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AN ALGEBRAIC CONCEPT OF SYMPLECTIC CURVATURE STRUCTURES

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1. Introduction

Given a pseudo-riemannian manifold $(\mathcal{M}, \sigma)$ Levi-Civitas unique torsionfree connection ∇ induces the canonical pseudo-riemannian curvature structure R by

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

for vector fields X, Y on $\mathcal{M}$. R is skew symmetric in X and Y, fulfills the first Bianchi identity (see (S.3) below) and defines a section $R(X, Y)$ of the pseudo-orthogonal Lie algebra bundle over $\mathcal{M}$. Singer and Thorpe [15] conversely defined a curvature structure on a pseudo-orthogonal vector space, here $(T_p\mathcal{M}, \sigma)$, as a (1, 3) tensor with these three axioms. The linear space spanned by these curvature structures can be taken as a typical fibre of a vector bundle over $\mathcal{M}$ in which the above canonical curvature structure is a section. The special types of pseudo-riemannian manifolds (of constant curvature, Einsteinian, etc.) usually are defined in terms of this section. Petrov first has given a basis in the space of curvature structures on a 4-dimensional Minkowski space.

In the following, on a symplectic vector space $(\mathbf{E}, \sigma)$ of finite dimension $2n$, an algebraic analog of such a pseudo-orthogonal curvature structure is developed, by changing the sign in the first axiom (S.1) and inserting the symplectic Lie algebra

$$\text{sp}(\mathbf{E}, \sigma) = \{Q \text{ in end } \mathbf{E}/\sigma(Qx, y) + \sigma(x, Qy) = 0 \text{ for all } x, y \text{ in } \mathbf{E}\}$$

for the pseudo-orthogonal one in (S.3). In Sections 2 and 3 it is shown that most results on pseudo-orthogonal curvature structures can be overtaken almost literally. Especially Weyl's conformal curvature again defines a projector and hence a decomposition of the curvature space which is invariant under the induced action of the symplectic group $\text{Sp}(\mathbf{E}, \sigma)$ on $(\mathbf{E}, \sigma)$ and its Lie algebra $\text{sp}(\mathbf{E}, \sigma)$. Nomizu's characterization [14] of the kernel of this projector by the (Jordan algebra of) σ-selfadjoint endomorphisms on $\mathbf{E}$,

$$JA(\mathbf{E}, \sigma) = \{A \text{ in end } \mathbf{E}/\sigma(Ax, y) = \sigma(x, Ay) \text{ for all } x, y \text{ in } \mathbf{E}),$$

can be proved as well. The proofs are essentially those of riemannian differential geometry, as described for instance in [5], [6] and [8]. The following treatment is based on the work of Kulkarni [10], [11], Kowalski [9] and Nomizu

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[14], a little different but equivalent notation is used by Gray [4], Marcus [13] and Singer and Thorpe [15].

Those pseudo-riemannian curvature structures which fulfill Cartan's condition, i.e., the pseudo-riemannian symmetric ones, are related to the pseudo-orthogonal Lie triples via left multiplication. A systematic treatment of Lie triples is given in the books of Loos [12] and Jacobson [7]. To keep this relation we introduce another kind of triple, where only the skew symmetry of Lie triples is changed. This kind of triple was introduced in [17] as the odd subtriples of a graded generalization of Lie triples, the even subtriples being exactly the ordinary Lie triples. As in the case of Lie triples and their graded generalizations there is a covariant functor into a category of algebras, called standard embedding. These algebras generalize a special type of graded Lie algebras which Djokovic and Hochschild [3] called symplectic sequence. Generalities on these graded Lie algebras are given in [2], [3] and [17]. A special symplectic triple of this new category occured first in the quantization of Bosons, i.e., in the Weyl algebras over a symplectic vector space [16].

Finally pseudo-unitary curvature-like-structures are considered where the Bianchi identity is dropped, and the pseudo-unitary subalgebra is inserted for the whole symplectic algebra. Note that the pseudo-riemannian curvature structures in [4], [13] and [15] are defined without the Bianchi identity.

