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POLYNOMIAL GROUP LAWS OVER VALUATION RINGS

BY

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Let A be a discrete valuation ring with fraction field K and residue field k. Let R be a finitely generated flat A-algebra, and suppose that $R \otimes K$ and $R \otimes k$ are polynomial rings. Must R be a polynomial ring? Proofs of this have been given only for one variable (Danilov, unpublished; Kambayashi-Miyanishi [5]) and for two variables if k is algebraically closed of characteristic zero (Kambayashi [4]). The situation is better, however, when R is the ring of functions $A[G]$ on an affine group scheme G. This was indeed the context in which Weisfeiler and Dolgachev [7] first raised the question, since, when char $(k) = 0$, the result for $A[G]$ is easily established by Lie theory. They were able to establish it also when char $(K) = p$ and k is perfect and the generic fiber G_K is G_a^n. The theorem was later proved [8] for all commutative G. In this paper it is proved for group schemes without restriction:

THEOREM. *Let G be a flat affine group scheme of finite type over a discrete valuation ring A. Assume the two fibers are represented by polynomial rings. Then $A[G]$ is a polynomial ring.*

As in [8] and [4], the proof is in outline an induction using Néron blow-ups. Some new results on the structure of polynomial groups over fields are needed for the argument and will be established first.

1. Review of Néron blow-ups

Let $G = \text{Spec } A[G]$ be a flat affine scheme of finite type over the discrete valuation ring A. Tensoring with the fraction field, we can by flatness identify $A[G]$ with a subalgebra of $K[G] = A[G] \otimes_A K$. Let X be a closed subscheme of the special fiber G_k, so X is defined by some ideal $J = (\pi, f_1, \ldots, f_n)$, where π is the uniformizer. The subalgebra

$$A[\pi^{-1}J] = A[G][\pi^{-1}f_1, \ldots, \pi^{-1}f_n]$$

represents a scheme G^X which one says is obtained by *blowing up* X in G. We will need the following properties, of which (b) is the crucial one (cf. [9, Theorem 1.4]).

(a) Let G' be any other such flat affine scheme. Any map $G' \to G$ sending G'_k into X factors through G^X.

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(b) Assume also that G' is of finite type and $G' \to G$ is an isomorphism on generic fibers. Let X be the smallest closed subscheme containing the image of G'_k. Then G' maps to G^X, and we can repeat the process. After finitely many such blow-up steps the map from G' will be an isomorphism.

(c) If G has a group scheme structure and X is a subgroup of G_k, then G^X is a group scheme.

(d) If in addition G has smooth connected fibers and X is a smooth connected subgroup, then $(G^X)_k$ maps onto X and the kernel is a vector group.

2. Polynomial groups over fields

The basic theorems are gathered together in [3, IV, Section 4]. We will need a few refinements, which are presented here. An affine group scheme U over a field k will be called *polynomial* if $k[U]$ is a (finitely generated) polynomial ring. This holds if U is smooth, connected, unipotent, and k-solvable. Any quotient of U inherits these properties and hence is again polynomial.

If U is a nontrivial polynomial group, it is known to contain a nontrivial central subgroup isomorphic to G_a^r for some r. Using this, we can prove a nonlinear version of the defining property of unipotence.

PROPOSITION. *Let V be isomorphic to G_a^s. Let G be a polynomial group acting as algebraic group automorphisms of V (not necessarily linear). Then there is a subgroup of V isomorphic to G_a on which G acts trivially.*

Proof. Suppose first $G \simeq G_a$. Let U be the semi-direct product of V and G, a polynomial group. It has then a central subgroup C isomorphic to G_a. If the projection of C to G is trivial, then C is $\subseteq V$ and has the desired property. If the projection is nontrivial, it is all of G. But C, being central, acts trivially on V by conjugation, so the G-action is trivial and the result is obvious.

