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INVARIANTLY COMPLEMENTED SUBSPACES OF $L_\infty(G)$ AND AMENABLE LOCALLY COMPACT GROUPS

BY

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1. Introduction

Let G be a locally compact group and let $L_\infty(G)$ be the W^*-algebra of essentially bounded measurable complex-valued functions on G with pointwise operations and essential sup norm. Let X be a weak*-closed left translation invariant subspace of $L_\infty(G)$. Then X is *invariantly complemented* in $L_\infty(G)$ if X admits a left translation invariant closed complement, or equivalently, X is the range of a continuous projection on $L_\infty(G)$ commuting with left translations.

In this paper, we are concerned with the following question: Under what condition will X be invariantly complemented?

H. Rosenthal proved in [14, Theorem 1.1] that if G is abelian and X is complemented in $L_\infty(G)$, then X is invariantly complemented in $L_\infty(G)$. Actually, Rosenthal's proof is valid for any locally compact group G which is amenable as discrete. However, we do not know whether the same conclusion still holds when G is an amenable locally compact group but G is not amenable as discrete (e.g. when G is the compact orthogonal group $SO(3, \mathbf{R})$ [8, p. 9]).

We prove in Section 3 (Theorem 3.3) that G is amenable if and only if each left translation invariant W^*-subalgebra of $L_\infty(G)$ is invariantly complemented. Proof of this theorem depends on an improvement of some recent results of P. K. Pathak and H. S. Shapiro [13] and G. Crombez and W. Govaerts [3] in associating translation invariant W^*-subalgebras of $L_\infty(G)$ with the set of functions in $L_\infty(G)$ fixed under translations by elements in a closed subgroup of G (see Lemma 3.2).[2] We also prove in Section 4 (Corollary 4.2) that G is amenable if and only if each weak*-closed weak*-complemented left translation invariant subspace of $L_\infty(G)$ is invariantly complemented. This is equivalent to Corollary 4.4: Each weak*-closed left translation invariant subspace of $UBC_r(G)$ which is complemented in $UBC_r(G)$ admits a left translation invariant closed complement in $UBC_r(G)$. Here $UBC_r(G)$ denotes the space of bounded right uniformly continuous complex-valued functions on G [9, p. 275]. Both Corollary 4.2 and Corollary 4.4 are direct consequences of characterizations of amenability of G

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in terms of certain invariant complemented subspace property on dual Banach spaces. Finally, in Section 5, we give conditions when a weak*-closed left translation invariant subspace in $L_\infty(G)$ of a compact group G is the range of a weak*-weak* continuous projection on $L_\infty(G)$ commuting with left translations.

2. Preliminaries

If E is a Banach space, then E^* denotes its continuous dual. Also if $\phi \in E^*$ and $x \in E$, then the value of ϕ at x will be written as $\phi(x)$ or $\langle \phi, x \rangle$. If F is another Banach space, then $\mathscr{B}(E, F)$ will denote the space of bounded continuous linear operators from E into F.

Throughout this paper, G denotes a locally compact group with a fixed left Haar measure. Let $C(G)$ denote the Banach algebra of bounded continuous complex-valued functions on G with the supremum norm, and let $C_0(G)$ be the closed subspace of $C(G)$ consisting of all functions in $C(G)$ which vanishes at infinity. The Banach spaces $L_p(G)$, $1 \leq p \leq \infty$, are as defined in [9]. If f is a complex-valued function defined locally almost everywhere on G, and if a, $t \in G$, then $l_a f(t) = f(a^{-1}t)$ and $r_a f(t) = f(ta)$ whenever this is defined. We say G is *amenable* if there exists $m \in L_\infty(G)^*$ such that $m \geq 0$, $\|m\| = 1$ and $m(l_a f) = m(f)$ for each $f \in L_\infty(G)$ and $a \in G$. Amenable locally compact groups include all compact groups and all solvable groups. However, the free group on two generators is not amenable.

