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NORMAL N.E.C. SIGNATURES

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1. Introduction

By a non-Euclidean crystallographic (N.E.C.) group, we shall mean a discrete subgroup Γ of isometries of the non-Euclidean plane with compact quotient space, including those reverse orientation, reflections and glide-reflections.

Let G denote the group of isometries of the upper-half plane D and let G^+ denote the subgroup of index 2 in G consisting of the conformal homeomorphisms. If Γ is an N.E.C. group we let $R(\Gamma, G)$ denote the set of isomorphisms $r: \Gamma \to G$ with the property that $r(\Gamma)$ is discrete and $D/r(\Gamma)$ is compact. $r_1, r_2 \in R(\Gamma, G)$ are called equivalent if for all $\gamma \in \Gamma$, $r_1(\gamma) = gr_2(\gamma)g^{-1}$ for some $g \in G$. The quotient space is denoted by $T(\Gamma, G)$, the Teichmüller space of Γ. It is homeomorphic to a cell of dimension $d(\Gamma)$. If Γ is a Fuchsian group with signature $(g; +; [m_1 \cdots m_\imath])$ then $d(\Gamma) = 6g - 6 + 2\imath$. Singerman [3] states that if Γ is a proper N.E.C. group, then $d(\Gamma) = \frac{1}{2}d(\Gamma^+)$.

Macbeath and Singerman [2] have proved that Mod (Γ) fails to be effective in its action on $T(\Gamma, G)$ if and only if there is an N.E.C. group Γ_1 with $\dim \Gamma_1 = \dim \Gamma$ and an embedding $\alpha: \Gamma \lhd \Gamma_1$.

We shall find here the signatures of all N.E.C. groups Γ such that Mod (Γ) fails to be effective in its action on $T(\Gamma, G)$. For this to be done we shall define in (Section 3) the concept of normal signature. The computation of these normal signatures on N.E.C. groups allows us to solve the problem (Section 4).

The corresponding problem for Fuchsian groups, was essentially solved in the work of Singerman [4].

2. N.E.C. signatures

N.E.C. groups are classified according to their signature. The signature of an N.E.C. group Γ is either of the form

$$(*) \qquad (g;\ +;\ [m_1 \cdots m_\imath];\ \{(n_{11} \cdots n_{1s_1}) \cdots (n_{k1} \cdots n_{ks_k})\})$$

or

$$(**) \qquad (g;\ -;\ [m_1 \cdots m_\imath];\ \{(n_{11} \cdots n_{1s_1}) \cdots (n_{k1} \cdots n_{ks_k})\})$$

The numbers m_i are the periods and the brackets $(n_{i1} \cdots n_{is_i})$ the period cycles.

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The group Γ with signature $(*)$ now has the presentation given by generators

$$\begin{array}{ll} x_i & i = 1, \ldots, \imath \\ e_i & i = 1, \ldots, k \\ c_{ij} & i = 1, \ldots, k, \quad j = 0, \ldots, s_i \\ a_j, b_j & j = 1, \ldots, g, \end{array}$$

and relations

$$x_i^{m_i} = 1, \quad i = 1, \ldots, \imath,$$

$$c_{is_i} = e_i^{-1} c_{i0} e_i, \quad i = 1, \ldots, k,$$

$$c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1} \cdot c_{ij})^{n_{ij}} = 1, \quad i = 1, \ldots, k, \quad j = 1, \ldots, s_i$$

$$x_1 \cdots x_\imath e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1$$

A group Γ with signature $(**)$ now has the presentation given by generators

$$\begin{array}{ll} x_i & i = 1, \ldots, \imath \\ e_i & i = 1, \ldots, k \\ c_{ij} & i = 1, \ldots, k, \quad j = 0, \ldots, s_i \\ d_j & j = 1, \ldots, g, \end{array}$$

and relations

$$x_i^{m_i} = 1, \quad i = 1, \ldots, \imath,$$

$$c_{is_i} = e_i^{-1} c_{i0} e_i, \quad i = 1, \ldots, k,$$

$$c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1} \cdot c_{ij})^{n_{ij}} = 1, \quad i = 1, \ldots, k, \quad j = 1, \ldots, s_i,$$

$$x_1 \cdots x_\imath e_1 \cdots e_k d_1^2 d_2^2 \cdots d_g^2 = 1.$$

(2.1) Definition. Let Γ be an N.E.C. group; we say that Γ is a proper N.E.C. group if it is not a Fuchsian group. We shall denote by Γ^+ the Fuchsian group $\Gamma \cap G^+$.