2. Symplectic curvature structures

Let $\mathbf{E}$ be a (necessarily even dimensional) real vector space and σ a non-degenerate skew bilinear form on $\mathbf{E}$, $\dim \mathbf{E} = 2n$. A *symplectic curvature structure* on $(\mathbf{E}, \sigma)$ is a bilinear mapping

$$S\colon \mathbf{E} \times \mathbf{E} \rightarrow \operatorname{end} \mathbf{E}$$

subject to

(S.1) $S(x, y) = S(y, x)$ (*Symmetry*),
(S.2) $\sigma(S(x, y)z, w) + \sigma(z, S(x, y)w) = 0$, i.e., $S(x, y) \in sp(\mathbf{E}, \sigma)$,
(S.3) $S(x, y)z + S(z, x)y + S(y, z)x = 0$ (*Bianchi identity*),

for x, y, z, w in $\mathbf{E}$. Using these axioms one has

(S.4) $\sigma(S(x, y)z, w) = \sigma(S(z, w)x, y)$.

The linear space of such curvature structures will be denoted by $\operatorname{curv}(\mathbf{E}, \sigma)$. It always contains the *trivial* curvature structure

$$S_0(x, y)z = \sigma(y, z)x + \sigma(x, z)y.$$

More generally, given A, B in $JA(\mathbf{E}, \sigma)$,

$$2S_0^{A,B}(x, y) = S_0(Ax, By) + S_0(Bx, Ay) = 2S_0^{B,A}(x, y)$$

defines a new element $S_0^{A,B}$ in curv $(\mathbf{E}, \sigma)$. The *Ricciform* of S

$$\rho_S(x, y) = \text{trace } (z \mapsto S(x, z)y)$$

is skew as well. The *Riccitransformation* L_S of S is defined by

$$\sigma(L_S x, y) = \rho_S(x, y).$$

It is σ- and ρ_S-selfadjoint, i.e., in $JA(\mathbf{E}, \sigma)$. The *curvature scalar* of S is

$$\text{Sc}(S) = \text{trace } L_S$$

Using trace $(z \mapsto \sigma(x, z)y) = \sigma(x, y)$ one gets

$$\text{(1)} \quad \begin{aligned} L_{S_0^{A,B}} &= \tfrac{1}{2}(AB + BA + A \text{ trace } B + B \text{ trace } A), \quad L_{S_0} = (2n + 1) \, \text{id}_{\mathbf{E}}, \\ \text{Sc } (S_0^{A,B}) &= \text{trace } AB + \text{trace } A \text{ trace } B, \quad \text{Sc } (S_0) = 2n(2n + 1), \end{aligned}$$

especially $\rho_{S_0}(x, y) = (2n + 1)\sigma(x, y)$. The *Ricci curvature structure* S^{ic} of S is given by $S_0^{L_S, \text{id}_{\mathbf{E}}}$, i.e., by

$$\text{(2)} \quad S^{ic}(x, y)z = \tfrac{1}{2}\{\rho_S(y, z)x + \rho_S(x, z)y + \sigma(y, z)L_S x + \sigma(x, z)L_S y\}.$$

In particular, we have

$$\text{(3)} \quad (S_0^{A,B})^{ic} = S_0^{C, \text{id}} \text{ with } C = L_{S_0^{A,B}}, \text{ and } S_0^{ic} = (2n + 1)S_0.$$

In general, one has

$$\text{(4)} \quad \begin{aligned} 2\rho_{S^{ic}}(x, y) &= (2n + 2)\rho_S(x, y) + \text{Sc}(S)\sigma(x, y) \\ 2L_{S^{ic}} &= (2n + 2)L_S + \text{Sc}(S) \, \text{id}_{\mathbf{E}} \\ \text{Sc}(S^{ic}) &= (2n + 1) \, \text{Sc}(S) \end{aligned}$$