We will need the following simple observation:

LEMMA 2.1. *Let X be a weak*-closed left translation invariant subspace of $L_\infty(G)$. Then $X \cap UBC_r(G)$ is weak*-dense in X.*

Proof. Let $\{\phi_\alpha\}$ be a bounded approximate identity in $L_1(G)$ and $f \in X$. Then the net $\{\phi_\alpha * f\}$ is in $X \cap UBC_r(G)$ (see [9, p. 295] and [12, Theorem 4.1]) and converges in the weak*-topology to f.

3. W^*-subalgebras of $L_\infty(G)$

A W^*-subalgebra X of $L_\infty(G)$ is a weak*-closed subalgebra of $L_\infty(G)$ such that $\bar{f} \in X$ whenever $f \in X$, where $\bar{f}(x) = \overline{f(x)}$, $x \in G$. In this section, we characterize the class of left translation invariant W^*-subalgebras of $L_\infty(G)$. This result is applied to show that if G is amenable, then any left translation invariant W^*-subalgebra of $L_\infty(G)$ is invariantly complemented.

LEMMA 3.1. *Let X be a left translation invariant W^*-subalgebra of $L_\infty(G)$ and $X \neq \{0\}$, then X contains constants.*

Proof. Indeed let h_0 be the identity of X (see [16, Proposition 1.6.1]). If $g \in G$, $f \in X$, we have $(l_g h_0)(f) = l_g[h_0(l_g^{-1} f)] = f$. Hence $l_g h_0 = h_0$. Let $\phi \in L_1(G)$ such that $\phi \geq 0$ and $\|\phi\| = 1$. Then $\phi * h_0 = h_0$ by Theorem 4.1 (b) [12]. In particular $h_0 \in UBC_r(G)$. Consequently $h_0 = 1$.

LEMMA 3.2. *X is a non-zero left translation invariant W^*-subalgebra of $L_\infty(G)$ if and only if there exists a unique closed subgroup N_X of G such that*

$$X = \{f \in L_\infty(G);\ r_g f = f \text{ for each } g \in N_X\}$$

Furthermore, X is translation invariant if and only if N_X is normal.

Proof. If N is a closed subgroup of G, it is easy to see that

$$\{f \in L_\infty(G),\ r_g f = f \text{ for each } g \in N\}$$

is a non-zero left translation invariant W^*-subalgebra of $L_\infty(G)$.

Conversely, if X is a left translation invariant W^*-subalgebra of $L_\infty(G)$, let

$$N = N_X = \{g \in G;\ r_g f = f \text{ for each } f \in X\}.$$

Then, as readily checked, N is a subgroup of G. Also, since the map

$$G \to (L_\infty(G), \text{weak*})$$

defined by $g \to r_g f$ ($f \in L_\infty(G)$) is continuous and X is weak*-closed, N is closed in G. Let

$$Y = \{f \in L_\infty(G);\ r_g f = f \text{ for each } g \in N\};$$

clearly $Y \supseteq X$. To prove equality, it suffices, by Lemma 2.1, to show that $Y \cap C(G) \subseteq X$.

Let G/N denote the homogeneous space of left cosets $\{gN;\ g \in G\}$. Then G/N is locally compact and Hausdorff [9, p. 38]. Also, each $f \in Y \cap C(G)$ can be regarded as a continuous function $\bar{f}$ on G/N. Let K be a compact subset of G, and $\bar{K}$ be its image in G/N under the quotient map. Then an application of Lemmas 2.1 and 3.1 shows that $\mathscr{A} = \{q(\bar{f});\ f \in X \cap C(G)\}$, where $q(\bar{f})$ is the restriction of $\bar{f}$ to $\bar{K}$, separate points on K, closed under conjugation and contains constants. So, by the Stone-Weierstrass Theorem, $\mathscr{A}$ is uniformly dense in $C(\bar{K})$.