Γ^+ is a normal subgroup of Γ with index two. Moreover if Γ and Γ' are N.E.C. groups and Γ is a subgroup of Γ' with index N, then Γ^+ is a subgroup of Γ'^+ with index N.

Singerman [5] proves the following theorem, which determines the signature of Γ^+ in terms of the signature of Γ.

(2.2) THEOREM. (a) *If Γ has signature $(*)$ then Γ^+ has signature*

$$(2g+k-1;\ +;\ [(m_1)^2 \cdots (m_r)^2 n_{11} \cdots n_{ks_k}]).$$

(b) *If Γ has signature $(**)$ then Γ^+ has signature*

$$(g+k-1;\ +;\ [(m_1)^2 \cdots (m_r)^2 n_{11} \cdots n_{ks_k}])$$

where $(\cdots)^2$ means that this proper period is repeated 2 times.

3. Normal N.E.C. signature

(3.1) DEFINITION. Let σ, σ' be the signatures of two N.E.C. group. We say that σ is normal in σ', and denote this $\sigma \lhd \sigma'$, if there is an N.E.C. group Γ of signature σ, and an N.E.C. subgroup Γ' with signature σ' such that $\Gamma \lhd \Gamma'$ and $d(\Gamma) = d(\Gamma')$; Γ, Γ' are called N.E.C. groups associated to σ, σ', respectively.

(3.2) PROPOSITION. *Let σ, σ' be the signatures of two N.E.C. groups such that $\sigma \lhd \sigma'$, and assume that Γ and Γ' are proper N.E.C. groups of signatures σ, σ', with $\Gamma \lhd \Gamma'$. Then, if σ^+ and σ'^+ are the signatures of Γ^+ and Γ'^+, we have $\sigma^+ \lhd \sigma'^+$.*

Proof. First we prove that $\Gamma^+ \lhd \Gamma'^+$; since $\Gamma \lhd \Gamma'$, for $t \in \Gamma^+$ and $q \in \Gamma'^+$, then qtq^{-1} preserves orientation, whence it belongs to Γ^+; thus $\Gamma^+ \lhd \Gamma'^+$. Now

$$d(\Gamma) = \tfrac{1}{2}(d(\Gamma^+))$$

$$d(\Gamma') = \tfrac{1}{2}(d(\Gamma'^+))$$

hence $d(\Gamma^+) = d(\Gamma'^+)$, and $\sigma^+ \lhd \sigma'^+$.

(3.3) PROPOSITION. *Let σ and σ' be two N.E.C. signatures; then $\sigma \lhd \sigma'$ if and only if there are groups Γ, Γ' with signatures σ and σ' such that $\Gamma \lhd \Gamma'$ and $\sigma^+ \lhd \sigma'^+$.*

Proof. If $\sigma \lhd \sigma'$, there are Γ and Γ' with $\Gamma \lhd \Gamma'$; thus (3.2) implies $\sigma^+ \lhd \sigma'^+$.

If there are groups Γ, Γ' of signatures σ, σ', satisfying $\Gamma \lhd \Gamma'$ and $\sigma^+ \lhd \sigma'^+$, then $d(\Gamma) = d(\Gamma')$ because $\sigma^+ \lhd \sigma'^+$; thus, $\sigma \lhd \sigma'$.

Let σ, σ' be N.E.C. signatures, and $\sigma \lhd \sigma'$, since every normal subgroup of a Fuchsian group is Fuchsian one, some of the following properties must be true:

(1) σ, σ' are the signatures of Fuchsian groups;
(2) σ, σ' are the signatures of proper N.E.C. groups;
(3) σ is the signature of Fuchsian group Γ and σ' that of a proper N.E.C. group Γ', with $d(\Gamma) = d(\Gamma') = 0$.

4. Computation of normal N.E.C. signatures

Now we shall give all pairs of signatures σ, σ' such that $\sigma \lhd \sigma'$.

First we consider the pairs corresponding to Fuchsian groups, thus were given by Singerman [4], and are the following.