The *sectional curvature* $\|S\|$ is defined as

$$\|S\|(x, y) = \sigma(S(x, y)y, x)$$

with $\|S_0^{A,B}\|(x, y) = -\sigma(Ax, y)\sigma(Bx, y)$, hence $\|S_0\|(x, y) = -\sigma(x, y)^2$ (which may be called the *discriminant* of σ) and $\|S^{ic}\|(x, y) = -\rho_S(x, y)\sigma(x, y)$. It is easy to show that Ricciform and Riccitransformation are linear in their indices, that the *Ricci map*

$$\mathcal{R}i: S \mapsto S^{ic}$$

is an endomorphism and Sc a linear form on curv$(\mathbf{E}, \sigma)$. In the following we use the injective linear mapping

$$\Omega: A \mapsto S_0^{A, \text{id}}, \quad \Omega: JA(\mathbf{E}, \sigma) \to \text{curv}(\mathbf{E}, \sigma).$$

Obviously $\Omega(L_S) = \mathcal{R}i(S)$.

3. The Weyl curvature structures

The *Weyl curvature structure* of $S \in \operatorname{curv}(\mathbf{E}, \sigma)$ is

$$\mathscr{W}S = S - \frac{2}{2n+2} S^{ic} + \frac{\operatorname{Sc}(S)}{(2n+1)(2n+2)} S_0 \quad \in \operatorname{curv}(\mathbf{E}, \sigma).$$

(5) LEMMA. (a) $\rho_{\mathscr{W}S} = 0$, $L_{\mathscr{W}S} = 0$, $\operatorname{Sc}(\mathscr{W}S) = 0$, $\mathscr{W}(S^{ic}) = 0$.
(b) $\mathscr{W}S_0^{A,\mathrm{id}} = 0$; i.e., $\mathscr{W} \circ \Omega = 0$. *In particular* $\mathscr{W}S_0 = 0$.
(c) $\mathscr{W}$ is a projector on curv $(\mathbf{E}, \sigma)$.
(d) $(\mathscr{W}S)^{ic} = 0$; i.e., $\mathscr{Ri} \circ \Omega = 0$.
The proof of (a) and (b) is elementary. Part (c) is a consequence of the linearity of $\mathscr{Ri}$ and Sc. Part (d) follows from (c): consider the complementary projector $\mathscr{W}^{\perp}$ of $\mathscr{W}$ onto Kern $\mathscr{W}$, given by

$$\mathscr{W}^{\perp}S = (\mathrm{id} - \mathscr{W})S = \frac{1}{2n+2}\left\{2S^{ic} - \frac{\operatorname{Sc}(S)}{2n+1} S_0\right\}.$$

Using (a) we get

$$\begin{aligned}\mathscr{W}^{\perp}S = \mathscr{W}^{\perp}\mathscr{W}^{\perp}S &= \frac{1}{2n+2}\left\{2((\mathrm{id} - \mathscr{W})S)^{ic} - \frac{\operatorname{Sc}((\mathrm{id} - \mathscr{W})S)}{2n+1} S_0\right\} \\ &= \mathscr{W}^{\perp}S - \frac{1}{2n+2}(\mathscr{W}S)^{ic}.\end{aligned}$$

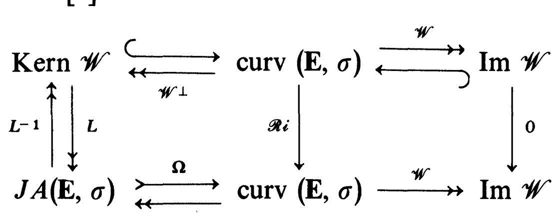
(6) LEMMA. (a) $\mathscr{W}S = 0$ *implies*

$$\Omega\left(\frac{1}{2n+2}\left\{2L_S - \frac{\operatorname{Sc}(S)}{2n+1} \mathrm{id}_{\mathbf{E}}\right\}\right) = S,$$

hence

$$\operatorname{Kern} \mathscr{W} \subset \operatorname{Im} \Omega.$$

(b) $\Omega \circ L \circ \mathscr{W}^{\perp} = \mathscr{Ri}$.
(c) $L \circ \Omega$ is injective, hence bijectiv.
(d) Ω is surjective onto Kern $\mathscr{W}$ and L is bijectiv.
The proof is technical. Summarizing we have the commutative diagram of short exact sequences [9]

