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CERTAIN SECONDARY OPERATIONS THAT DETECT INCOMPRESSIBLE MAPS

BY

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Introduction

Let $f\colon X \to K(Z_2, n)$ be a non-trivial map and let $\operatorname{im} f^*$ be the subset of $H^*(X; Z_2)$ induced by f. If all primary Steenrod operations act trivially on $\operatorname{im} f^*$ then the incompressibility of such maps cannot be detected by such primary operations. In this paper it is shown that if $H^*(X; Z_2)$ satisfies certain conditions, there is a sequence of secondary operations that can be used to detect the incompressibility of f. These secondary operations are of the form $Sq^{Q_k}\Phi$ where Φ is a secondary operation and $Q_k = (q_k, \ldots, q_1)$ is an admissible sequence.

Let Φ be a secondary operation associated to the relation $\alpha \circ \beta = 0$. If $x = f^*(\iota_n)$ and $\Phi(x)$ is defined, there is a fibration

$$\Omega K_1 \xrightarrow{i} E \xrightarrow{\pi} K(Z_2, n),$$

a map $\tilde{f}\colon X \to E$ and an element $w \in H^*(E; Z_2)$ such that $\Phi(x) = \tilde{f}^*(w)$ mod indeterminacy. The following are proved:

THEOREM. *Let $f\colon X \to K(Z_2, n)$ classify $x \in H^n(X; Z_2)$, $Q_k = (q_k, \ldots, q_1)$ be an admissible sequence and Φ be a secondary operation associated with $\alpha \circ \beta = 0$ and defined on x. If for all positive integers k,*

(a) $Sq^{Q_k}(i^*(w)) = \sum_j Sq^{p_{j,k}} v_{j,k}$ *where* $0 < p_{j,k} \le q_k$ *for all* j,
(b) $v_{j,k}$ *transgresses to a non-zero element for all* j,
(c) $M_k - n < q_k \le M_k$ *where* $M_k = \deg w + \sum_{i=1}^{k-1} q_i$,
(d) $Sq^{Q_k}\Phi(x) \neq 0 \bmod (\operatorname{im} Sq^{Q_k}\alpha + \sum_{j,k} \operatorname{im} Sq^{p_{j,k}})$,

then f is incompressible.

COROLLARY. *Let $\pi_n\colon E_n \to K(Z_2, n)$ be the universal fibration classifying $x \in H^n(X; Z_2)$ for which x is annihilated by A_2, the* mod 2 *Steenrod algebra. π_n is incompressible.*

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COROLLARY. *Let $f\colon X \to K(Z_2, n)$ classify $x \in H^n(X; Z_2)$. If x is annihilated by A_2 and for every $k \geq 1$,*

$$Sq^{2^k n} \circ \cdots \circ Sq^n \beta_{(2)}(x) \neq 0 \mod (\text{im } Sq^{2^{k+1}} + \text{im } Sq^{2^k n} \cdots Sq^n Sq^1),$$

then f is incompressible.

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Notation and conventions

All cohomology is assumed to have Z_2 coefficients. If $f\colon X \to Y$ then f^* always denotes the induced homomorphism in cohomology. If $A = (a_t, \ldots, a_1)$ then Sq^A means $Sq^{a_t} \cdots Sq^{a_1}$. The generator of $H^n(K(Z_2, n))$ will be denoted by ι_n, or more simply ι if the context is clear.

If $Y = \prod_{s=1}^k Y_s$, then $p_j\colon Y \to Y_j$ denotes projection and $\text{inc}_j\colon Y_j \to Y$ denotes inclusion.

All diagrams are homotopy commutative and all spectral sequences are assumed to be Z_2 cohomology spectral sequences.

Spectral operations

Let

$$F \xrightarrow{i} E \xrightarrow{\pi} B$$

be a fibration and let $\{E_r, d_r\}$ be its spectral sequence. For information about spectral sequences, see [3]. If there is no confusion we will write $v \in E_r$ and $v \in E_{r+1}$ if $d_r v = 0$ and there is no u such that $d_r u = v$.

In particular, there is the injection $I\colon E_\infty^{0,n} \to E_2^{0,n} = H^n(F)$ such that $I(v) = v$.

Because Z_2 is a field, there is a map $\theta\colon H^n(E) \to \bigoplus_{i=0}^n E_\infty^{i,n-i}$ which is a Z_2 module isomorphism.

Let $F^k H^n(E) = \bigoplus_{i=k}^n E_\infty^{i,n-i}$ and let $\theta_1\colon H^n(E) \to E_\infty^{0,n} \oplus F^1 H^n(E)$ be the isomorphism induced by θ. If $p\colon E_\infty^{0,n} \oplus F^1 H^n(E) \to E_\infty^{0,n}$ is projection, then $i^* = I \circ p \circ \theta_1$.

