ILLINOIS JOURNAL OF MATHEMATICS
Volume 26, Number 2, Summer 1982

A BOUND ON THE RANK OF $\pi_q(S^n)$

BY
PAUL SELICK[1]

Let p be an odd prime. In [7], Toda constructs two fibrations which he uses to give a bound on the exponent of $\pi_q(S^n)$. We show here how these fibrations can be used to give a bound on the rank of $\pi_q(S^n)$. These groups are known to be finitely generated from the work of Serre [6]. It suffices to consider odd n since, again from Serre [5],

$$\pi_q(S^{2n})_{(p)} \cong \pi_{q-1}(S^{2n-1})_{(p)} \oplus \pi_q(S^{4n-1})_{(p)}.$$

Although analogues of Toda's fibrations exist for the prime 2, (James [2]), the arguments given here fail because unlike the situation for odd primes, $\pi_*(X; Z/2Z)$ fails to be a $Z/2Z$ vector space.

Let X be a compactly generated topological space with basepoint "*". Let $J_k(X)$ denote the kth stage of the James Construction on X. That is, $J_k(X) = X^k/\sim$ where

$$(x_1, \ldots, x_{j-1}, *, x_{j+1}, \ldots, x_k) \sim (x_1, \ldots, x_{j-1}, x_{j+1}, *, x_{j+2}, \ldots, x_k).$$

After localizing to p, there are fibrations up to homotopy:

$$J_{p-1}(S^{2n}) \xrightarrow{i} \Omega S^{2n+1} \xrightarrow{H} \Omega S^{2pn+1}$$

and

$$S^{2n-1} \xrightarrow{j} \Omega J_{p-1}(S^{2n}) \xrightarrow{T} \Omega S^{2pn-1}.$$

We use mod-p homotopy in order to be able to make use of vector space arguments. Of course,

$$\dim \left(\pi_q\left(S^{2n+1}; \frac{Z}{pZ}\right)\right) = \operatorname{rank}\,(\pi_q(S^{2n+1})) + \operatorname{rank}(\pi_{q-1}(S^{2n+1})) \quad \text{for } q > 2n+2.$$

Let x be a non-zero element of $\pi_q(S^{2n+1}; Z/pZ)$ with $n > 0$. Construct a non-zero element x', in some mod-p homotopy group of some sphere as follows:

If $H_\#(x) \neq 0$, let $x' = H_\#(x)$. If $H_\#(x) = 0$, select a non-zero element,

$$y \in \pi_{q-1}(J_{p-1}(S^{2n})) = \pi_{q-2}(\Omega J_{p-1}(S^{2n})),$$

Received May 27, 1980.

[1] Partially supported by a National Science Foundation grant.

such that $i_{\#}(y) = x$. If $T_{\#}(y) \neq 0$, let $x' = T_{\#}(y)$. If $T_{\#}(y) = 0$, select $x' \in \pi_{q-2}(S^{2n-1})$ such that $j_{\#}(x') = y$.

In this manner, construct a sequence $(x, x', \ldots, x^{(s)})$ in which each term is a non-zero element of some mod-p homotopy group of some sphere and is produced from the preceding term by the above procedure. Observe that throughout such a sequence, q never increases and must eventually decrease since for degree reasons one cannot continue indefinitely to get non-zero elements upon applying $H_{\#}$. Thus the sequence must eventually reach an element of $\pi_1(S^1; Z/pZ)$ at which point it terminates.

The sequence $(x, x', \ldots, x^{(s)})$ is referred to as "a sequence belonging to x". The sequence of groups to which the terms of the sequence belong, beginning with $\pi_q(S^{2n+1}; Z/pZ)$ and ending with $\pi_1(S^1; Z/pZ)$, is referred to as the "trace sequence". Note that neither the sequence nor the trace sequence is uniquely determined by x.

THEOREM 1. *There exists a basis $v_1, \ldots, v_r$ for $\pi_q(S^{2n+1}; Z/pZ)$ and sequences belonging to $v_1, \ldots, v_r$ having the property that for any subset of the v_i's which have the first k terms of their trace sequences identical, the kth terms of the sequences for these elements form a linearly independent set in that kth group.*

Proof. Select a basis $w_1, \ldots, w_r$ for $\pi_q(S^{2n+1}; Z/pZ)$ and select a sequence

$$(w_i, \ldots, w_i^{(r_i)})$$

for each w_i. Suppose by induction that the basis $w_1, \ldots, w_r$ and the chosen sequences have the desired property for all $k \leq t$.

