is generated by r elements. If we invert any element of X', we may reduce the size of the matrix and by the induction obtain R_q is integrally closed, which will imply (f) for such q. Thus it is enough to show (f) holds for $m = (x_{ij})$. However, the

$$\text{height } m = rs - (r - t + 1)(s - t + 1)$$

and this is greater than $r + 2$, unless $t = 1$ in which case there is nothing to prove. This shows (f) and finishes the example.

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