

THE EXPONENTIAL FUNCTION CHARACTERIZED BY AN APPROXIMATE FUNCTIONAL EQUATION

BY

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THEOREM. *Let p be real, $p > 1$. Let the complex-valued function $f(x)$ belong to the Lebesgue class $L^p(0, z)$ for each real $z > 0$, and satisfy*

$$(1) \quad \lim_{z \rightarrow \infty} e^{-\varepsilon z} \int_0^z \int_0^z |f(x+y) - f(x)f(y)|^p dx dy = 0$$

for each fixed $\varepsilon > 0$.

Then either there is a (possibly complex) constant β so that

$$(2) \quad f(x) = e^{\beta x}$$

almost surely for $x \geq 0$, or

$$(3) \quad \lim_{z \rightarrow \infty} e^{-\varepsilon z} \int_0^z |f(x)|^p dx = 0$$

for each fixed $\varepsilon > 0$.

Each of the conditions (2), (3) is sufficient to guarantee the validity of that at (1).

The proof of this theorem depends upon the possibility of analytically continuing the solution of certain Riccati differential equations in the complex plane.

Since $\|a\| - \|b\| \leq \|a - b\|$ condition (1) is also satisfied by $|f(x)|$. For the time being we shall assume that $f(x)$ is real and non-negative.

It is convenient to define

$$J(z) = \int_0^z f(x) dx$$

for $z \geq 0$.

LEMMA 1. *There is a non-negative constant B so that*

$$J(z) \leq e^{Bz}$$

holds for $z \geq 3$.

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Proof. The proof is by induction on z , and it is convenient to establish a stronger inequality

$$(4) \quad J(z) \leq e^{B(z-2)}.$$

We begin with the inequalities

$$\begin{aligned} \left| \int_{z-1}^z \{J(z+y) - J(y)\} dy - \int_{z-1}^z f(y) dy \cdot J(z) \right| \\ = \left| \int_{y=z-1}^z \int_{x=0}^z \{f(x+y) - f(x)f(y)\} dx dy \right| \\ \leq \int_0^z \int_0^z |f(x+y) - f(x)f(y)| dx dy \leq c_1 e^z, \end{aligned}$$

the last following, with a certain constant c_1 , from the hypothesis (1). In particular

$$\begin{aligned} J(2z-1) &\leq \int_{z-1}^z J(z+y) dy \\ &\leq \int_{z-1}^z \{J(y) + f(y)J(z)\} dy + c_1 e^z \\ &\leq J(z) + J(z)\{J(z) - J(z-1)\} + c_1 e^z. \end{aligned}$$

We may assume that for all large w , $J(w) \geq 1$, otherwise the lemma is already established. Thus

$$\begin{aligned} J(2z-1) &\leq (J(z))^2 + c_1 e^z, \\ J(z) &\leq (J(z+1)/2)^2 + c_1 e^z \end{aligned}$$

for all $z \geq 1$.

Suppose now that inequality (4) is valid over an interval $3 \leq z \leq z_0$, and that $z_0 < z \leq 3z_0/2$. Then $(z+1)/2 \leq z_0$, and applying our inductive estimate,

$$J(z) \leq e^{B(z-2)} \{e^{-B} + c_1 e^{2B+(1-B)z}\}.$$

If $z_0 \geq 5$ and B is sufficiently large then the expression in the curly brackets is less than 1 in value, and the induction will proceed.

By fixing B at a value which is also large enough that

$$J(z) \leq e^B (\leq e^{B(z-2)})$$

for $3 \leq z \leq 5$, we obtain the inequality (4) for all $z \geq 3$.

This completes the proof of Lemma 1.

It is convenient here to deduce the following:

LEMMA 2. Let $\theta = \min(1 - p^{-1}, 1/2)$. Then

$$J(z + h) - J(z) = O(h^\theta e^{Bz})$$

holds uniformly for $z \geq 0$ and $0 \leq h \leq 1$.

Proof. After an application of Hölder's inequality with $p^{-1} + q^{-1} = 1$, we deduce from our hypothesis (1) that

$$\int_z^{z+h} \int_z^{z+h} (f(x)f(y) - f(x + y)) dx dy = O(h^{2/q} e^{Bz/p}).$$

Set $\varepsilon = Bp$ and note that

$$\int_z^{z+h} \int_z^{z+h} f(x + y) dx dy = \int_z^{z+h} (J(y + z + h) - J(z)) dy = O(h e^{2Bz}),$$

the last step by Lemma 1.