(7) 

which shows how L and $\mathscr{R}i$ are related to each other via Ω and $\mathscr{W}^{\perp}$. Obviously

$$L^{-1}\colon A \mapsto \frac{1}{2n+2}\left\{2S_0^{A,\mathrm{id}} - \frac{\operatorname{trace} A}{2n+1} S_0\right\},\quad L^{-1}\colon JA(\mathbf{E},\sigma) \to \operatorname{Kern} \mathscr{W},$$

$$\Omega^{-1}\colon S \mapsto \frac{1}{2n+1}\left\{2L_S - \frac{\operatorname{Sc}(S)}{2n+1} \mathrm{id}_{\mathbf{E}}\right\},\quad \Omega^{-1}\colon \operatorname{Kern} \mathscr{W} \to JA(\mathbf{E},\sigma),$$

with $\Omega \circ \Omega^{-1} = \mathscr{W}^{\perp}$ and $L^{-1} \circ L = \mathscr{W}^{\perp}$. Ω^{-1} is called deviation map in [9]. $JA(\mathbf{E}, \sigma)$ can be decomposed uniquely in the space of multiples of $\mathrm{id}_{\mathbf{E}}$ and the $(n(2n-1)-1)$-dimensional space of traceless endomorphisms. Hence Kern $\mathscr{W}$ can be decomposed directly into $\mathbf{R}S_0$ and the Ω-image $\mathbf{L}_2$ of these traceless matrices, altogether

$$\operatorname{curv}(\mathbf{E}, \sigma) = \mathbf{R}S_0 \oplus \mathbf{L}_2 \oplus \operatorname{Im} \mathscr{W}. \tag{8}$$

(9) THEOREM. (a) *S is in $\mathbf{R}S_0$ if and only if the sectional curvature of S is constant.*

(b) *S is in* Im $\mathscr{W}$ *if and only if the Ricciform of S is zero.*

(c) *S is in $\mathbf{R}S_0 \oplus$* Im $\mathscr{W}$ *if and only if the Riccitransformation of S is a multiple of* $\mathrm{id}_{\mathbf{E}}$ *(Einsteinian curvature structures).*

(d) *S is in $\mathbf{L}_2 \oplus$* Im $\mathscr{W}$ *if and only if the scalar curvature of S vanishes.*

Proof. (a) (S.1), (S.2) and (S.4) imply that $\sigma(S(x, z)z, y)$ is symmetric in x and y. From

$$\sigma(S(x,z)z, y) = \tfrac{1}{2}\{\|S\|(x+y, z) - \|S\|(x, z) - \|S\|(y, z)\}$$

and

$$S(x,y)z = S(x, y+z)(y+z) + S(x+z, y)(x+z) - S(x,y)y - S(x,y)x - S(x,z)x - S(z,y)z,$$

$\sigma(S(x, y)z, w)$ can be written as a linear combination of some $\|S\|(\cdots)$. Therefore $\|S\| = 0$ is equivalent to $S = 0$. More generally, $\|S\| = \lambda\|S_0\|$ implies $\|S - \lambda S_0\| = 0$, hence $S = \lambda S_0$. Part (b) is clear from $\rho_{\mathscr{W}S} = 0$ and the fact that this implies $S \in \operatorname{Kern} L = \operatorname{Im} \mathscr{W}$. Parts (c) and (d) are simple consequences of the above results.