Let $f_0 \in Y \cap C(G)$ be fixed. For any finite subset $\sigma = \{\psi_1, \dots, \psi_n\} \subseteq C_{00}(G)$ (continuous functions with compact support) with $\|\psi_i\| \le 1$, let K be a compact subset of G such that $\psi_i(x) = 0$ if $x \notin K$, $i = 1, \dots, n$. Then for each $0 < \varepsilon < 1$, there exists (by density of $\mathscr{A}$ in $C(\bar{K})$) $h \in X \cap C(G)$ such that

$$\|f_0 - h\|_K = \sup\{|(f_0 - h)(x)|;\ x \in K\} < \varepsilon.$$

In particular, $\|h\|_K \le \|f_0\|_\infty + 1$. Let $\phi\colon C \to C$ be a continuous function such that $\phi(z) = z$ if $|z| \le \|f_0\|_\infty + 1$ and $|\phi(z)| \le \|f_0\|_\infty + 1$ if $|z| > \|f_0\|_\infty + 1$. Let $k_{(\sigma,\varepsilon)} = k = \phi \circ h \in X$. Then $k \in X$ (by [16, Proposition 1.18.1]), $\|k\| \le \|f_0\|_\infty + 1$ and $\|f_0 - k\|_K < \varepsilon$. Hence

$$\left|\int (f_0 - k)(t)\psi_i(t)dt\right| < \varepsilon$$

for each $i = 1, \ldots, n$. Consequently the bounded net $\{k_{(\sigma,\varepsilon)}\}$ in X converges to f_0 in the topology $\mathscr{T}$ determined by the seminorms

$$\{p_\psi;\ \psi \in C_{00}(G)\},\ \|\psi\|_1 \leq 1\},\quad \text{where } p_\psi(f) = \left|\int f(t)\psi(t)dt\right|, f \in L_\infty(G).$$

Since the topology $\mathscr{T}$ and the weak*-topology coincide on bounded spheres of $L_\infty(G)$, $f_0 \in X$.

Suppose N_0 is another closed subgroup of G such that

$$X = \{f \in L_\infty(G);\ l_g f = f \text{ for each } g \in N_0\};$$

then $N_0 \subseteq N$. If $g \in N$ and $g \notin N_0$, there exists an $f \in C(G)$, f is constant on the left cosets $\{tN_0;\ t \in G\}$ and $f(g) \neq f(e)$. Hence $f \in X$, but $r_g f \neq f$, which is impossible. Hence $N_0 = N$.

Finally, if X is translation invariant, $g \in G$ and $a \in N$, then

$$r_{g^{-1}ag}(f) = r_{g^{-1}} r_a(r_g f) = r_{g^{-1}} r_g f = f$$

since $r_g f \in X$. Hence N is normal. Conversely, if N is normal $f \in X$ and $g \in G$, then for each $a \in N$, $r_a(r_g f) = r_{ga} f = r_{bg} f = r_g f$ where $b = gag^{-1} \in N$. In particular $r_g f \in X$.

Remark. Note that if X is translation invariant, then

$$N_X = \{a \in G;\ r_a f = l_a f = f \text{ for all } f \in X\}$$

In particular, Lemma 3.2 implies the main theorem in [3]. Indeed, let $a \in N_X$. If $f \in X \cap C(G)$, then f is constant on the coset $aN_X = N_X a$ (by normality) and $l_a f = f$. Consequently $l_a f = f$ for each $f \in X$ by Lemma 2.1.