Kristenson [2] has defined spectral operations Sq^i which are compatible with Steenrod operations. One feature of these operations is that they can increase base filtration. In particular:

LEMMA 1. *Let $u \in F^p H^{p+q}(E)$ and let $q < i \leq p + q$. Then*

$$Sq^i u \in F^{p+i-q} H^{p+q+i}(E).$$

Proof. See Theorem 7.9 of [2].

One consequence of Lemma 1 is that we can deduce that Steenrod operations annihilate elements in $H^*(E)$ by observing that their spectral counterparts push base filtration into regions where E_∞ is zero. We summarize the necessary results:

LEMMA 2. *Suppose* $E_\infty^{p,q} = 0$ *for all* $1 \leq p < n$ *and all* $p > N$ *and let* $u \in H^M(E)$ *be such that* $u \in F^1H^M(E)$. *If for each* $k \geq 1$, $Q_k = (q_k, \ldots, q_1)$ *is a sequence of positive integers such that*

(a) $M - n < q_1 \leq M$,
(b) $2^{k-1}(M - n) < q_k \leq M + \sum_{i=1}^{k-1} q_i$, *for* $k > 1$,

then there is an integer k *such that* $Sq^{Q_k}u = 0$.

Proof. Let $s_k = q_k - 2^{k-1}(M - n)$, $t_k = 1 + \sum_{i=1}^{k} s_k$ and $r_k = M + \sum_{i=1}^{k} q_i$. By Lemma 1,

$$Sq^{Q_1}u \in F^{t_1}H^{r_1}(E) \text{ and in general } Sq^{Q_k}u \in F^{t_k}H^{r_k}(E).$$

For some k, $t_k > N$, and since $E_\infty^{p,q} = 0$ for all $p > N$, $F^{t_k}H^*(E) = 0$. Hence for this k, $Sq^{Q_k}u = 0$. ∎

A relation between secondary operations and primary operations mod filtration

Let $Sq^Q\iota_n = \sum_i Sq^{A_i}Sq^{B_i}\iota_n$ and suppose $Sq^Q\iota_n = 0$. We will define a secondary operation Φ associated to this decomposition of Sq^Q.

Let

$$K_1 = \prod_i K(Z_2, n + \deg B_i) \quad \text{and} \quad K_2 = K(Z_2, n + \deg Q).$$

Let β: $K(Z_2, n) \to K_1$ be such that $p_i \circ \beta$ classifies $Sq^{B_i}\iota_n$. Let α: $K_1 \to K_2$ be such that $\alpha \circ \mathrm{inc}_i$: $K(Z_2, n + \deg B_i) \to K_2$ classifies $Sq^{A_i}\iota$. Since $\alpha \circ \beta \simeq 0$ we can define the following diagram

$$\begin{array}{ccccc}
\Omega K_1 & \xrightarrow{\alpha} & \Omega K_2 & & \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathrm{inc}_2} & & \\
E & \xrightarrow{j} & K(Z_2, n) \times \Omega K_2 & \xrightarrow{p_2} & \Omega K_2 \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_1} & & \\
K(Z_2, n) & \longrightarrow & K(Z_2, n) & & \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha\ \beta} & & \\
K_1 & \xrightarrow{\alpha} & K_2 & &
\end{array}$$

where E is the fibre of β. Let $w \in H^*(E)$ be classified by $p_2 \circ j$. Let $f\colon X \to K(Z_2, n)$ and $x = f^*(\iota_n)$. If $\beta \circ f \simeq 0$ then f has a lifting $\tilde{f}\colon X \to E$ and we define $\Phi(x) = \tilde{f}^*(w)$. Φ is well defined mod (image α).

Let $K^N(Z_2, n) \to K(Z_2, n)$ be the inclusion of the N skeleton and let

$$\Omega K_1 \xrightarrow{i_N} E_N \longrightarrow K^N(Z_2, n)$$

be the pullback fibration associated to this inclusion.

Hence we have a diagram

$$\begin{array}{ccc} \Omega K_1 & \longrightarrow & \Omega K_1 \\ \downarrow{\scriptstyle i_N} & & \downarrow{\scriptstyle i} \\ E_N & \xrightarrow{j_N} & E \\ \downarrow{\scriptstyle \pi_N} & & \downarrow{\scriptstyle \pi} \\ K^N(Z_2, n) & \longrightarrow & K(Z_2, n). \end{array}$$

Since $\beta \circ \pi \circ j_N \simeq 0$, Φ is defined on $y_N = j_N^* \pi^*(\iota_n) \in H^*(E_N)$.

The following lemma establishes a relationship between $\Phi(y_N)$ and primary operations on $H^*(E_N)$.