The finestructure of Im $\mathscr{W}$ seems to be more complicated than that of Kern $\mathscr{W}$. We give a class of Weyl curvature structures for $n > 1$: let M be a skew $n \times n$ matrix:

$$A = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \left(\text{resp. } A = \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}\right)$$

are traceless nilpotent matrices in $JA(\mathbf{E}, \sigma)$ such that $S_0^{A,A} = \mathscr{W}S_0^{A,A} \neq 0$.

4. The action of Sp(E, σ) on curv(E, σ)

Let S be in curv(E, σ). For G in $Sp(\mathbf{E}, \sigma)$ (resp. Q in $sp(\mathbf{E}, \sigma)$), define

$$G \cdot S(x, y) = GS(G^{-1}x, G^{-1}y)G^{-1} \tag{10}$$

$$(\text{resp. } Q \cdot S(x, y) = [Q, S(x, y)] - S(Qx, y) - S(x, Qy)).$$

This makes curv(E, σ) a $Sp(\mathbf{E}, \sigma)$-module (resp. a $sp(\mathbf{E}, \sigma)$-module). Writing $GAG^{-1} = G \cdot A$ (resp. $[Q, A] = Q \cdot A$) (these are elements of $JA(\mathbf{E}, \sigma)$ for $A \in JA(\mathbf{E}, \sigma)$) we get

$$G \cdot S_0^{A,B} = S_0^{G \cdot A, G \cdot B} (\text{resp. } Q \cdot S_0^{A,B} = S_0^{Q \cdot A, B} + S_0^{A, Q \cdot B}), \tag{11}$$

and as a special case of this

$$G \cdot \Omega(A) = \Omega(G \cdot A) \text{ and } G \cdot S_0 = S_0$$
$$(\text{resp. } Q \cdot \Omega(A) = \Omega(Q \cdot A) \text{ and } Q \cdot S_0 = 0). \tag{12}$$

It is easy to see that

$$\rho_S(G^{-1}x, G^{-1}y) = \rho_{G \cdot S}(x, y)$$

(resp. $-\rho_S(Qx, y) - \rho_S(x, Qy) = \rho_{Q \cdot S}(x, y)$) from which

$$G \cdot L_S = L_{G \cdot S} \text{ and Sc } (G \cdot S) = \text{Sc}(S)$$
$$(\text{resp. } Q \cdot L_S = L_{Q \cdot S} \text{ and Sc } (Q \cdot S) = 0) \tag{13}$$

results. With the aid of these equations one proves that multiplication with G (resp. Q) commutes with $\mathscr{Ri}$ and $\mathscr{W}$. Hence the decomposition (8) is invariant under the action of the symplectic group and its Lie algebra. More general given two symplectic vector spaces $(\mathbf{E}', \sigma')$ and $(\mathbf{E}, \sigma)$ and a symplectic isomorphism $\Phi: \mathbf{E}' \to \mathbf{E}$,

$$\Phi^* S(x, y) = \Phi S(\Phi^{-1}x, \Phi^{-1}y)\Phi^{-1}$$

defines a new curvature structure. Whence curv is a covariant functor from the category of symplectic vector spaces onto the category of curvature spaces whose morphisms are the curvature preserving mappings. For S in curv $(\mathbf{E}, \sigma)$ define int S as the linear span of the elements $S(x, y)$ in $sp(\mathbf{E}, \sigma)$. From (10) we see that int S is a Lie subalgebra of $sp(\mathbf{E}, \sigma)$ if int S leaves S invariant, i.e., if $S(x, y) \cdot S = 0$ for all x, y in $\mathbf{E}$. Let us call such S *selfinvariant*. Now it is easy to see that S selfinvariant implies that $G \cdot S$ is selfinvariant for all G in $Sp(\mathbf{E}, \sigma)$. Consequently, selfinvariant curvature structures lie in selfinvariant $Sp(\mathbf{E}, \sigma)$-orbits in curv $(\mathbf{E}, \sigma)$. From (12) and the injectivity of Ω we see that $\Omega(A) = S_0^{A, \mathrm{id}}$ is selfinvariant if and only if $S_0^{A, \mathrm{id}}(x, y) \cdot A = 0$ for all x, y if and only if $A^2 = \lambda \, \mathrm{id}_{\mathbf{E}}$. This shows that the elements in $\mathbf{R}S_0$ are selfinvariant. The $Sp(\mathbf{E}, \sigma)$-orbits in $\mathbf{R}S_0$ are exactly the points. The above examples in Im $\mathscr{W}$ are selfinvariant as well. A complex structure J in $Sp(\mathbf{E}, \sigma)$ induces an involutive transformation in curv$(\mathbf{E}, \sigma)$, i.e., $J \cdot J \cdot S = S$ for all S. Then $\mathbf{L}_2$ and Im $\mathscr{W}$