Then,

$$(J(z + h) - J(z))^2 = O(e^{2Bz}(h + h^{2/q}))$$

from which the asserted result follows.

This completes the proof of Lemma 2.

For complex $s = \sigma + it$, $\sigma = \text{Re}(s)$, define the Laplace transform

$$w = w(s) = \int_0^\infty e^{-sx} f(x) dx.$$

In view of Lemma 1 an integration by parts shows that this function is well defined in the half-plane $\sigma > B$.

If the integral

$$\int_0^\infty e^{-\sigma x} f(x) dx$$

converges for every positive value of σ then we shall already have the result

$$\lim_{z \rightarrow \infty} e^{-\varepsilon z} \int_0^z f(x) dx = 0$$

for every $\varepsilon > 0$. As we shall show later, this already leads to the condition (3) in the statement of the theorem. We shall therefore assume that there is a positive α so that the integral converges for $\sigma > \alpha$ but not for $\sigma = \alpha$. In particular the integral converges uniformly absolutely in each half-plane $\sigma \geq \alpha + \delta$, $\delta > 0$, and so defines an analytic function $w(s)$ in the half-plane $\sigma > \alpha$.

Let

$$\eta(x, y) = f(x + y) - f(x)f(y),$$

and define

$$g(s) = \int_0^\infty \int_0^\infty e^{-s(x+y)} \eta(x, y) \, dx \, dy.$$

For $u > 0, \varepsilon > 0$ we deduce from (1) that

$$\begin{aligned} \iint_{u \leq x+y \leq 2u} e^{-\sigma(x+y)} |\eta(x, y)| \, dx \, dy &\leq e^{-\sigma u} \int_0^{2u} \int_0^{2u} |\eta(x, y)| \, dx \, dy \\ &= O(e^{(\varepsilon - \sigma)u}). \end{aligned}$$

It follows readily that $g(s)$ is analytic in $\sigma > 0$, and bounded above uniformly in every half-plane $\sigma \geq \delta$. Of course, we expect this bound to get worse as δ decreases in size. We need the following, better estimate.

LEMMA 3.

$$\lim_{|\tau| \rightarrow \infty} g(\sigma + i\tau) = 0$$

uniformly in every strip $(0 <) \sigma_1 \leq \sigma \leq \sigma_2 (< \infty)$.

Proof. Let $\varepsilon > 0$ be given. For any particular $\sigma > 0$, the integral defining $g(s)$ converges absolutely and we can find a number v so that

$$\iint_{x+y > v} |\eta(x, y)| e^{-\sigma(x+y)} \, dx \, dy < \varepsilon.$$

The finite Fourier integral

$$\iint_{x+y \leq v} \eta(x, y) e^{-(\sigma + i\tau)(x+y)} \, dx \, dy$$

approaches zero as $|\tau| \rightarrow \infty$, with the usual Riemann-Lebesgue argument. Hence

$$\limsup_{|\tau| \rightarrow \infty} |g(s)| \leq \varepsilon,$$

and since ε was arbitrary positive we obtain the result on each line $\operatorname{Re}(s) > 0$.

The uniformity may now be deduced by means of a Phragmén-Lindelöf argument (e.g., Titchmarsh [4], p. 180).

This completes the proof of lemma 3.

LEMMA 4. w satisfies the Riccati differential equation

$$(5) \quad w' + w^2 = -g(s)$$

in the half-plane $\sigma > \alpha$, and $g(s)$ is analytic in $\sigma > 0$. Here $'$ denotes differentiation with respect to s .

Proof. Multiply both sides of the identity $f(x+y) - f(x)f(y) = \eta(x, y)$ by $e^{-s(x+y)}$ and integrate over the positive quadrant of the (x, y) -plane. The change of variables $x = r(\cos \theta)^2$, $y = r(\sin \theta)^2$, with associated Jacobian $r \sin 2\theta$, transforms the integral involving $f(x+y)$ into

$$\int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-sr} f(r) \cdot r \sin 2\theta \cdot dr d\theta = -w'(s).$$

This completes the proof of Lemma 4.

We seek to analytically continue the solution w to this differential equation into the strip $0 < \sigma \leq \alpha$.

Let $s_0 = \sigma_0 + i\tau_0$ be a point in the half-plane $\sigma > \alpha$. One may regard σ_0 as being "near" to α . Let c_1, c_2, t_1 be positive numbers and define a rectangle

$$\Delta: \alpha - c_1 \leq \sigma \leq \alpha + c_2, \quad |\tau - \tau_0| \leq t_1.$$

We shall assume that $\sigma_0 < \alpha + c_2$ so that s_0 lies inside Δ .