Now in general let H be a central subgroup of G isomorphic to G_a. There are then subgroups of V isomorphic to G_a on which H acts trivially. The composite W of all such is a polynomial group isomorphic to some G_a^r. Since G centralizes H, it maps W to itself, and the action on W factors through the quotient G/H. This quotient is polynomial, and the proposition follows by induction on dim (G). ■

COROLLARY. *Let G be a polynomial group and F a nontrivial normal polynomial subgroup. Then F contains a subgroup isomorphic to G_a and central in G.*

Proof. Let V be the composite of all subgroups central in F and isomorphic to G_a. ■

DEFINITION. A system of coordinates $x_1, \ldots, x_n$ on a polynomial group is called *primitive* if each $x_r(gh) - x_r(g) - x_r(h)$ depends only on the first $r - 1$ coordinates of g and h, and the identity e is the origin.

COROLLARY. *Let G be a polynomial group and F a normal polynomial subgroup. Then there is a primitive coordinate system in which F is given by $x_1 = \cdots = x_r = 0$.*

Proof. Assuming F is nontrivial, take a $C \simeq G_a$ central in G and contained in F. (If F is trivial, take C central in G and replace F by C.) By induction we have primitive coordinates on G/C in which F/C is given by $x_1 = \cdots = x_r = 0$. The central extension $1 \to C \to G \to G/C \to 1$ is known to have a scheme-theoretic section, so we can write G as $(G/C) \times C$ with

$$(h, x) \cdot (h', x') = (hh', x + x' + f(h, h'))$$

for some cocycle f. The coordinates on G/C followed by the additive coordinate on C are a primitive system with the desired property. ∎

If $x_1, \ldots, x_n$ are primitive coordinates, the subgroup C defined by $x_1 = \cdots = x_{n-1} = 0$ is central, because for g in G and c in C we have

$$\begin{aligned} x_i(gc) - x_i(g) - x_i(c) &= x_i(ge) - x_i(g) - x_i(e) \\ &= 0 = x_i(eg) - x_i(e) - x_i(g) \\ &= x_i(cg) - x_i(c) - x_i(g). \end{aligned}$$

By induction, the subgroups $x_1 = \cdots = x_r = 0$ form a central series. In particular they are all normal, and thus an arbitrary polynomial subgroup E cannot be of that form. This is essentially why the proof in [8] required G to be commutative. The next section will show how to push through the proof in general using the following weaker result.

COROLLARY. *Let G be a polynomial group and E a proper polynomial subgroup. Then there are coordinate systems $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ on G such that $x_1, \ldots, x_n$ is primitive, $y_1 = x_1$, and E is defined by $y_1 = \cdots = y_r = 0$.*

Proof. We first need a basic result:

LEMMA. *There is a polynomial subgroup one dimension larger than E and normalizing E.*

Proof. Induction gives us a chain of normal polynomial subgroups

$$1 = C_0 < C_1 < \cdots < C_n = G$$

with $C_i/C_{i-1} \simeq G_a$ central in G/C_{i-1}. If C_{s-1} is the largest one contained in E, then C_s normalizes E. Hence $C_s E$ is a group and normalizes E. As a scheme it is a quotient of $C_s \times E$; this makes it connected and reduced over $\bar{k}$ (hence smooth) and k-split, so it is polynomial. Clearly $\dim C_s E = 1 + \dim E$. ∎

Now to prove the corollary, take a chain $E = E_r \lhd E_{r+1} \lhd \cdots \lhd E_n = G$ with E_i polynomial of dimension i. We have then extensions $1 \to E_i \to E_{i+1} \to G_a \to 1$ which we know must split as schemes. Start with coordinates on E and extend step by step, taking a coordinate system $y_{n-i+1}, \ldots, y_n$ on E_i and extending it to a system $y_{n-i}, \ldots, y_n$ on E_{i+1} with y_{n-i} the projection of E_{i+1} to G_a. These will not in general be primitive coordinates on E_{i+1}. But at the last stage E_{n-1} is normal in G, and the previous corollary gives primitive coordinates $x_1, \ldots, x_n$ with $x_1 = 0$ defining E_{n-1}. Now, x_1 and y_1 are homomorphisms of G onto G_a with the same kernel, so they are constant multiples of each other, and we can change x_1 to equal y_1. ■

3. Proof of the theorem

Let us say for the moment that a group scheme $H = \operatorname{Spec} R$ over A *has good coordinates* if R can be written as a polynomial ring $A[X_1, \ldots, X_n]$ where the X_i are in the augmentation ideal and reduce to primitive coordinates on the special fiber H_k.