THEOREM 3.3. *G is amenable if and only if every left translation invariant W^*-subalgebra of $L^\infty(G)$ is invariantly complemented.*

Proof. Let X be a left translation invariant W^*-subalgebra of $L^\infty(G)$. We may assume that $X \neq \{0\}$. By Lemma 3.2, there exists a closed subgroup N of G such that

$$X = \{f \in L^\infty(G);\ r_g f = f \text{ for all } g \in N\}.$$

If G is amenable, then N is also amenable. For each $f \in L^\infty(G)$, let K_f denote the weak*-closure of the convex hull of $\{r_g f;\ g \in N\}$. Then the affine action $N \times K_f \to K_f$, defined by $(g, k) \to r_g(k)$ is separately (hence jointly) continuous when K_f has the weak*-topology. By the Day's fixed point theorem [4, Theorem 3], there exists $k \in K_f$; $r_g k = k$ for each $g \in N$. Consequently $K_f \cap X \neq \emptyset$. Now applying the proposition of Yeadon [21] (see also [11]), there exists a continuous projection P from $L_\infty(G)$ onto X such that P commutes with any weak*-weak* continuous linear operators from $L_\infty(G)$ into $L_\infty(G)$ which commutes with the right translations $\{r_g;\ g \in H\}$. In particular, P commutes with the left translations on $L_\infty(G)$.

To prove the converse, we consider the subalgebra X of $L_\infty(G)$ consisting of constant functions. If $P\colon L_\infty(G) \to X$ is continuous projection of $L_\infty(G)$ onto X commuting with left translations, define $\phi(f) = \lambda$ if $P(f) = \lambda \cdot 1$. Then ϕ is a non-zero left translation invariant linear functional on $L_\infty(G)$. Let $\psi = \frac{1}{2}(\phi + \phi^*)$, where $\phi^*(f) = \overline{\phi(\bar{f})}$, $f \in L_\infty(G)$; then ψ is non-zero (since $\psi(1) = 1$), self-adjoint, and left translation invariant. Write $\psi = \psi^+ - \psi^-$, its unique decomposition as difference of two positive linear functionals ψ^+, ψ^- such that $\|\psi\| = \|\psi^+\| + \|\psi^-\|$ (see [16, Theorem 1.14.3]). Then, as readily checked, for each $a \in G$, $l_a^*\psi^+$ and $l_a^*\psi^-$ are positive and have the same norm as ψ^+ and ψ^- respectively. Hence $l_a^*\psi^+ = \psi^+$ and $l_a^*\psi^- = \psi^-$. Consequently, if $\psi^+ \neq 0$ (say), then $m = \psi^+/\psi^+(1)$ is a left invariant mean on $L_\infty(G)$.

4. Weak* invariant complemented subspace property

Let E be a dual Banach space with a fixed predual E_*. We say that E has the *weak* G-invariant complemented subspace property* if the following condition holds: Whenever $\mathscr{G} = \{T_g : g \in G\}$ is a representation of G as linear isometries from E onto E such that the map $(g, x) \to T_g x$ is a separately continuous linear map from $G \times E$ into E when E has the weak* topology, if X is a weak*-closed invariant subspace of E and if there exists a weak*-weak* continuous projection Q from E onto X with $\|Q\| \leq \alpha$, then there exists a continuous projection P from E onto X such that $T_g P = P T_g$ for each $g \in G$ with $\|P\| \leq \alpha$.

THEOREM 4.1. *G is amenable if and only if any dual Banach space has the weak* G-invariant complemented subspace property.*

Proof. If any dual Banach space has the weak* G-invariant complemented subspace property, we consider the representation $\mathscr{G} = \{l_g;\ g \in G\}$ of G on $L_\infty(G)$ and the one-dimensional subspace consisting of constant functions, then an argument similar to that for Theorem 3.3 shows that G is amenable.

Conversely, if G is amenable, $\mathscr{G} = \{T_g;\ g \in G\}$ is a representation of G as linear isometries from a dual Banach space E such that $(g, x) \to T_g(x)$ is a separately continuous linear map when E has the weak*-topology, let X be a weak*-closed invariant subspace of E and Q be a weak*-weak* continuous projection of E onto X such that $\|Q\| \leq \alpha$. Following an idea of Rosenthal in [14, Lemma 3.1], let $\mathscr{P}$ denote the set of continuous projections P from E onto X such that $\|P\| \leq \alpha$. Then $\mathscr{P}$ is a non-empty subset of $\mathscr{B}(E, X)$. Let τ denote the weak*-operator topology on $\mathscr{B}(E, X)$ determined by the family of seminorms

$$\{p_{z,\phi};\ z \in E,\ \phi \in E_*\}, \quad \text{where } p_{z,\phi}(T) = |\phi(Tz)|,\ T \in \mathscr{B}(E, X).$$