It is convenient to use L to denote $c_1 + c_2 + 2t_1$, so that $2L$ is the length of the perimeter of Δ .

Let M be a positive number. The space S of all functions $h(x, y)$, considered as functions of the pair (x, y) , which are continuous and bounded by M on Δ , is complete with respect to the sup norm

$$\|h\| = \sup |h(x, y)|, \quad (x, y) \in \Delta.$$

Consider the map T , of S into S , given by

$$Th = w_0 + \int_{s_0}^s \{-h^2(s) - g(s)\} ds$$

where w_0 is a complex number to be specified presently, and the integration is along the half-rectangle $\Gamma: s_0$ to $\sigma_0 + i\tau$ to s . Clearly

$$\|Th\| \leq |w_0| + L(\|h^2\| + \|g\|) \leq M$$

if (say)

$$(6) \quad 3|w_0| \leq M, \quad 3LM \leq 1, \quad 3L\|g\| \leq M,$$

so that Th belongs to S .

Moreover, if the conditions (6) are satisfied, then for any two members h_1, h_2 of S

$$\begin{aligned} \|Th_1 - Th_2\| &\leq L\|h_1^2 - h_2^2\| \\ &\leq L\|(h_1 + h_2)(h_1 - h_2)\| \\ &\leq 2LM\|h_1 - h_2\|. \end{aligned}$$

Since $2LM < 1$, T is a contraction map and has a unique fixed point w ,

$$w = w_0 - \int_{s_0}^s \{w^2 + g(s)\} ds.$$

If, instead, we integrate along the (other) half-rectangle $\tilde{\Gamma}$, s_0 to $\sigma + i\tau_0$ to s , then we obtain a fixed point \tilde{w} of the corresponding map \tilde{T} .

Since $T^n(M)$, $\tilde{T}^n(M)$, $n = 1, 2, \dots$, are analytic inside Δ ,

$$w = \lim_{n \rightarrow \infty} T^n(M) = \lim_{n \rightarrow \infty} \tilde{T}^n(M) = \tilde{w}.$$

In particular,

$$\int_{\Gamma} (w^2 + g) ds = \int_{\tilde{\Gamma}} (w^2 + g) ds.$$

It is now straightforward to prove that $\partial w/\partial \sigma$ and $-i\partial w/\partial \tau$ exist, are equal, and are continuous, at points inside Δ . Thus w is analytic in the interior of Δ , and there (uniquely) satisfies the Riccati equation (5) with the boundary condition $w(s_0) = w_0$.

In order to obtain in this way an analytic continuation of our function $w(s)$ from the half-plane $\sigma > \alpha$ into the interior of Δ , we need to satisfy the conditions (6).

Regarding $\sigma_0 (> \alpha)$ as fixed, just as we argued in the proof of lemma 3 we can prove that

$$\lim_{|\tau| \rightarrow \infty} w(\sigma_0 + i\tau) = \lim_{|\tau| \rightarrow \infty} \int_0^{\infty} e^{-(\sigma_0 + i\tau)x} f(x) dx = 0.$$

Let $\varepsilon > 0$, $\delta > 0$ be given. We shall set $c_1 = \alpha - \delta$, $c_2 = \sigma_0 - \alpha + \delta$, $t_1 = 1$, so that Δ becomes

$$\delta \leq \sigma \leq \sigma_0 + \delta, |\tau - \tau_0| \leq 1.$$

If τ_0 is sufficiently large then from lemma 3 we deduce that $\|g\| < \varepsilon$. Since L is now fixed, $M = (3L)^{-1}$ will satisfy the middle condition at (6). With this value, $3L\|g\| \leq M$ will certainly hold if ε is sufficiently small. Similarly $3|w(s_0)| < M$ will hold for all large τ_0 .

Since we have proved that there is a unique solution to the differential equation (5) in the interior of Δ , which has the value $w(s_0)$ at s_0 , and since Δ overlaps the half-plane $\sigma > \alpha$ in the strip $\alpha < \sigma \leq \sigma_0 + \delta$, $|\tau - \tau_0| \leq 1$, our solution to the Riccati equation is an analytic continuation of $\{w(s), \sigma > \alpha\}$.

By keeping σ_0 fixed, and sliding τ_0 up (or down as the case may be), we can cover the semi-infinite strips $\delta \leq \sigma \leq \sigma_0 + \delta$, $|\tau| > \tau_0 - 1$ and so arrive at the following result.