Divide the w_i's into sets having the first t terms of their trace sequences identical. Renumbering, let $w_1, \ldots, w_m$ form such a set. Suppose that the non-zero elements of the set

$$\{H_{\#}(w_1^{(t)}), \ldots, H_{\#}(w_m^{(t)})\}$$

are linearly dependent. Renumbering, if necessary, there is a relation of the form

$$H_{\#}(w_1^{(t)}) = \sum_{j=2}^{m} \lambda_j H_{\#}(w_j^{(t)}).$$

Replace w_1 in the original basis by

$$\hat{w}_1 = w_1 - \sum_{j=2}^{m} \lambda_j w_j.$$

There is a natural choice for the first t terms of the sequence for $\hat{w}_1$ using the sequences for $w_1, \ldots, w_m$. Choose any completion of the sequence. By the induction hypothesis the first t terms of the trace sequence for $\hat{w}_1$ are identical with those of $w_2, \ldots, w_m$. However, by construction, the $(t+1)$st term is different. By continuing to change the original basis in this way, we may arrange for the non-zero elements of the set

$$\{H_\#(w_1^{(t)}), \ldots, H_\#(w_m^{(t)})\}$$

to be linearly independent.

Forming the $(t+1)$st terms of the sequences of those w_i's for which $H_\#(w_i) = 0$ involves taking inverse images under $i_\#$. Taking inverse images always preserves linear independence, so the resulting set is linearly independent. Proceeding as before, change the original basis so that the non-zero elements of the set of images under $T_\#$ are linearly independent. At this point, the remaining elements will have the $(t+1)$st terms of their sequences formed by taking inverse images under $j_\#$ which again preserves linear independence. This completes the induction step.

COROLLARY 2. *There is a basis for* $\pi_q(S^{2n+1}; Z/pZ)$ *whose elements have distinct trace sequences.*

Proof. Select a basis having sequences as in Theorem 1. Since all sequences terminate in $\pi_1(S^1; Z/pZ)$, which has dimension 1, the trace sequences must be distinct.

COROLLARY 3. dim $\pi_q(S^{2n+1}; Z/pZ)$ *is less than or equal to the number of possible trace sequences beginning with* $\pi_q(S^{2n+1}; Z/pZ)$.

The exact number of possible trace sequences beginning with $\pi_q(S^{2n+1}; Z/pZ)$ is not easy to compute and would not give the exact dimension of $\pi_q(S^{2n+1}; Z/pZ)$ in any event. However, it is easy to give an upper bound. Here is one method:

From the construction,

$$\text{number of possible trace sequences} \leq 3^{(\text{maximum length of a sequence})};$$

clearly, any trace sequence beginning with $\pi_q(S^{2n+1}; Z/pZ)$ has less than q^2 terms since no sequence can go through the same group twice and there are only q^2 groups under consideration. Thus dim $\pi_q(S^{2n+1}; Z/pZ) \leq 3^{q^2}$.

REFERENCES

1. I. JAMES, *Reduced product spaces*, Ann. of Math., vol. 62 (1955), pp. 170–197.
2. ———, *On the suspension sequence*, Ann. of Math., vol. 65 (1957), pp. 74–107.
3. F. PETERSON, Thesis, Princeton University, 1955.
4. P. SELICK, Thesis, Princeton University, 1977.
5. J.-P. SERRE, *Homologie singulière des espaces fibres. Applications*, Ann. of Math., vol. 54 (1951), pp. 425–505.
6. ———, *Groupes d'homotopie et classes de groupes abeliens*, Ann. of Math., vol. 58 (1953), pp. 258–294.
7. H. TODA, *On the double suspension E^2*, J. Inst. Polytech. Osaka City Univ., Ser. A, vol. 7 (1956), pp. 103–145.

UNIVERSITY OF WESTERN ONTARIO
LONDON, ONTARIO, CANADA