An application of the Theorem in [10, p. 973] shows that $(\mathscr{P}, \tau)$ is also compact. Consider now the affine action $G \times (\mathscr{P}, \tau) \to (\mathscr{P}, \tau)$ defined by the map

$$(g, P) \to T_{g^{-1}} P T_g, \quad g \in G,\ P \in \mathscr{P}.$$

Clearly, if $g \in G$, that map $P \to T_{g^{-1}}PT_g$ from $(\mathscr{P}, \tau)$ into $(\mathscr{P}, \tau)$ is continuous. Also, the map $g \to T_{g^{-1}}PT_g$ from G into $(\mathscr{P}, \tau)$ is continuous at $P = Q$. Indeed, by assumption, Φ: $(g, x) \to T_g x$ is a separately continuous map from $G \times B$ into B when B has the weak*-topology and $B = \{x \in E;\ \|x\| \leq \alpha\}$. Also, since $(B, \text{weak*})$ is compact, Ellis' result [6, Theorem 1] implies that Φ is even jointly continuous. Hence if $\{g_n\}$ is a net in G, $g_n \to g$, $g \in G$, $x \in E$, $\|x\| \leq 1$, then $Q(T_{g_n}(x))$ is a net in B converging to $Q(T_g(x))$ in the weak*-topology. In particular, the net $T_{g_n^{-1}}QT_{g_n}(x)$ converges to $T_{g^{-1}}QT_g(x)$ in the weak* topology of E also. By Day's fixed point theorem [5, Theorem 4], there exists $P \in \mathscr{P}$ such that $T_{g^{-1}}PT_g = P$ for each $g \in G$.

Remark. Theorem 4.1 implies Lemma 3.1 in [14].

A weak*-closed subspace X of $L_\infty(G)$ is weak*-complemented if there exists a weak*-weak* continuous projection from $L_\infty(G)$ onto X.

COROLLARY 4.2. *G is amenable if and only if any weak*-complemented left translation invariant weak*-closed subspace of $L_\infty(G)$ is invariantly complemented.*

We do not know if a weak*-closed left translation invariant complemented subspace of $L_\infty(G)$ of an amenable group G is necessarily invariantly complemented unless G is abelian (or more generally when G is amenable as discrete). However, we have the following:

THEOREM 4.3. *G is amenable if and only if G has the following property:*
(C) *Whenever $\mathscr{G} = \{T_g;\ g \in G\}$ is a representation of G as weak*-weak* continuous linear isometries from E onto E and A is a closed invariant subspace of E such that the maps $(g, x) \to T_g x$ is a continuous linear map of the product subspace $G \times A$ into A, if X is a weak*-closed invariant subspace of E contained in A and if there exists a continuous projection Q from A onto X with $\|Q\| \leq \alpha$, then there exists a continuous projection P from A onto X such that $T_g P = PT_g$ for each $g \in G$* with $\|P\| \leq \alpha$.

Proof. If property (C) holds, consider the representation $\mathscr{G} = \{l_g;\ g \in G\}$ of G on $L_\infty(G)$ and the closed subspaces $A = UBC_r(G)$, $X = \{\lambda 1;\ \lambda \in C\}$. Then A and X satisfies the conditions of (C) (see [9, p. 275]). Let Q be the continuous projection from A onto X defined by $Q(f) = f(e)1$, where e is the identity of G. Then $\|Q\| = 1$. Hence there exists a continuous projection P from A onto X commuting with each l_g; $g \in G$ and $\|P\| \leq 1$. Define $m(f) = P(f)(e)$ for each $f \in A$. Then $m(1) = \|m\| = 1$ and $m(l_g f) = m(f)$ for each $g \in G$, $f \in A$. Hence G is amenable by [8, Theorem 2.2.1].