split into the eigenspaces of J of eigenvalues 1 and -1. The subgroup of $Sp(\mathbf{E}, \sigma)$ commuting with J is a pseudo-unitary group whose signature depends on J, and orbits of $Sp(\mathbf{E}, \sigma)$ split into pseudo-unitary ones.

It is straightforward to prove that the above curvature representations of the symplectic group has the kernel $\{\pm \mathrm{id}_{\mathbf{E}}\}$.

5. Curvature structures, triples and graded algebras

A *triple* is a trilinear composition $[,,]: \mathbf{E} \times \mathbf{E} \times \mathbf{E} \to \mathbf{E}$. Moreover we assume

(T.1) $[x, y, z] = [y, x, z]$ (*Symmetry*),
(T.2) $[x, y, z] + [y, z, x] + [z, x, y] = 0$ (*Jacobi identity*),
(T.3) $[x, y, [u, v, z]] = [[x, y, u], v, z] + [u, [x, y, v], z] + [u, v, [x, y, z]]$

for x, y, z, u, v in $\mathbf{E}$. The *left multiplication* is defined as $\mathrm{ad}(x, y)z = [x, y, z]$. (T.3) shows that $ad(x, y)$ is a (*inner*) derivation of $(\mathbf{E}, [,,])$. In terms of left multiplications (T.3) reads

(14) $$[\mathrm{ad}(x, y), \mathrm{ad}(u, v)] = \mathrm{ad}(\mathrm{ad}(x, y)u, v) + \mathrm{ad}(u, \mathrm{ad}(x, y)v).$$

The space of inner derivations $\mathrm{int}(\mathbf{E}, [,,])$ is an ideal in the Lie algebra of all derivations of $(\mathbf{E}, [,,])$.

An elementary example is the *Bose triple*, defined by $[x, y, z] = S_0(x, y)z$ on a symplectic vector space.

(15) PROPOSITION. $\mathrm{int}(\mathbf{E},[,,]) \oplus \mathbf{E} = \mathbf{V}_0 \oplus \mathbf{V}_1$ *is a* $\mathbf{Z}_2$*-Lie-graded algebra with respect to the composition*

$$[A \oplus x, B \oplus y] = [A, B] + \mathrm{ad}(x, y) \oplus Ay - Bx.$$

The graded skew symmetry is obvious; one easily verifies the second axiom of a Lie-graded algebra, the graded Jacobi identity

$$(-1)^{km}[[x_k, y_l], z_m] + (-1)^{lm}[[z_m, x_k], y_l] + (-1)^{kl}[[y_1, z_m], x_k] = 0$$

for x_k in $\mathbf{V}_k$, y_1 in $\mathbf{V}_l$, z_m in $\mathbf{V}_m$. Let us call the algebra $\mathbf{V}_0 \oplus \mathbf{V}_1$ the *standard embedding* of $(\mathbf{E}, [,,])$. Obviously $[[x, y], z] = [x, y, z]$ for x, y, z in $\mathbf{E}$. $\mathbf{V}_0$(resp. $\mathbf{V}_l$) is the eigenspace of eigenvalue 1 (resp. -1) in this algebra for the *standard* involutive automorphism $A \oplus x \mapsto A \oplus -x$. Given a morphism $\Phi: \mathbf{E} \to \mathbf{E}'$ of such triples, i.e., $\Phi[x, y, z] = [\Phi(x), \Phi(y), \Phi(z)]'$, one proves with (T.1), (T.2) and (14) that