Γ'	Γ	$[\Gamma' : \Gamma]$
$(0;\ +;\ [2, 2, 2, 2, 2, 2])$	$(2;\ +;\ [—])$	2
$(0;\ +;\ [2, 2, 2, 2, t])$	$(1;\ +;\ [t, t])$	2
$(0;\ +;\ [2, 2, 2, 2t])$	$(1;\ +;\ [t])$	2
$(0;\ +;\ [2, 2, 2, t])$	$(0;\ +;\ [t, t, t, t])$ $(t \geq 3)$	4
$(0;\ +;\ [2, 2, t, u])$	$(0;\ +;\ [t, t, u, u])$ $(\max(t, u) \geq 3)$	2
$(0;\ +;\ [3, 3, t])$	$(0;\ +;\ [t, t, t])$ $(t \geq 4)$	3
$(0;\ +;\ [2, 3, 2t])$	$(0;\ +;\ [t, t, t])$ $(t \geq 4)$	6
$(0;\ +;\ [2, t, 2u])$	$(0;\ +;\ [t, t, u])$ $(t \geq 3, t + u \geq 7)$	2

We shall use Proposition (3.3) to compute the remaining pairs of normal signatures, which correspond to the cases (2) and (3) of Section 3.

Let $\sigma^+ = (2;\ +;\ [—])$ and assume that in σ the plus sign appears. Then, accordingly to (2.2), we have $2g + k - 1 = 2$; hence $g = 0$, $k = 3$, or $g = 1$, $k = 1$; therefore, the possible signatures are

$$\sigma_1 = (0;\ +;\ [—];\ \{(—)(—)(—)\})$$

$$\sigma_2 = (1;\ +;\ [—];\ \{(—)\}).$$

If in σ appears the minus sign, we have $g + k - 1 = 2$; hence

$$g = 1 \quad k = 2$$

$$g = 2 \quad k = 1$$

$$g = 3 \quad k = 0;$$

therefore we respectively have

$$\sigma_3 = (1;\ -;\ [—];\ \{(—)(—)\})$$

$$\sigma_4 = (2;\ -;\ [—];\ \{(—)\})$$

$$\sigma_5 = (3;\ -;\ [—])$$

Proposition (2.2) now gives us the N.E.C. signatures σ' when $\sigma'^+ = (0;\ +;\ [2, 2, 2, 2, 2, 2])$. If the plus sign appears in σ, we have $2g + k - 1 = 0$; hence $g = 0$, $k = 1$, and therefore we respectively have

$$\sigma'_1 = (0;\ +;\ [—];\ \{(2, 2, 2, 2, 2, 2)\})$$

$$\sigma'_2 = (0;\ +;\ [2];\ \{(2, 2, 2, 2)\})$$

$$\sigma'_3 = (0;\ +;\ [2, 2];\ \{(2, 2)\})$$

$$\sigma'_4 = (0;\ +;\ [2, 2, 2];\ \{(—)\})$$

If the minus sign appears in σ, then $g + k - 1 = 0$; hence $g = 1$, $k = 0$, and therefore we respectively have

$$\sigma'_5 = (1;\ -;\ [2, 2, 2])$$

Now, assume that Γ_i and Γ'_j are proper N.E.C. groups associated to σ_i and σ'_j, when

$$\sigma_i^+ = (2;\ +;\ [—]) \quad \text{and} \quad \sigma_j'^+ = (0;\ +;\ [2, 2, 2, 2, 2, 2]),$$

such that $\Gamma_i \lhd \Gamma'_j$. Not all the preceding computed pairs σ_i, σ'_j are admissible; the results of (1) make it possible to restrict our study to the pairs

$$\begin{array}{ccc} \sigma'_1\ \sigma_1 & \sigma'_3\ \sigma_4 & \sigma'_4\ \sigma_4 \\ \sigma'_2\ \sigma_3 & \sigma'_4\ \sigma_2 & \sigma'_4\ \sigma_5 \\ \sigma'_3\ \sigma_2 & \sigma'_4\ \sigma_3 & \sigma'_5\ \sigma_5 \end{array}$$

The following propositions will tell us what are the possible ones among them.

(4.1) Proposition. *Given σ'_1 and σ_1, there is an N.E.C. group Γ'_1 of signature σ'_1, and an N.E.C. group Γ_1 of signature σ_1, such that $\Gamma_1 \lhd \Gamma'_1$.*

Proof. Let Γ'_1 be an N.E.C. group with signature σ'_1. By (Section 2) we already know that a set of generators and relations of Γ'_1 is given by generators

$$c_1, c_2, c_3, c_4, c_5, c_6, c_7$$

and relations

$$(c_1 c_2)^2 = (c_2 c_3)^2 = (c_3 c_4)^2 = (c_4 c_5)^2 = 1,$$
$$(c_5 c_6)^2 = (c_6 c_7)^2 = 1,$$
$$c_1^2 = c_2^2 = c_3^2 = c_4^2 = c_5^2 = c_6^2 = c_7^2 = 1,$$
$$c_1 \cdot c_7 = 1.$$