LEMMA 5. *For each $\delta > 0$ there is a number τ_1 so that the function $w(s)$, initially defined in $\sigma > \alpha$, may be analytically continued into the semi-infinite strips, $\delta < \sigma < \alpha + 1$, $|\tau| > \tau_1$.*

Moreover, $w(s)$ is uniformly bounded in these strips.

Proof. Only the last assertion of Lemma 4 calls for comment. In our construction of the solution w to the differential equation we obtained the bound $\|w(s)\| = \sup |w(s)| \leq M = (3L)^{-1}$ on each Δ . Here L did not depend upon the value τ_0 associated with the rectangle Δ immediately under consideration.

This ends the proof of lemma 5.

We cannot give a precisely similar treatment of the Riccati equation in the box $\delta \leq \sigma \leq \alpha + 1, |\tau| \leq \tau_1$, since we expect a singularity at the point $s = \alpha$. Even if we didn't, the above argument does not apply directly since larger values of $\|g\|$ and $|w(s_0)|$ are to be expected, and these will only be compatible with the inequalities (6) if L is small. Thus we would not be able to reach as far left as the line $\sigma = \delta$.

Consider now the situation when the rectangle Δ has associated point s_0 with $|\tau_0| \leq \tau_1$, and $w(s_0)$, which is given to us in advance, is "large".

Let λ be a positive number. We look for a solution w_1 to the differential equation which is analytic in Δ , and satisfies $w_1(s_0) = \lambda$. For the moment we abandon hopes of it analytically continuing $\{w(s), \sigma > \alpha\}$.

Let $|g(s)| \leq c_0, c_0 > 0$, hold for $\sigma \geq \delta$. We apply our above space S with a box Δ_1 defined by

$$\sigma_0 - \frac{1}{12\sqrt{c_0}} \leq \sigma \leq \sigma_0 + \frac{1}{12\sqrt{c_0}}, \quad |\tau - \tau_0| \leq \frac{1}{12\sqrt{c_0}},$$

that is with (in our old notation)

$$c_1 = -\sigma_0 + \alpha + \frac{1}{12\sqrt{c_0}}, \quad c_2 = \sigma_0 - \alpha + \frac{1}{12\sqrt{c_0}}, \quad t_1 = \frac{1}{12\sqrt{c_0}}.$$

We shall assume that $\sigma_0 - \alpha \leq (24\sqrt{c_0})^{-1}$. Since c_0 is fixed, this can be safely arranged.

For our upper bound M in the definition of the space S we take $\sqrt{c_0}$. Then $3L = c_0^{-1/2}$ so that $3LM = 1$. Moreover, $3L\|g\| \leq c_0^{-1/2}c_0 = M$.

If $3|\lambda| \leq \sqrt{c_0}$ then the conditions (6) will all be satisfied and we shall obtain a (unique) solution w_1 to the Riccati equation, valid in the box Δ_1 , analytic in the interior of Δ_1 , and satisfying the boundary condition $w(s_0) = \lambda$.

Let w_1, w_2, w_3 be three such solutions, obtained by the restrictions $w(s_0) = \lambda, 2\lambda, 3\lambda$, where λ is (for the moment) any number for which $0 < 9|\lambda| \leq \sqrt{c_0}$.

It is a property of the Riccati equation (for example, see Ince [3], Chapter I, Section 12, pp. 22-23) that any further solution ω must satisfy the cross-ratio condition

$$(7) \quad \frac{\omega - w_2}{\omega - w_1} \frac{w_3 - w_1}{w_3 - w_2} = \mu$$

for some constant μ . Thus the functions ω may be parametrized by μ . In another form (7) becomes

$$\omega = \frac{-\mu w_1(w_3 - w_2) + w_2(w_3 - w_1)}{w_3 - w_1 - \mu(w_3 - w_2)} = \frac{l}{m}$$

say, where l and m are analytic in Δ_1 . The function m has the value $\lambda(2 - \mu)$ at s_0 , and so will not be identically zero unless $\mu = 2$.

We now look for a solution ω so that $\omega(s_0) = w(s_0)$. From (7) this determines μ by

$$\mu = 3 \left(\frac{w(s_0) - 2\lambda}{w(s_0) - \lambda} \right);$$

and conversely. This will certainly be possible if λ is chosen so that $\lambda \neq w(s_0)$, $\lambda \neq w(s_0)/4$ (this last ensuring that $\mu \neq 2$).

In this way we obtain a meromorphic continuation ω^* of $w(s)$ into the interior of Δ_1 .