LEMMA 3. *Let $Q_k = (q_k, \ldots, q_1)$ and let $y_N = j_N^* \pi^*(\iota_n)$. If, for all $k \geq 1$,*

(a) $Sq^{Q_k}(i^*(w)) = \sum_j Sq^{p_{j,k}} v_{j,k}$ *where $0 < p_{j,k} \leq q_k$ for all j,*
(b) $v_{j,k}$ *transgresses to a non-zero element for all j,*

then for every positive integer N, there is a k such that

$$Sq^{Q_k}\Phi(y_N) \bmod (\text{im } Sq^{Q_k}\alpha) = \sum_J Sq^{p_{j,k}} v_{j,k} \bmod F^1 H^*(E_N).$$

Proof. Let k be such that $q_{k-1} + \cdots + q_1 + \deg w > N$. Then $\deg v_{j,k} > N$ and since $v_{j,k}$ transgresses in the spectral sequence of

$$\Omega K_1 \to E \to K(Z_2, n),$$

it follows that $v_{j,k}$ transgresses to 0 in the spectral sequence of

$$\Omega K_1 \to E_N \to K^N(Z_2, n).$$

Since $i_N^*(Sq^{Q_k}\Phi(y_N)) = \sum Sq^{p_{j,k}} v_{j,k}$, it follows that

$$Sq^{Q_k}\Phi(x) \bmod (\text{im } Sq^{Q_k}\alpha) = \sum_j Sq^{p_{j,k}} v_{j,k} \bmod F^1 H^*(E_N).$$

The main results

THEOREM. *Let* $f\colon X \to K(Z_2, n)$ *classify* $x \in H^n(X; Z_2)$, $Q_k = (q_k, \ldots, q_1)$ *be an admissible sequence and* Φ *be a secondary operation associated with* $\alpha \circ \beta = 0$ *and defined on x. If for all positive integers k,*

(a) $Sq^{Q_k}(i^*(w)) = \sum_j Sq^{p_{j,k}}v_{j,k}$ *where* $0 < p_{j,k} \leq q_k$ *for all j,*
(b) $v_{j,k}$ *transgresses to a non-zero element for all j,*
(c) $M_k - n < q_k \leq M_k$ *where* $M_k = \deg w + \sum_{i=1}^{k-1} q_i$,
(d) $Sq^{Q_k}\Phi(x) \neq 0 \bmod (\mathrm{im}\ Sq^{Q_k}\alpha + \sum_{j,k} \mathrm{im}\ Sq^{p_{j,k}})$,

then f is incompressible.

Proof. Suppose f compresses into $K^N(Z_2, n)$ so that there exists

$$f_N\colon X \to K^N(Z_2, n)$$

such that $\mathrm{inc} \circ f_N \simeq f$. By Proposition 1.2 of [1], there exists a map

$$\tilde{f}_N\colon X \to E_N$$

such that $j_N \circ \tilde{f}_N \simeq \tilde{f}$. Let $w_N = j_N^*(w) = \Phi(y_N)$ where $y_N = j_N^* \pi^*(\iota_n)$. By Lemma 3, there is an integer k such that

$$Sq^{Q_k}w_N \bmod (\mathrm{im}\ Sq^{Q_k}\alpha) = \sum_j Sq^{p_{j,k}}v_{j,k} + u$$

where $u \in F^1H^*(E_N)$.

Let $M = \deg u$ and let $\bar{q}_1 = q_{k+1}, \ldots, \bar{q}_s = q_{k+s}$. It is easy to show that for every $s \geq 1$, $\bar{Q}_s = (\bar{q}_s, \ldots, \bar{q}_1)$ satisfies the conditions of Lemma 2 and since $E_\infty^{p,q} = 0$ for all $1 \leq p < n$ and all $p > N$, there is an s such that

$$Sq^{\bar{Q}_s}u = 0.$$

Hence

$$Sq^{Q_{k+s}}w_N = Sq^{\bar{Q}_s}Sq^{Q_k}w_N = \sum Sq^{p_{j,k+s}}v_{j,k+s}$$

Hence

$$\begin{aligned} Sq^{Q_{k+s}}\Phi(x) &= Sq^{Q_{k+s}}\tilde{f}^*(w) = Sq^{Q_{k+s}}\tilde{f}_N^* j_N^*(w) \\ &= Sq^{Q_{k+s}}\tilde{f}_N^*(w_N) = \tilde{f}_N^* \left(\sum Sq^{p_{j,k+s}}v_{j,k+s}\right) \\ &= \sum Sq^{p_{j,k+s}}\tilde{f}_N^*(v_{j,k+s}) \end{aligned}$$

This contradicts hypothesis (d) so that f is incompressible. ▮

Let

$$\Omega K_0 \xrightarrow{i_n} E_n \xrightarrow{\pi_n} K(Z_2, n)$$

be the universal fibration classifying $x \in H^n(X; Z_2)$ for which x is annihilated by A_2, the mod 2 Steenrod algebra. If t is the largest integer such that $2^t \leq n$ and $K_0 = \prod_{s=0}^t K(Z_2, n + 2^s)$, then E_n is the fibre of the map $g\colon K(Z_2, n) \to K_0$ where $p_s \circ g$ classifies $Sq^{2^s}\iota_n$.