Proof of the converse is very similar to that for Theorem 4.1, observing that the affine action $G \times (\mathscr{P}, \tau) \to (\mathscr{P}, \tau)$ is even continuous in this case when $G \times (\mathscr{P}, \tau)$ has the product topology by continuity of the map $G \times A \to A$ and Ellis' Theorem [6, Theorem 1]. We safely suppress the details.

Remark. Theorem 4.4 implies Theorem 1.1 and Lemma 3.1 in [14].

COROLLARY 4.4. *G is amenable if and only if every weak*-closed left translation subspace of $L_\infty(G)$ which is contained and complemented in $UBC_r(G)$ admits a left translation invariant complement in $UBC_r(G)$.*

Remarks (1) Let G be a (discrete) semigroup. An argument similar to that of Theorem 3.3 shows that if G has the weak* G-invariant complemented subspace property, then G is left amenable. However, we do not know whether a left amenable semigroup G has the weak* G-invariant complemented subspace property even when G is commutative. Our proof of Theorem 4.1 (and Theorem 3.3) depends heavily on the fact that G is a group.

(2) It follows from Theorem 9 in [17] and Theorem 3 in [18] of Silverman and their proofs that if G is a left amenable semigroup, then G has a certain montone projection property (MPP):

(MPP) Let $\mathcal{G} = \{T_g; g \in G\}$ be an antirepresentation of G as linear operators on an ordered linear space Y with an invariant cone C. Let V be a vector subspace of Y such that the induced cone in V is sharp, V considered as an order linear space is a boundedly complete vector lattice (see [19, p. 75]), $y + V \cap C \neq \emptyset$ for all $y \in V$ and $T_g(v) = v$ for all $g \in G, v \in V$. Then there exists a montone projection P from Y onto V such that $PT_g = P$ for all $g \in G$.

Conversely if G has the (MPP), then G is left amenable. In fact, let Y be the space of bounded real valued functions on G, $V = \{\alpha 1; \alpha \in \mathbf{R}\}$, and $(T_g f)(x) = f(gx), f \in Y, g, x \in G$. If P is a montone projection from Y onto V such that $PT_g = P$ for $g \in G$, let $a \in G$ be fixed, then $m \in Y^*$ defined by $m(f) = (Pf)(a), f \in Y$, is a left invariant mean on Y.

(We thank the referee for bringing our attention to the work of Silverman).

5. Compact groups

A Banach space E is said to have the *G-invariant complemented subspace property* if: Whenever $\mathcal{G} = \{T_g; g \in G\}$ is a representation of G as linear isometries from E onto E such that the map $(g, x) \to T_g x$ is a continuous map of the product space $G \times E$ into E, if X is a closed invariant subspace of E and if there exists a continuous projection Q from E onto X with $\|Q\| \leq \alpha$, then there exists a continuous projection P from E onto X commuting with each T_g, $g \in G$, with $\|P\| \leq \alpha$.

Rudin [15, Theorem 1] proved that if G is compact, then any Banach space has that G-invariant complemented subspace property. The following observation shows that the converse is also true:

PROPOSITION 5.1. *G is compact if and only if every Banach space has the G-invariant complemented subspace property.*

Proof. Consider the representation $\{l_g;\, g \in G\}$ of G as linear isometries on $L_1(G)$. If $L_1(G)$ has the G-invariant complemented subspace property, let

$$X = \left\{ f \in L_1(G);\ \int f(x)dx = 0 \right\}.$$

Then X is a closed invariant subspace of $L_1(G)$ with codimension 1, hence complemented. By assumption, there exists a continuous projection P from $L_1(G)$ onto X such that $l_{g^{-1}} P l_g = P$ for all $g \in G$. Then $K = \{f \in L_1(G);\ P(f) = 0\}$ is also invariant and one-dimensional. Pick $f \in K$ and $f \neq 0$. If $a \in G$, then $l_a f = \lambda f$ for some $\lambda \in C$. Hence

$$\lambda \int f(x)dx = \int l_a f(x)dx = \int f(x)dx.$$

Since $\int f(x)dx \neq 0$, $\lambda = 1$. In particular $l_a f = f$ for all $a \in G$. Hence G is compact.