We may now slide our box Δ_1 to the left and continue with this argument. We first note that function m obtained above has only finitely many zeros in the interior of Δ_1 , say N . Thus one of the strips

$$\sigma_0 - \frac{1}{12\sqrt{c_0}} + v\varepsilon_2 < \sigma \leq \sigma_0 - \frac{1}{12\sqrt{c_0}} + (v + 1)\varepsilon_2,$$

$v = 0, 1, \dots, N$, must be free of them. With a small enough value of ε_2 we can choose such a strip Λ as near to the line $\text{Re}(s) = \sigma_0 - (12\sqrt{c_0})^{-1}$ as we like. If we replace $|\tau - \tau_1| \leq (12\sqrt{c_0})^{-1}$ in the definition of Δ_1 by

$$|\tau - \tau_1| \leq (12\sqrt{c_0})^{-1} - \varepsilon_2,$$

then our solution ω^* will be analytic in our (τ -truncated) strip and we can apply the above argument to a space of continuous functions defined on a new rectangle Δ_2 , with Λ at its right-hand end (overlapping it). Any point in $\Lambda \cap \Delta_2$ may play the rôle previously assigned to s_0 , and the only change is a slightly smaller value of L .

It is clear that in this manner we can meromorphically continue $\{w(s), \sigma > \alpha\}$ into the half-plane $\sigma > \delta$.

LEMMA 6. *In the notation of lemma 4, $w(s)$ may be meromorphically continued into the strip $\delta < \sigma < \alpha + 1, |\tau| < \tau_1 + 1$.*

If $w(s)$ has a pole in this region it must be simple, and have residue 1.

For any $\varepsilon_3 > 0$ there is a line-segment $\text{Re}(s) = \sigma_1, |\tau| < \tau_1 + 1, \delta < \sigma_1 < \delta + \varepsilon_3$, on which $w(s)$ is uniformly bounded.

Proof. The third assertion of Lemma 6 may be obtained by constructing zero-free strips in the manner used during the meromorphic continuation of $w(s)$.

Concerning the second assertion, near a pole ρ of $w(s)$ we shall have

$$w(s) = \frac{D}{(s - \rho)^k} + \dots$$

say. Since $w(s)$ satisfies the Riccati equation (5),

$$\frac{-kD}{(s - \rho)^{k+1}} + \dots \left(\frac{D}{(s - \rho)^k} + \dots \right)^2 = -g(s)$$

must be analytic in a neighbourhood of ρ . Hence $2k \leq k + 1$ must hold, and this allows only $k = 1$. Then $-D + D^2 = 0$ also holds, which leads to $D = 1$.

This completes the proof of Lemma 6.

Define

$$J_1(z) = \int_0^z J(u) du, \quad z \geq 0.$$

Integration by parts shows that $J_1(z)$ has the Laplace transform $w(s)s^{-2}$, and a Fourier inversion gives

$$J_1(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sz} \frac{w(s)}{s^2} ds$$

for every real $c > \alpha$.

Let $\varepsilon > 0$ be given, and choose δ so that $\varepsilon/2 \leq \delta \leq 2\varepsilon/3$ and $w(\delta + i\tau)$ is bounded uniformly for all τ . Let ρ_1, \dots, ρ_n denote the poles of $w(s)$ in the half-plane $\sigma > \delta$. Moving the above contour to $\text{Re}(s) = \delta$ we pass over these poles and obtain the estimate

$$(8) \quad J_1(z) = \sum_{j=1}^n \rho_j^{-2} e^{\rho_j z} + O(e^{\varepsilon z}), \quad z \geq 0.$$

To estimate $J(z)$ let $t > \tau_1$ (see Lemma 6) be a real number to be chosen presently. Then starting from

$$J(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{sz} w(s)}{s} ds$$

we deform the contour into the line-segments

$$c - i\infty \rightarrow c - it \rightarrow \delta - it \rightarrow \delta + it \rightarrow c + it \rightarrow c + i\infty.$$

The integrals over the line-segments $\text{Re}(s) = c, |\tau| \geq t$ we put together to obtain

$$(9) \quad \frac{1}{2\pi} \int_t^\infty e^{cz} \left(e^{itz} \frac{w(c + i\tau)}{c + i\tau} + e^{-itz} \frac{w(c - i\tau)}{c - i\tau} \right) d\tau.$$

If $t \geq 2c$ then we can replace $c \pm i\tau$ in the denominators by $\pm i\tau$ at the expense of an amount

$$O\left(\int_t^\infty e^{cz}\tau^{-2} \sup_{\sigma=c} |w(s)| d\tau\right) = O(t^{-1}e^{cz}).$$

The remaining integral is then

$$\begin{aligned} \lim_{u \rightarrow \infty} \frac{e^{cz}}{\pi} \int_t^u \left(\int_0^\infty f(x)e^{-cx} \operatorname{Sin} \tau(z-x) dx \right) \frac{d\tau}{\tau} \\ = \frac{e^{cz}}{\pi} \int_0^\infty f(x)e^{-cx} \left(\int_t^\infty \frac{\operatorname{Sin} \tau(z-x)}{\tau} d\tau \right) dx, \end{aligned}$$

since $f(x)e^{-cx}$ belongs to the class $L(0, \infty)$.

