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CRITERIA FOR APPROXIMATION BY HARMONIC FUNCTIONS

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1. Summary

In [1], P. R. Ahern gives "geometric" conditions which ensure that every continuous function on K, harmonic in the interior of K, can be approximated uniformly on K by functions harmonic in a neighborhood of K. Here we observe that Ahern's conditions can be sharpened to yield necessary and sufficient conditions for such approximation to obtain. The proof depends on a simple characterization of stable boundary points, which facilitates the evaluation of certain logarithmic potentials.

2. Statement of the Theorem

Let K be a compact plane set. By $D(K)$ we denote the space of continuous functions on K which are harmonic on the interior $\mathring{K}$ of K. The Choquet boundary of $D(K)$ coincides with the set R of regular boundary points of $\mathring{K}$. For each $p \in K$ there is a unique probability measure μ_p carried on R, called the **harmonic measure for** p, which satisfies $\int u \, d\mu_p = u(p)$ for all $u \in D(K)$.

Let $H(K)$ denote the space of functions harmonic in a neighborhood of K, and let $\overline{\mathbf{H(K)}}$ denote the uniform closure of $H(K)$ in $C(K)$. A point $q \in \partial K$ is a **stable boundary point** for K if there is an open set containing $K \backslash \{q\}$ which has q as a regular boundary point. The set of stable boundary points, denoted by P, coincides with the Choquet boundary of $\overline{H(K)}$. Since $\overline{H(K)} \subseteq D(K)$, evidently $P \subseteq R$. For more details and further references, see [3] and [6].

The **outer boundary** of a compact set K is the union of the boundaries of the components of the complement K^c of K. The remaining boundary points comprise the **inner boundary** of K. Ahern [1] proves that the following two "geometric" conditions together imply that $\overline{H(K)} = D(K)$:

(1) For all $p \in \mathring{K}$, μ_p is supported on the outer boundary of K.
(2) The inner boundary of ∂K (not of K!) has zero area.

Our object is to establish the following sharpened version of Ahern's theorem. The geometric nature of Ahern's conditions are sacrified for conditions that are both necessary and sufficient for approximation.

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THEOREM. *In order that $\overline{H(K)} = D(K)$, it is necessary and sufficient that the following two conditions hold:*

(3) *μ_p is carried on P for all $p \in \mathring{K}$.*

(4) *$\overline{H(\partial K)} = C(\partial K)$.*

It is well known that outer boundary points of K are stable, (cf. [3]), so that (1) implies (3). It is also well known that if $K \backslash P$ has zero area, then $\overline{H(K)} = C(K)$. (We will presently give a proof of this fact.) Applying this result to ∂K instead of K, we see that (2) implies (4). Hence the theorem indeed represents a generalization of Ahern's theorem. The examples cited by Ahern show that neither (3) nor (4) in itself is sufficient to guarantee that $\overline{H(K)} = D(K)$.

Proof of necessity. Suppose $\overline{H(K)} = D(K)$, so that $R = P$. Then (3) is valid. By Kellogg's theorem, (cf. [6]), $(\partial K) \backslash K$ has zero logarithmic capacity, hence zero area. Since every point of P is a point of stability of the boundary of ∂K, the unstable boundary points of ∂K have zero area. Consequently $\overline{H(\partial K)} = C(\partial K)$, and the necessity of (3) and (4) is established.

3. A characterization of stable boundary points

Before proving the sufficiency, we give an elementary characterization of stable boundary points. A special case of the following lemma was used by T. McCullough [4, Theorem 1]. See also [5, Section 3].

LEMMA. *A point $p \in K$ is a stable boundary point of K if and only if there is a sequence of probability measures $\{\sigma_n\}_{n=1}^{\infty}$ with the following properties:*

(5) *The closed supports of the σ_n are compact subsets of K^c which converge to the singleton $\{p\}$.*

(6) *If u is defined and subharmonic in a neighborhood of p, then*

$$u(p) = \lim_{n \to \infty} \int u d\sigma_n.$$

Proof. Suppose first that p is stable. Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of compact sets such that $\mathring{K}_n \supseteq K_{n+1}$, and $\bigcap_{n=1}^{\infty} K_n = K$. Let λ_n be the harmonic measure for p on ∂K_n. Since p is stable the λ_n converge weak-star to the point mass at p. Consequently there are discs $\Delta_n = \{|z - p| \le r_n\}$ such that $r_n \downarrow 0$ and $\lambda_n(\Delta_n) \to 1$. Define σ_n to be the restriction of λ_n to Δ_n, normalized to have unit mass. That is, $\lambda_n(\Delta_n)\sigma_n = \lambda_n|_{\Delta_n}$. The measures σ_n evidently satisfy (5).

Let u be subharmonic near p. Its Laplacian Δu is defined near p as a positive measure, say $\Delta u = \sigma$ near p. If

$$h(z) = \frac{1}{2\pi} \int \log |z - w| \, d\sigma(w),$$

then h is subharmonic on the complex plane and $\Delta h = \sigma$. Thus $\Delta(u - h) = 0$ near p, and $u - h$ is harmonic near p. Replacing u by h, we may assume that u is defined and subharmonic on the complex plane. Subtracting a constant from u, we may furthermore assume that $u \leq 0$ on K_1. Then

$$u(p) \leq \int u d\lambda_n \leq \lambda_n(\Delta_n) \int u\, d\sigma_n,$$

so that

$$u(p) \leq \liminf_{n\to\infty} \int u\, d\sigma_n.$$

On the other hand, since u is upper semi-continuous, and the supports of the σ_n's cluster at p, we have

$$\limsup_{n\to\infty} \int u\, d\sigma_n \leq u(p).$$

Thus (6) is valid.