$$\Phi^*: \mathrm{ad}(x, y) \mapsto \mathrm{ad}'(\Phi(x), \Phi(y))$$

is a morphism of the Lie algebras of inner derivations, and that diag (Φ^*, Φ) is a morphism of $\mathbf{Z}_2$-Lie-graded algebras which commutes with the standard involutions. This establishes a functor between the two categories. A triple is called *symplectic* if all $\mathrm{ad}(x, y)$ are in $sp(\mathbf{E}, \sigma)$. The Bose triple is symplectic.

More general

(16) PROPOSITION. *A triple* (**E**, [, ,]) *with* (T.1), (T.2) *and* (T.3) *is symplectic if and only if* ad *is in* curv(**E**, σ) *and*

$$[Q, \text{ad}(x, y)] = \text{ad}(Qx, y) + \text{ad}(x, Qy) \quad (\textit{Cartan condition for } Q)$$

for all x, y *in* **E** *and all* Q *in* int(**E**, [,,]).

For the S_0-Bose case Cartan's condition are the commutation relations of the symplectic Lie algebra. Its standard embedding was called the "symplectic sequence" in [3]. Proposition (16) shows that selfinvariance, (T.3) and Cartan's condition coincide.

The Bose triple can be realized by the canonical Weyl algebra weyl(**E**, σ) = ten **E**/(($x \otimes y - \otimes x - \sigma(x, y)1$)), which occurs in the quantization of Bosons: **E** is embedded in weyl(**E**, σ) and the space Λ^2**E** in weyl^y(**E**, σ) generated by the symmetrized elements of second power $\Lambda xy = \frac{1}{2}(xy + yx)$ is isomorphic as a Lie algebra to sp(**E**, σ) [16]. One has

(17) $[\Lambda xy, z] = \frac{1}{2}(x[y, z] + [y, z]x + y[x, z] + [x, z]y) = \sigma(y, z)x + \sigma(x, z)y$

and the corresponding standard embedding $\mathbf{V}_0 \oplus \mathbf{V}_1$ is $\Lambda^2\mathbf{E} \oplus \mathbf{E}$ in weyl (**E**, σ). In the same way Fermion quantization leads to an ordinary Lie triple in a Clifford algebra.

Another class of generalized Bose triples are subspaces of associative algebras which are closed under the triple composition

(18) $$[x, y, z] = \tfrac{1}{2}[xy + yx, z] = \tfrac{1}{2}(xyz + yxz - zxy - zyx).$$

This allows the definition of a representation of a generalized Bose triple and of the universal envelopping algebra

$$\text{ten } \mathbf{E}/((2[x, y, z] - x \otimes y \otimes z - y \otimes x \otimes z + z \otimes x \otimes y + z \otimes y \otimes x))$$

of (**E**, [, ,]) in which (**E**, [, ,]) is injectively embedded. Here ((x)) means the two-sided ideal generated by the element x. Taking traces one shows that the Bose triple has no faithful finite-dimensional representation. Its universal envelopping algebra obviously is the Weyl algebra.

The above composition [, ,] on an associative algebra establishes via the standard embedding a functor of the category of associative algebras into that of the graded Lie algebras.