We define an epimorphism

$$\theta\colon \Gamma'_1 \to \frac{Z}{(2)} = \{\bar{0}, \bar{1}\}$$

in the following way:

$$\theta(c_1) = \bar{1}, \quad \theta(c_2) = \bar{0}, \quad \theta(c_3) = \bar{1}, \quad \theta(c_4) = \bar{0},$$
$$\theta(c_5) = \bar{1}, \quad \theta(c_6) = \bar{0}, \quad \theta(c_7) = \bar{1}.$$

Since θ is onto,

$$\frac{\Gamma'_1}{\ker \theta} \approx \frac{Z}{(2)},$$

hence ker θ is a normal N.E.C. subgroup of Γ'_1. From a fundamental region of Γ'_1, we get one for ker θ, and from the last one we have, for ker θ, generators

$$e'_2 = c_3 c_5, \quad e'_1 = c_1 c_3, \quad e'_3 = c_5 c_7, \quad c_2, \quad c_4, \quad c_6,$$

and relations

$$e'_1 e'_2 e'_3 = 1, \quad e'^{-1}_1 c_2 e'_1 c_2 = 1, \quad e'^{-1}_2 c_4 e'_2 c_4 = 1, \quad e'^{-1}_3 c_6 e'_3 c_6 = 1.$$

Therefore the signature of ker θ is

$$\sigma_1 = (0;\ +;\ [—];\ \{(—)(—)(—)\});$$

thus, ker θ is the group Γ_1 we wanted.

(4.2) PROPOSITION. *Given σ'_2 and σ_3, there is an N.E.C. group Γ'_2 of signature σ'_2, and an N.E.C. group Γ_3 of signature σ_3, such that $\Gamma_3 \lhd \Gamma'_2$.*

Proof. Let Γ'_2 be an N.E.C. group with signature σ'_2. By Section 2 we already know that Γ'_2 has generators

$$x_1, e_1, c_1, c_2, c_3, c_4, c_5,$$

and relations

$$x_1 \cdot e_1 = 1, \quad x_1^2 = 1, \quad e_1^{-1} c_1 e_1 c_5 = 1,$$

$$(c_1 c_2)^2 = (c_2 c_3)^2 = (c_3 c_4)^2 = (c_4 c_5)^2 = c_1^2 = c_2^2 = c_3^2 = c_4^2 = c_5^2.$$

We define an epimorphism $\theta\colon \Gamma'_2 \to Z/(2)$ in the following way:

$$\theta(x_1) = \bar{1}, \quad \theta(c_1) = \bar{0}, \quad \theta(c_2) = \bar{1}, \quad \theta(c_3) = \bar{0},$$

$$\theta(c_4) = \bar{1}, \quad \theta(c_5) = \bar{0}, \quad \theta(e_1) = \bar{1}.$$

Since θ is onto,

$$\frac{\Gamma'_2}{\ker \theta} \approx \frac{Z}{(2)},$$

hence ker θ is a normal N.E.C. subgroup of Γ'_2. From a fundamental region of Γ'_2, we get one for ker θ, and from the last one we have, for ker θ, generators

$$e'_1 = c_2 c_4, \quad d_1 = x_1 c_2, \quad e'_2 = c_4 x_1 c_2 x_1, \quad c_3, \quad c_5,$$

and relations

$$e'^{-1}_1 c_3 e'_1 c_3 = 1, \quad e'^{-1}_2 c_5 e'_2 c_5 = 1, \quad e'_1 e'_2 d_1^2 = 1.$$

Therefore the signature of ker θ is

$$\sigma_3 = (1;\ -;\ [—];\ \{(—)(—)\});$$

thus, ker θ is the group Γ_3 we wanted.

(4.3) PROPOSITION. *Given σ'_3 and σ_2, there are no N.E.C. groups Γ'_3 and Γ_2 with signatures σ'_3 and σ_2 respectively, such that $\Gamma_2 \lhd \Gamma'_3$.*

Proof. Assume there is such a pair of N.E.C. groups Γ_2, Γ'_3. Then

$$\frac{\Gamma'_3}{\Gamma_2} \approx \frac{Z}{(2)},$$

because $[\Gamma'_3 : \Gamma_2] = 2$. Therefore, there is an epimorphism

$$\theta\colon \Gamma'_3 \to Z/(2)$$

satisfying $\ker \theta = \Gamma_2$.