Let H be a group with good coordinates, and let E be a proper polynomial subgroup of H_k. Choose coordinates $y_1, \ldots, y_n$ on H_k so that E is given by $y_1 = \cdots = y_r = 0$ and y_1 is additive. Let F be $\ker(y_1)$, and let $y_1 = x_1, x_2, \ldots, x_n$ be a primitive coordinate system. By [8, Theorem 1], a change from one primitive coordinate system to another arises by a sequence of changes, each of which either multiplies some x_i by a constant or adds to x_i some polynomial in the other variables. All such changes obviously lift to changes of variable over A. Thus we may assume that the coordinates $X_1, \ldots, X_n$ reduce to $x_1, \ldots, x_n$. Then

$$A[H^F] = A[H][\pi^{-1}X_1] = A[X_2, \ldots, X_n, \pi^{-1}X_1].$$

Simple computation as in [8, Theorem 2] shows that $X_2, \ldots, X_n, \pi^{-1}X_1$ (in this order) are good coordinates on H^F.

Now $A[H^E] = A[G][\pi^{-1}X_1, \pi^{-1}Y_2, \ldots, \pi^{-2}Y_r]$, where the $Y_i(X)$ are polynomials reducing to y_i. The kernel of the map $(H^E)_k \to (H^F)_k$ has ring of functions

$$A[H][\pi^{-1}X_1, \pi^{-1}Y_2, \ldots, \pi^{-1}Y_r]/(\pi, X_2, \ldots, X_n, \pi^{-1}X_1).$$

This is generated by the images of $\pi^{-1}Y_2, \ldots, \pi^{-1}Y_r$, and thus the kernel has dimension at most $r - 1$. By flatness $(H^E)_k$ has dimension n, so its image in $(H^F)_k$ has dimension at least $n - (r - 1) = 1 + \dim(E)$. Since $(H^E)_k$ is an extension of E by a vector group, it is polynomial, and hence its image D in $(H^F)_k$ is polynomial. Clearly $H^E = (H^F)^D$, as we have natural maps both ways. We can now replace H and E by H^F and D and iterate the construction. Since the image dimension is increasing, we get a surjection on the special fibers after $\dim(H) - \dim(E)$ steps. By [9, 1.3], the map from H^E is then an isomorphism. Since there are good coordinates at each stage, we have shown that H^E has good coordinates.

Now take any G as in the theorem. Choose primitive coordinates $X_1, \ldots, X_n$ on the generic fiber G_K. By proper scaling of them as in [8, Theorem 3] we may assume that they are in $A[G]$, and

$$\Delta X_i \equiv X_i \otimes 1 + 1 \otimes X_i \bmod \pi A[X_1, \ldots, X_n].$$

In particular they are good coordinates on $H = \operatorname{Spec} A[X_1, \ldots, X_n]$. There is an obvious map $G \to H$, and the image E of G_k is a polynomial subgroup of H_k. Then H^E again has good coordinates, and G maps to H^E. After finitely many such steps we reach G itself. Thus G has good coordinates, and in particular $A[G]$ is a polynomial ring. ■

In the theorem we do not really need to assume G affine, since that follows from the other hypotheses [1]. We can also easily extend to the number-theoretic case:

COROLLARY. *Let B be a Dedekind domain with generic characteristic zero and perfect residue fields. Let G be an affine group scheme of finite type over B. If G is smooth with unipotent connected fibers, then $B[G]$ is the symmetric algebra of a projective B-module (and conversely).*

Proof. If B is a valuation ring, this follows from the theorem, since smooth connected unipotent groups over perfect fields are automatically k-solvable and hence polynomial. The globalization then is a general result of commutative algebra [2, 6]. ■

COROLLARY. *A smooth affine group of finite type over $\mathbf{Z}$ with connected unipotent fibers can be nothing but a group law on affine n-space.* ■

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