To prove the converse, suppose that such measures σ_n exist. Let U be an open set containing $K\backslash\{p\}$, such that $\overline{U}$ is disjoint from the closed supports of the σ_n. If p is not in the closure of any single component of U, then p is a regular boundary point for U, (cf. [3]), and hence a stable boundary point for K. So we can suppose that p is in the closure of some component V of U. If $q \in V$, and λ_q is harmonic measure on ∂V for q, then

$$v(z) = \int \log |z - \zeta|\, d\lambda_q(\zeta) - \log |z - q|$$

is subharmonic on $C\backslash\{q\}$. Moreover, v vanishes off $\overline{V}$, and by upper semi-continuity, $v \geq 0$ on ∂V. Since λ_q is supported on ∂V, from [6, Theorem III, 1] we have $\liminf v(z) \geq 0$ as $z \to \partial V$. Now v is harmonic on $V\backslash\{q\}$, and $v(z) \to +\infty$ as $z \to q$, so that $v > 0$ on V. Since

$$\limsup_{z\to p} v(z) = v(p) = \lim_{n\to\infty} \int v\, d\sigma_n = 0,$$

v is a "barrier" at p. So p is a regular boundary point of V, hence of U (cf. [3]), and p is a stable boundary point of K. This proves the lemma.

COROLLARY. *Let p be a stable boundary point of K, and let α be a finite real measure on the complex plane, such that*

$$\int \log |\zeta - p|\, d|\alpha|(\zeta) < \infty.$$

Then

$$\lim_{z\in K^c,\, z\to p} \int \log |\zeta - z|\, d\alpha(\zeta) = \int \log |\zeta - p|\, d\alpha(\zeta), \tag{7}$$

providing the limit exists.

Proof. Write $\alpha = \alpha^+ - \alpha^-$, where $\alpha^+ \geq 0$ and $\alpha^- \geq 0$ are mutually singular, and apply the lemma to the logarithmic potentials of α^+ and α^- separately. This leads to the identity

$$\lim_{n\to\infty} \int \left[\int \log |\zeta - z| \, d\alpha(\zeta)\right] d\sigma_n(z) = \int \log |\zeta - p| \, d\alpha(\zeta), \tag{8}$$

which holds even when the limit in the left hand side of (7) fails to exist. When the limit in (7) does exist, then the property (5) of the σ_n's shows that the limit coincides with the limit in (8). This proves the corollary.

For α a measure on K_1 define

$$V_\alpha(z) = \int \log |z - w| \, d\alpha(w), \tag{9}$$

wherever the integral converges absolutely. Since V_α is a convolution of a locally integrable function and a measure, V_α exists a.e. $(dx\, dy)$, and V_α is itself locally integrable. If $V_\alpha = 0$ a.e. $(dx\, dy)$, then $\alpha = 0$. The importance of V_α in approximation theory stems from the following easily proven fact: the measure α is orthogonal to $H(K)$ if and only if $V_\alpha = 0$ on K^c. We immediately obtain the following corollary to the corollary.

COROLLARY. *If α is a measure on K orthogonal to $H(K)$, then $V_\alpha = 0$ on K^c, and $V_\alpha(p) = 0$ at every stable boundary point p at which the defining integral* (9) *converges absolutely.*

In particular, it is clear now that $\overline{H(K)} = C(K)$ as soon as $K \backslash P$ has zero area. In this case, if α is orthogonal to $H(K)$, then $V_\alpha = 0$ a.e. $(dx\, dy)$, and $\alpha = 0$.

Completion of proof of the theorem. To prove the sufficiency of the conditions (3) and (4), we follow closely the proof given by Ahern, using the preceding corollary.

Assume (3) and (4) are valid. By Choquet theory, it suffices to show that the only real measure α on R orthogonal to $H(K)$ is zero. We define V_α as in (9).

Let W be a component of $\mathring{K}$, and let $q \in W$. Then

$$\int \log |\zeta - z| \, d\mu_q(z) = \log |\zeta - q|$$

providing either $\zeta \notin \overline{W}$, or ζ is a regular boundary point of W. Since

$$\iint \log |\zeta - z| \, d\mu_q(z) \, d|\alpha|(\zeta) \leq \int \log |\zeta - q| \, d|\alpha|(\zeta) < \infty,$$

we can apply Fubini's theorem to obtain

$$\begin{aligned} V_\alpha(q) &= \iint \log |\zeta - z| \, d\mu_q(z) \, d\alpha(\zeta) \\ &= \int V_\alpha(z) \, d\mu_q(z). \end{aligned}$$

Since μ_q is carried on P, and $V_\alpha = 0$ wherever defined on P, the latter integral is zero, and $V_\alpha(q) = 0$. Hence $V_\alpha = 0$ off ∂K, α is orthogonal to $H(\partial K) = C(\partial K)$, and $\alpha = 0$. This completes the proof of the theorem.

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