An integration by parts shows that the innermost integral in the last expression is $O(\min(1, (t|z-x|)^{-1}))$. Then

$$\int_{z-1/t}^{z+1/t} |f(x)| e^{-cx} dx = O(e^{-cz}t^{-\theta}e^{Bz})$$

by Lemma 2, whilst

$$\int_0^{z-1/t} |f(x)| e^{-cx}(t|z-x|)^{-1} dx = O\left(t^{-1} \int_{1/t}^z |f(z-u)| u^{-1} du\right).$$

Define the function

$$F(v) = \int_0^v f(z-u) du,$$

for real v , $0 \leq v \leq z$. Applying Lemma 2 gives

$$F(v) = J(z) - J(z-v) = O(v^\theta e^{Bz}).$$

Integrating by parts we have

$$\begin{aligned} \int_{1/t}^z f(z-u)u^{-1} du &= [-u^{-1}F(u)]_{1/t}^z - \int_{1/t}^z u^{-2}F(u) du \\ &= O(t^{1-\theta}e^{Bz}). \end{aligned}$$

Hence

$$\int_0^{z-1/t} f(x)e^{-cx}(t|z-x|)^{-1} dx = O(t^{-\theta}e^{Bz}),$$

and a similar estimate holds for the corresponding integral over the range $x \geq z + t^{-1}$.

Assuming (as we may) that $c \geq B$, we obtain for the integral at (9) the upper bound $O(t^{-\theta}e^{cz})$.

The remaining integrals in the representation of $J(z)$ are easily estimated to contribute

$$O\left(e^{\delta z} \int_0^t \frac{d\tau}{|s|}\right) + O(t^{-1}e^{cz})$$

and we arrive at

$$(10) \quad J(z) = \sum_{j=1}^n \rho_j^{-1} e^{\rho_j z} + O(e^{\delta z} \log t + t^{-\theta} e^{cz})$$

valid for all $t \geq \max(2, 2c)$. Choosing t to be $\exp(cz/\theta)$ we obtain for the error term the bound $O(e^{\delta z} \log z)$, which is $O(e^{\epsilon z})$.

We now study the possible values for the poles ρ_j by means of the following result.

LEMMA 7. *Let*

$$\left| \sum_{j=1}^k c_j e^{d_j z} \right| \leq A e^{wz}$$

hold for some real A, w , and all real $z \geq 0$.

Then for each j with $c_j \neq 0$, $\text{Re}(d_j) \leq w$ must hold.

Proof. We write

$$\sum_{j=1}^k c_j e^{d_j z} = \sum_{l=1}^m e^{zr_l} P_l(z)$$

where the functions $P_l(z)$ have the form

$$P_l(z) = \sum_{h=1}^{h_l} a_{lh} e^{i\theta_{lh} z},$$

the θ_{lh} are real, so are the r_l and $r_1 > r_2 > \dots$. Without loss of generality the r_l and w may be assumed non-negative.

We may assume that $r_1 = \text{Re}(d_1) \geq \text{Re}(d_j)$ for every j , and that $c_1 \neq 0$. Otherwise there is nothing to prove. The function $P_1(z)$ is then not identically zero for real z .

We assert that

$$\limsup_{z \rightarrow \infty} |P_1(z)| = d > 0.$$

For otherwise $P_1(z) \rightarrow 0$ as (real) $z \rightarrow \infty$, so that

$$\sum_{h=1}^{h_1} |a_{1h}|^2 = \lim_{T \rightarrow \infty} T^{-1} \int_0^T |P_1(z)|^2 dz = 0,$$

which possibility has been ruled out.

Then

$$e^{zr_1} |P_1(z)| \leq Ae^{wz} + \sum_{i=2}^m e^{zr_i} |P_i(z)|$$

and by letting z approach infinity through a suitable sequence of values, we shall obtain a contradiction unless $r_1 \leq w$.