LEMMA 5.2. *Let X be a weak*-closed invariantly complemented subspace of $L_\infty(G)$. Then there exists a weak*-weak* continuous projection from $L_\infty(G)$ onto X commuting with left translations if and only if $X \cap C_0(G)$ is weak*-dense in X.*

Proof. Let P be a continuous projection from $L_\infty(G)$ onto X commuting with left translations. If $f \in UBC_r(G)$, then an argument similar to that of de Leeuw in the proof of Theorem 4.1 [7] shows that $P(f) \in UBC_r(G)$. Let $\mu \in M(G)$ such that

$$P(f)(e) = \int f(x)d\mu(x).$$

Then, as readily checked,

$$P(f) = \mu_l(f), f \in C_0(G), \quad \text{where } \mu_l(f)(g) = \int f(gx)d\mu(x).$$

Define $S\colon L_1(G) \to L_1(G)$ by $S(h) = h * \mu$, $h \in L_1(G)$. Hence $Q = S^*$ commutes with left translations on $L_\infty(G)$ also. Furthermore $Q(f) = f$ for each $f \in C_0(G) \cap X$. Now if $C_0(G) \cap X$ is weak*-dense in X, then $Q(f) = f$ for each $f \in X$ by weak*-continuity of Q. Also if $f \in C_0(G)$ and $h \in X^\perp$, $h \in L_1(G)$, then $\langle Q(f), h\rangle = \langle h, \mu_l(f)\rangle = 0$. Consequently $\langle Q(f), h\rangle = 0$ for each $f \in L_\infty(G)$ by weak* density of $C_0(G)$ in $L_\infty(G)$; i.e., $Q(f) \in X^{\perp\perp} = X$. Hence Q is a projection of $L_\infty(G)$ onto X.

Conversely if Q is a weak*-weak* continuous projection from $L_\infty(G)$ onto X commuting with left translation, by Wendel's result [17, Theorem 1], there exists $\mu \in M(G)$ such that $Q^*(h) = h * \mu$, $h \in L_1(G)$. If $f \in C_0(G)$, then $Q(f) = \mu_l(f)$, which is also in $C_0(G)$. Since $C_0(G)$ is weak*-dense in $L_\infty(G)$, it follows that $C_0(G) \cap X$ is weak*-dense in $X = \{Q(f); f \in L_\infty(G)\}$.

Rosenthal [14, p. 19] proved that if G is a compact abelian group, then any translation invariant weak*-closed complemented subspace of $L_\infty(G)$ is the range of a weak*-weak* continuous projection commuting with translations. Our next result is an improvement of this fact.

THEOREM 5.3. *Let G be a compact group. Let X be a weak*-closed left translation invariant subspace of $L_\infty(G)$. Then there exists a weak*-weak* continuous projection from $L_\infty(G)$ onto X provided any one of the following conditions hold:*

(i) *X is also closed under multiplication and conjugation.*
(ii) *X is weak*-complemented.*
(iii) *X is complemented and G amenable as discrete.*

Proof. Using Lemmas 2.1 and 5.2, each of the three cases follow directly from Theorem 3.3, Corollary 4.2 and Theorem 4.3 respectively.

Remark. It follows from Lemma 5.2 and [1, Theorem 1] that if G abelian and $\hat{G}$ is connected, then there exists no non-trivial weak*-weak* continuous projection from $L_\infty(G)$ into $L_\infty(G)$ commuting with translations. This result also follows easily from Cohen's theorem on idempotent measure [2, Theorem 3] and Wendel's theorem [20, Theorem 1].

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