6. Pseudo-unitary curvature-like structures

Given a complex structure J in $Sp(\mathbf{E}, \sigma)$, i.e., $J^2 = -\text{id}_{\mathbf{E}}$, $\tau(x, z) = \sigma(Jx, z)$ is a symmetric non-degenerate bilinear form and J is in the pseudo-orthogonal group $O(\mathbf{E}, \tau)$. In addition $\tau(Jx, z) = -\sigma(x, z)$, $\tau(x, Jz) = \sigma(x, z)$ and $\sigma(x, Jz) = -\tau(x, Jz)$. For S in curv (**E**, σ), define

$$\begin{aligned}\mathscr{U}S(x, y)z &= (S(x, y) + JS(x, y)J^{-1})z \\ &= \sigma(y, z)x + \sigma(x, z)y + \tau(y, z)Jx + \tau(x, z)Jy.\end{aligned}$$

$\mathcal{U}S$ is a curvature-like structure which fulfills (S.1) and (S.2) but not the Bianchi identity [even if $S = S_0$). Polarizing $x \mapsto x + y$ and $z \mapsto z + w$ in

$$(19) \quad [\mathcal{U}S_0(x, x), \mathcal{U}S_0(z, z)] = 4\sigma(x, z)\mathcal{U}S_0(x, z) + 4\tau(x, z)\mathcal{U}S_0(Jx, z),$$

we get the cartan condition for $\mathcal{U}S_0(x, y)$ of $\mathcal{U}S_0$, i.e., selfinvariance for $\mathcal{U}S_0$. Note that if we drop the Bianchi identity the Ricciform no longer is skew and the Riccitransformation no longer in $JA(\mathbf{E}, \sigma)$. The standard embedding then is a graded skew algebra without the graded Jacobi identity.

To show that (19) are the commutation relations of a pseudo-unitary algebra consider the $n \times n$ matrix $I = \operatorname{diag}(\mathrm{id}_p, -\mathrm{id}_q)$ and a basis in $\mathbf{E}$ in which σ has the matrix

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

$$J = \begin{pmatrix} 0 & -\mathrm{id}_n \\ \mathrm{id}_n & 0 \end{pmatrix}$$

is such a complex structure and the matrix of τ becomes diag (I, I). Then

$$M + iN \mapsto \begin{pmatrix} M & -N \\ N & M \end{pmatrix}$$

is an isomorphism of the pseudo-unitary Lie algebra $u(p, q)$ onto the Lie algebra $sp(\mathbf{E}, \sigma) \cap so(\mathbf{E}, \tau)$, which is the eigenspace with eigenvalue 1 of the Cartan decomposition of $sp(\mathbf{E}, \sigma)$ defined by J.

7. Remarks on classification and globalization

It remains to give the $Sp(\mathbf{E}, \sigma)$-orbits in $\mathbf{L}_2$ and Im $\mathcal{W}$. The classification of selfinvariant orbits leads to the classification of the symplectic generalized Bose triples. The structure theory of the generalized Bose triples should be started as that for Lie triples [12] by introducing the Ricciform and relating it to the graded Killing form [2] of its standard embedding. Then semi-simple triples should be defined as having a non-degenerate skew Ricciform and one expects that this is equivalent to the semi-simplicity of the standard embedding [2], [3]. Semi-simple symplectic triples give rise to semi-simple orbits in the curvature space. For a general generalized Bose triple one can try to define the radical and to prove then a Levi-Malcev decomposition into a semi-direct sum of the radical and a semi-simple subtriple. More general one should try to develop a second cohomology of triples with values in arbitrary triples of the same kind. To include the Lie triple case this should be tried for the graded generalizations of Lie triples described in [17].

A certainly more difficult problem is the globalization of the above. Is there a category of manifolds with an internal composition such that the tangent functor maps onto the category of generalized Bose triples? This corresponds to the analogous relation of the categories of symmetric spaces and Lie triples. A similar question is whether a symplectic manifold admits a canonical Levi-Civita-type connection which induces a section in the bundle of curvature spaces over each tangent space. And is there a combination of these two concepts as for pseudo-riemannian symmetric spaces by identifying left multiplication in the triple with canonical curvature structures. In this connection it is interesting to see that Bertram Kostant in a forthcoming work developpes a graded generalization of differential geometry and Lie theory, in which these concepts eventually are included as specializations to the odd substructures.

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