This completes the proof of Lemma 7.

Returning to our hypothesis (1), with p replaced by 1, we have

$$J_1(2z) - 2J_1(z) - (J(z))^2 = O(e^{\varepsilon z})$$

for each fixed $\varepsilon > 0$, and in particular for a value not exceeding $\alpha/3$.

In view of our estimates (8) and (10),

$$\sum_{j=1}^n \rho_j^{-2} (e^{2\rho_j z} - 2e^{\rho_j z}) - \left(\sum_{j=1}^n \rho_j^{-1} e^{\rho_j z} + O(e^{\varepsilon z}) \right)^2 = O(e^{2\varepsilon z})$$

for all real $z \geq 0$.

Suppose that amongst the $\text{Re}(\rho_j)$ at least two distinct values $r_1 > r_2 > \varepsilon$ occur. Then from the squared bracket a non-zero term involving (say)

$$e^{(\rho_1 + \rho_2)z}, \quad \text{Re}(\rho_1) = r_1, \quad \text{Re}(\rho_2) = r_2,$$

will arise which cannot be cancelled by any term of the form $e^{2\rho_j z}$ or $e^{\rho_j z}$. Applying Lemma 7 we deduce that $r_1 + r_2 \leq \max(2\varepsilon, r_1 + \varepsilon)$, so that $r_2 \leq \varepsilon$.

Without loss of generality we may therefore assume that the ρ_j all have the same real part.

In our present circumstances $w(s)$ is the Laplace/Fourier transform of a non-negative function, and since its defining integral diverges when $\sigma = \alpha$, we must have $\text{Re}(\rho_1) = \alpha$. (See, for example, Elliott [2], Chapter 2 concluding remarks.)

Let $\rho_j = \alpha + i\theta_j$, θ_j real, $j = 1, \dots, n$. Then

$$\sum_{j=1}^n \rho_j^{-2} e^{2i\theta_j z} - \left(\sum_{j=1}^n \rho_j^{-1} e^{i\theta_j z} \right)^2 = O(e^{(\varepsilon - \alpha)z})$$

for all $z \geq 0$. The expression $P(z)$ on the left-hand side of this estimate is of the same form as the $P_i(z)$ considered during the proof of lemma 7. Moreover, it approaches zero as $z \rightarrow \infty$. It must therefore vanish identically for all real $z \geq 0$.

We can regard $P(z)$ as a function of the complex variable z . It is then an integral function which vanishes on the positive half of the real axis, and so over the whole complex z -plane. In particular it vanishes on the imaginary axis. However, considering $P(iy)$ for real y , after the manner of lemma 7, we see that this is possible only if there is not more than one number θ_1 .

We have now arrived at an estimate

$$(11) \quad J(z) = \alpha^{-1} e^{\alpha z} + O(e^{\varepsilon z})$$

valid for every fixed $\varepsilon > 0$.

From our hypothesis (1),

$$\int_{x=0}^z \left| \int_{y=0}^{2z} f(x+y) dy - f(x) \int_0^{2z} f(y) dy \right| dx = O(e^{\varepsilon z}),$$

$$\int_0^z J(2z) \left| \left(\frac{J(2z+x) - J(x)}{J(2z)} \right) - f(x) \right| dx = O(e^{\varepsilon z}).$$

From our estimate (11),

$$(J(2z+x) - J(x))/J(2z) = e^{\alpha x} \{1 + O(e^{(3\varepsilon - 2\alpha)z})\}$$

so that

$$\int_0^z J(2z) |e^{\alpha x} - f(x)| dx = O(e^{(3\varepsilon + \alpha)z})$$

and

$$\int_0^z |e^{\alpha x} - f(x)| dx = O(e^{(3\varepsilon - \alpha)z}).$$

Letting $z \rightarrow \infty$ we obtain the almost sure representation $f(x) = e^{\alpha x}$.

At this stage we recall the earlier alternative that

$$\lim_{z \rightarrow \infty} e^{-\varepsilon z} \int_0^z f(x) dx = 0$$

for each fixed $\varepsilon > 0$.

From our initial hypothesis (1), using the inequality $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ and Holder's inequality, we obtain the bound

$$\begin{aligned} (J(z))^p \int_0^z f(x)^p dx &\leq 2^p \int_0^z |J(z)f(x) - \{J(z+x) - J(x)\}|^p dx + 2^p \int_0^z |J(z+x) - J(x)|^p dx \\ &\leq 2^p \int_0^z z^{p-1} \int_0^z |f(x)f(y) - f(x+y)|^p dy dx + O(ze^{p\varepsilon z}) \\ &= O(z^{p+1} e^{p\varepsilon z}). \end{aligned}$$

If $J(z)$ is zero for all $z \geq 0$ then conclusion (3) of the theorem is trivially valid. Otherwise

$$\int_0^z f(x)^p dx = O(e^{2pe\epsilon z})$$

for every fixed $\epsilon > 0$ and we obtain conclusion (3) anyway.