We shall show the impossibility of this. Since Γ'_3 has generators

$$x_1, x_2, c_1, c_2, c_3, e_1,$$

and relations

$$e_1^{-1} c_1 e_1 c_3 = 1, \quad x_1 x_2 e_1 = 1,$$
$$(c_1 c_2)^2 = (c_2 c_3)^2 = c_1^2 = c_2^2 = c_3^2 = 1, \quad x_1^2 = x_2^2 = 1,$$

and Γ_2 has the signature $(1; +; [—]; \{(—)\})$, $\theta(x_1) = \bar{1}$ and $\theta(x_2) = \bar{1}$. Since $e_1 x_1 x_2 = 1$, $\theta(e_1) = \bar{0}$. Thus Γ_2 has an empty period cycle, and no period; and for the remaining generators we must have

$$(1) \qquad \theta(c_1) = \bar{0}, \quad \theta(c_2) = \bar{0}, \quad \theta(c_3) = \bar{0},$$

or

$$(2) \qquad \theta(c_1) = \bar{1}, \quad \theta(c_2) = \bar{0}, \quad \theta(c_3) = \bar{1}.$$

For case (1), we shall see in (4.4) that $\ker \theta$ has σ_4 as signature. In case (2), $\ker \theta$ has generators

$$d_1 = c_3 x_2, \quad d_2 = x_1 c_1, \quad e'_1 = c_1 c_3, \quad c_2,$$

and relations

$$e_1'^{-1} c_2 e'_1 c_2 = 1, \quad e'_1 d_1^2 d_2^2 = 1, \quad c_2^2 = 1.$$

Therefore, the signature of $\ker \theta$ also is σ_4, whence it cannot be Γ_2.

(4.4) Proposition. *Given σ'_3 and σ_4, there is an N.E.C. group Γ'_3 of signature σ'_3, and an N.E.C. group Γ_4 of signature σ_4, such that $\Gamma_4 \lhd \Gamma'_3$.*

Proof. Let Γ'_3 be an N.E.C. group with signature σ'_3. By Section 2 we already know that Γ'_3 has generators

$$x_1, x_2, c_1, c_2, c_3, e_1,$$

and relations

$$e_1^{-1} c_1 e_1 c_3 = 1, \quad x_1 x_2 e_1 = 1,$$
$$(c_1 c_2)^2 = (c_2 c_3)^2 = c_1^2 = c_2^2 = c_3^2 = 1,$$
$$x_1^2 = 1, \quad x_2^2 = 1.$$

We define an epimorphism $\theta\colon \Gamma'_3 \to Z/(2)$ in the following way:

$$\theta(x_1) = \bar{1}, \quad \theta(x_2) = \bar{1}, \quad \theta(c_1) = \bar{0},$$
$$\theta(c_2) = \bar{1}, \quad \theta(c_3) = \bar{0}, \quad \theta(e_1) = \bar{0}.$$

Since θ is onto,

$$\frac{\Gamma'_3}{\ker\theta} \approx \frac{Z}{(2)}$$

hence $\ker\theta$ is a normal N.E.C. subgroup of Γ'_3. From a fundamental region of Γ'_3, we get one for $\ker\theta$, and from the last one we have, for $\ker\theta$, generators

$$d_1 = x_1 c_2, \quad d_2 = c_2 x_2, \quad e'_1 = x_2 c_2 x_2 x_1 c_2 x_1, \quad c'_1 = x_1 c_3 x_1,$$

and relations

$$e'_1 d_1^2 d_2^2 = 1, \quad e_1'^{-1} c'_1 e'_1 c'_1 = 1, \quad c_1'^2 = 1.$$

Therefore the signature of $\ker\theta$ is

$$\sigma_4 = (2;\ -;\ [\text{—}];\ \{(\text{—})\});$$

thus, $\ker\theta$ is the group Γ_4 we wanted.