We now abandon our assumption that $f(x)$ be real and non-negative and summarise our results in the following form

LEMMA 8. *Either there is a positive α so that $|f(x)| = e^{\alpha x}$ almost surely for $x \geq 0$, or for each $\epsilon > 0$*

$$\lim_{z \rightarrow \infty} e^{-\epsilon z} \int_0^z |f(x)|^p dx = 0$$

is satisfied.

For the remainder of the proof of the theorem we shall assume that the first alternative of this lemma holds.

We can carry out analogues of the above arguments with (the implicit) $|f(x)|$ everywhere replaced by a now complex-valued $f(x)$. We shall then either reach an almost sure representation $f(x) = e^{\beta x}$ for $x \geq 0$ or the estimate

$$(12) \quad \int_0^z f(x) dx = O(e^{\epsilon z}), \quad z \geq 0,$$

for each fixed positive ϵ .

We shall show that this last possibility can only occur if $f(x)$ has the form (2) or is almost surely zero, and thus complete the proof of the theorem.

We modify the argument given following the estimate (11), by

$$\begin{aligned} & \left| \int_{x=0}^z \left(\int_{y=0}^{2z} f(x+y)\psi(y) dy - f(x) \int_0^{2z} f(y)\psi(y) dy \right) dx \right| \\ & \leq \int_{x=0}^z \int_{y=0}^{2z} |f(z+y) - f(x)f(y)| dx dy = O(e^{\epsilon z}), \end{aligned}$$

where

$$\psi(y) = \begin{cases} \exp(-i \arg f(y)) & \text{if } f(y) \neq 0, \\ 0 & \text{if } f(y) = 0. \end{cases}$$

In this definition we take any value of the argument of $f(y)$. Thus $|\psi(y)| \leq 1$ and $f(y)\psi(y) = |f(y)|$.

Hence

$$\int_0^{2z} |f(y)| dy \cdot \int_0^z f(x) dx = \int_{y=0}^{2z} \psi(y) \left(\int_{x=0}^z f(x+y) dx \right) dy + O(e^{\varepsilon z}),$$

$$= O(e^{4\varepsilon z}),$$

this last step by means of (12). In view of Lemma 8 (see (11)) we have

$$(13) \quad J(z) = \int_0^z f(x) dx = O(e^{(4\varepsilon - 2\alpha)z}).$$

The Laplace transform

$$w(s) = \int_0^\infty f(x)e^{-sx} dx = s \int_0^\infty J(x)e^{-sx} dx$$

is now well defined and analytic in the half plane $\sigma > -2\alpha$, and (letting $z \rightarrow \infty$ in the estimate of (13)) satisfies $w(0) = 0$.

For each real τ we can carry out the whole of the above argument with $e^{i\tau x}f(x)$ in place of $f(x)$. Unless we obtain a desired estimate of the type (2) in the statement of the theorem this leads to $w(i\tau) = 0$.

The function $w(s)$ then vanishes on the imaginary axis, and so in the half-plane $\sigma > -2\alpha$. A Fourier transform inside the half-plane $\sigma > \alpha$ now shows that $J(z) = 0$ for $z \geq 0$, so that $f(x)$ vanishes almost surely for $x \geq 0$.

This completes the proof of the theorem.

Concluding Remarks. If we assume only that

$$\lim_{z \rightarrow \infty} z^{-2} \int_0^z \int_0^z |f(x+y) - f(x)f(y)|^p dx dy = 0$$

must $f(x) = e^{\beta x}$ almost surely, or

$$\lim_{z \rightarrow \infty} z^{-1} \int_0^z |f(x)|^p dx = 0$$

hold?

If we strengthen the hypothesis to

$$\lim_{z \rightarrow \infty} z^{-1} \int_0^z \int_0^z |f(x+y) - f(x)f(y)|^p dx dy = 0$$

must $f(x) = e^{\beta x}$ hold almost surely?

Analogues of these results are certainly valid if the equation which is implicitly under consideration is that of Cauchy (see the author's paper [1]).

What if $0 < p \leq 1$ is allowed? Will it help if $f(x)$ is assumed real?

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