(4.5) Proposition. *Given σ'_4 and σ_2, there is an N.E.C. group Γ'_4 of signature σ'_4, and an N.E.C. group Γ_2 of signature σ_2, such that $\Gamma_2 \lhd \Gamma'_4$.*

Proof. Let Γ'_4 be an N.E.C. group with signature σ'_4. By Section 2 we already know that Γ'_4 has generators

$$x_1, x_2, x_3, e_1, c_1,$$

and relations

$$x_1 x_2 x_3 e_1 = 1, \quad e_1^{-1} c_1 e_1 c_1 = 1, \quad x_1^2 = x_2^2 = x_3^2 = 1, \quad c_1^2 = 1.$$

We define an epimorphism $\theta\colon \Gamma'_4 \to Z/(2)$ in the following way:

$$\theta(x_1) = \bar{1}, \quad \theta(x_2) = \bar{1}, \quad \theta(x_3) = \bar{1}, \quad \theta(e_1) = \bar{1}, \quad \theta(c_1) = \bar{0}.$$

Since θ is onto,

$$\frac{\Gamma'_4}{\ker\theta} \approx \frac{Z}{(2)},$$

hence $\ker\theta$ is a normal N.E.C. subgroup of Γ'_4. From a fundamental region of Γ'_4, we get one for $\ker\theta$, and from the last one we have, for $\ker\theta$, generators

$$a_1 = x_1 x_2, \quad b_1 = x_2 x_3, \quad e'_1 = x_1 e_1^{-2} x_1, \quad c'_1 = x_1 c_1 x_1,$$

and relations

$$e'_1 a_1 b_1 a_1^{-1} b_1^{-1} = 1, \quad e_1'^{-1} c'_1 e'_1 c'_1 = 1, \quad c_1'^2 = 1.$$

Therefore, the signature of ker θ is

$$\sigma_2 = (1;\ +;\ [-];\ \{(-)\});$$

thus, ker θ is the group Γ_2 we wanted.

(4.6) PROPOSITION. *Given σ'_4 and σ_3, there are no N.E.C. groups Γ'_4 and Γ_3 with signatures σ'_4 and σ_3 respectively, such that $\Gamma_3 \lhd \Gamma_4$.*

Proof. Assume there is such a pair of N.E.C. groups Γ_3, Γ'_4. Then

$$\frac{\Gamma'_4}{\Gamma_3} \approx \frac{Z}{(2)}$$

because $[\Gamma'_4\colon \Gamma_3] = 2$; thus, there is an epimorphism

$$\theta\colon \Gamma'_4 \to Z/(2)$$

satisfying ker $\theta = \Gamma_3$.

We shall show the impossibility of this. Since Γ'_4 has generators

$$x_1, x_2, x_3, e_1, c_1,$$

and relations

$$x_1 x_2 x_3 e_1 = 1, \quad e_1^{-1} c_1 e_1 c_1 = 1, \quad x_1^2 = x_2^2 = x_3^2 = 1, \quad c_1^2 = 1,$$

and Γ_3 has the signature $(1;\ -;\ [-];\ \{(-)(-)\})$, $\theta(x_1) = \bar{1}$, $\theta(x_2) = \bar{1}$ and $\theta(x_3) = \bar{1}$. Since $x_1 x_2 x_3 e_1 = 1$, $\theta(e_1) = \bar{1}$. Now, Γ_3 has two empty period cycles, and no period; thus $\theta(c_1) = \bar{0}$. But for this case we have already shown that ker θ has signature σ_2, whence ker θ cannot be Γ_3.

(4.7) PROPOSITION. *Given σ'_4 and σ_4, there are no N.E.C. groups Γ'_4 and Γ_4 with signatures σ'_4 and σ_4 respectively, such that $\Gamma_4 \lhd \Gamma'_4$.*

Proof. Assume there is such a pair of N.E.C. groups Γ_4, Γ'_4. Then

$$\frac{\Gamma'_4}{\Gamma_4} \approx \frac{Z}{(2)}$$

because $[\Gamma'_4\colon \Gamma_4] = 2$; thus, there is an epimorphism

$$\theta\colon \Gamma'_4 \to Z/(2)$$

satisfying ker $\theta = \Gamma_4$.

We shall show impossibility of this. Since Γ'_4 has generators

$$x_1, x_2, x_3, e_1, c_1,$$

and relations

$$x_1 x_2 x_3 e_1 = 1, \quad e_1^{-1} c_1 e_1 c_1 = 1, \quad x_1^2 = x_2^2 = x_3^2 = 1, \quad c_1^2 = 1,$$

and Γ_4 has the signature $(2; -; [—]; \{(—)\})$, $\theta(x_1) = \bar{1}$, $\theta(x_2) = \bar{1}$ and $\theta(x_3) = \bar{1}$. Since $x_1 x_2 x_3 e_1 = 1$, $\theta(e_1) = \bar{1}$. Now Γ_4 has an empty period cycle, and no period; thus $\theta(c_1) = \bar{0}$. But for this case we have already shown that ker θ has signature σ_2, whence ker θ cannot be Γ_4.

(4.8) PROPOSITION. *Given σ'_4 and σ_5, there is an N.E.C. group Γ'_4 of signature σ'_4, and an N.E.C. group Γ_5 of signature σ_5, such that $\Gamma_5 \lhd \Gamma'_4$.*

Proof. Let Γ'_4 be an N.E.C. group with signature σ'_4. By Section 2, we already know that Γ'_4 has generators

$$x_1, x_2, x_3, e_1, c_1,$$

and relations

$$x_1 x_2 x_3 e_1 = 1, \quad e_1^{-1} c_1 e_1 c_1 = 1, \quad x_1^2 = x_2^2 = x_3^2 = 1, \quad c_1^2 = 1.$$

We define an epimorphism $\theta\colon \Gamma'_4 \to Z/(2)$ in the following way:

$$\theta(x_1) = \bar{1}, \quad \theta(x_2) = \bar{1}, \quad \theta(x_3) = \bar{1}, \quad \theta(c_1) = \bar{1}, \quad \theta(e_1) = \bar{1}.$$

Since θ is onto,

$$\frac{\Gamma'_4}{\ker \theta} \approx \frac{Z}{(2)},$$

hence ker θ is a normal N.E.C. subgroup of Γ'_4, and it has generators

$$d_1 = c_1 x_1, \quad d_2 = x_1 c_1 x_1 x_2, \quad d_3 = x_2 x_1 c_1 x_1 x_2 x_3,$$

and relations

$$d_1^2 d_2^2 d_3^2 = 1.$$

Therefore, the signature of ker θ is

$$\sigma_5 = (3; -; [—]),$$

thus, ker θ is the group Γ_5 we wanted.

(4.9) PROPOSITION. *Given σ'_5 and σ_5, there are no N.E.C. group Γ'_5 and Γ_5 with signatures σ'_5 and σ_5 respectively, such that $\Gamma_5 \lhd \Gamma'_5$.*

Proof. Assume there is such a pair of N.E.C. groups Γ_5, Γ'_5. Then

$$\frac{\Gamma'_5}{\Gamma_5} \approx \frac{Z}{(2)}$$

because $[\Gamma'_5 : \Gamma_5] = 2$; thus, there would be an epimorphism

$$\theta\colon \Gamma'_5 \to Z/(2)$$

satisfying ker $\theta = \Gamma_5$.

We shall show the impossibility of this. Since Γ'_5 has generators

$$x_1, x_2, x_3, d_1,$$

and relations

$$x_1^2 = x_2^2 = x_3^2 = 1, \quad x_1 x_2 x_3 d_1^2 = 1,$$

and Γ_5 has the signature $(1; -; [—])$, $\theta(x_1) = \bar{1}$, $\theta(x_2) = \bar{1}$ and $\theta(x_3) = \bar{1}$. Since $x_1 x_2 x_3 d_1^2 = 1$, $\theta(d_1) \neq \bar{1}, \bar{0}$, whence it cannot be an epimorphism θ with ker $\theta = \Gamma_5$.

Similar reasoning upon the remaining pairs of signatures that appear in (2) and (3) of Section 3, leads to the following list of pairs of normal signatures:

Γ'	Γ	$[\Gamma' : \Gamma]$
$(0; +; [2, 2]; \{(t)\})$	$(0; +; [—]; \{(t)(t)\})$	2
$(0; +; [2, 2]; \{(t)\})$	$(2; -; [t])$	2
$(0; +; [2]; \{(2, 2, t)\})$	$(1; -; [t]; \{(—)\})$	2
$(0; +; [2]; \{(2, 2, t)\})$	$(1; -; [—]; \{(t, t)\})$	2
$(0; +; [—]; \{(2, 2, 2, 2, t)\})$	$(0; +; [t]; \{(—)(—)\})$	2
$(0; +; [—]; \{(2, 2, 2, 2, t)\})$	$(0; +; [—]; \{(t, t)(—)\})$	2
$(0; +; [—]; \{(2, 2, 2, 2, 2, 2)\})$	$(0; +; [—]; \{(—)(—)(—)\})$	2
$(0; +; [2]; \{(2, 2, 2, 2)\})$	$(1; -; [—]; \{(—)(—)\})$	2
$(0; +; [2, 2]; \{(2, 2)\})$	$(2; -; [—]; \{(—)\})$	2
$(0; +; [2, 2, 2]; \{(—)\})$	$(1; +; [—]; \{(—)\})$	2
$(0; +; [2, 2, 2]; \{(—)\})$	$(3; -; [—])$	2
$(0; +; [2]; \{(2, 2t)\})$	$(1; -; [—]; \{(t)\})$	2
$(0; +; [—]; \{(2, 2, 2, 2t)\})$	$(0; +; [—]; \{(t)(—)\})$	2
$(0; +; [—]; \{(2, 2, 2, t)\})$	$(0; +; [t, t]; \{(—)\})(t \geq 3)$	4
$(0; +; [—]; \{(2, 2, 2, t)\})$	$(0; +; [—]; \{(t, t, t, t)\})(t \geq 3)$	4
$(0; +; [—]; \{(2, 2, 2, t)\})$	$(1; -; [t, t])(t \geq 3)$	4
$(0; +; [2]; \{(t, u)\})$	$(0; +; [—]; \{(t, u, t, u)\})$ $(\max(t, u) \geq 3)$	2
$(0; +; [2]; \{(t, u)\})$	$(1; -; [t, u])(\max(t, u) \geq 3)$	2
$(0; +; [—]; \{(2, 2, t, u)\})$	$(0; +; [t, u]; \{(—)\})$ $(\max(t, u) \geq 3)$	2
$(0; +; [—]; \{(2, 2, t, u)\})$	$(0; +; [—]; \{(t, t, u, u)\})$ $(\max(t, u) \geq 3)$	2
$(0; +; [—]; \{(2, t, 2, u)\})$	$(0; +; [t]; \{(u, u)\})$ $(\max(t, u) \geq 3)$	2
$(0; +; [t]; \{(2, 2)\})$	$(0; +; [t, t]; \{(—)\})(t \geq 3)$	2
$(0; +; [t, 2]; \{(—)\})$	$(0; +; [t, t]; \{(—)\})(t \geq 3)$	2
$(0; +; [t, 2]; \{(—)\})$	$(1; -; [t, t])(t \geq 3)$	2
$(0; +; [3]; \{(t)\})$	$(0; +; [—]; \{(t, t, t)\})(t \geq 4)$	3
$(0; +; [—]; \{(2, 3, 2t)\})$	$(0; +; [—]; \{(t, t, t)\})(t \geq 4)$	6

Γ'	Γ	$[\Gamma' : \Gamma]$
$(0; +; [-]; \{(2, t, 2u)\})$	$(0; +; [t]; \{(u)\})(t \geq 3)$ $(t + u \geq 7)$	2
$(0; +; [-]; \{(2, t, 2u)\})$	$(0; +; [-]; \{(t, t, u)\})(t \geq 3)$ $(t + u \geq 7)$	2
$(0; +; [-]; \{(t, t, u)\})$	$(0; +; [t, t, u])(t \geq 3)$ $(t + u \geq 7)$	2
$(0; +; [t]; \{(u)\})$	$(0; +; [t, t, u])(t \geq 3)$ $(t + u \geq 7)$	2
$(0; +; [3]; \{(t)\})$	$(0; +; [t, t, t])(t \geq 4)$	6
$(0; +; [-]; \{(3, 3, t)\})$	$(0; +; [t, t, t])(t \geq 4)$	6
$(0; +; [-]; \{(m, t, u)\})$	$(0; +; [m, t, u])(m \neq t \neq u \neq m)$ $(t, u \geq 4$ or $t \geq 7)$	2
$(0; +; [-]; \{(2, t, 2u)\})$	$(0; +; [t, t, u])(t \geq 3)$ $(t + u \geq 7)$	4
$(0; +; [-]; \{(2, 3, 2t)\})$	$(0; +; [t, t, t])(t \geq 4)$	12

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REFERENCES

1. E. BUJALANCE, *Sobre los subgrupos normales N.E.C.*, Tesis Doctoral, U. Complutense, 1980.
2. A. M. MACBEATH and D. SINGERMAN, *Space of subgroups and Teichmüller space*, Proc. London Math. Soc., vol. XXI (1975), pp. 211–256.
3. D. SINGERMAN, *Symetries of Riemann surfaces with large automorphism group*, Math. Ann., vol. 210 (1974), pp. 17–32.
4. ———, *Finitely maximal Fuchsian groups*, J. London Math. Soc. (2), vol. 6 (1972), pp. 29–38.
5. ———, *On the structure of non-Euclidean crystallographic groups*, Proc. Cambridge Philos. Soc., vol. 76 (1974), pp. 223–240.

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