LEMMA 4. *Let* $y = \pi_n^*(\iota)$ *and* $Q_k = (2^k n, \ldots, 2n, n)$. *Then*

$$Sq^{Q_k}\beta_{(2)}y \neq 0 \quad \text{mod (im } Sq^{2^{k+1}} + \text{im } Sq^{Q_k}Sq^1).$$

Proof. Suppose to the contrary that $Sq^{Q_k}\beta_{(2)}y \in (\text{im } Sq^{2^{k+1}} + \text{im } Sq^{Q_k}Sq^1)$. Since $Sq^{Q_k}Sq^1$ annihilates $H^n(E_n)$ this implies that

$$Sq^{Q_k}\beta_{(2)}y = Sq^{2^{k+1}}u$$

Let $V_k = (2^k n - 2^k, \ldots, 2n - 2, n - 1)$ and note that $i_n^*(\beta_{(2)}y) = Sq^1\iota_n$. Then

$$Sq^{Q_k}i_n^*(\beta_{(2)}y) = Sq^{Q_k}Sq^1\iota_n = Sq^{2^{k+1}}Sq^{V_k}\iota_n$$

Hence $Sq^{V_k}\iota_n = i_n^*(u) + v$ where $Sq^{2^{k+1}}v = 0$.

But $i_n^*(u)$ and $Sq^{V_k}\iota_n$ transgress and moreover, $Sq^{V_k}\iota_n$ transgresses to the non-zero class $Sq^{V_k}Sq^1\iota$. Hence v transgresses and $v = i_n^*(u)$ in E_∞. But this implies that

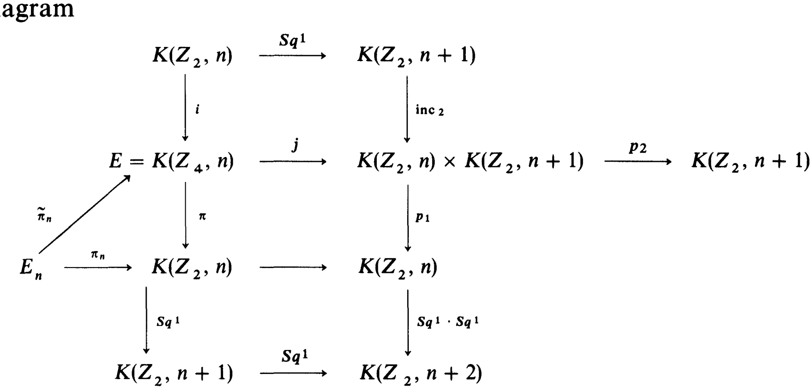
$$0 \neq Sq^{Q_k}Sq^1\iota_n = i_n^*(Sq^{Q_k}\beta_{(2)}y) = i_n^*(Sq^{2^{k+1}}u) = Sq^{2^{k+1}}v = 0.$$

Hence

$$Sq^{Q_k}\beta_{(2)}y \notin (\text{im } Sq^{2^{k+1}} + \text{im } Sq^{Q_k}Sq^1)$$

COROLLARY 1. π_n *is incompressible.*

Proof. Let Q_k and V_k be as in Lemma 4. Let Φ be the secondary operation $\beta_{(2)}$. That is, Φ is defined by the relation $Sq^1Sq^1 = 0$. Then we have the diagram



It suffices to show that hypotheses (a), (b), (c), and (d) of the theorem are satisfied. To check (a) we identify w as $\beta_{(2)}(\iota)$ and $i^*(w)$ as $Sq^1\iota_n$ and use the Adem relations to deduce

$$Sq^{Q_k}Sq^1\iota_n = Sq^{2^{k+1}}Sq^{V_k}\iota_n.$$

Then there is one summand with $p_{1,k} = 2^{k+1}$, $v_{1,k} = Sq^{V_k}\iota_n$.

Condition (b) follows since $Sq^{V_k}\iota_n$ transgresses to $Sq^{V_k}Sq^1\iota \neq 0$.

Condition (c) follows by observing that

$$M_k = n + 1 + \sum_{j=0}^{k-1} 2^j n = 2^k n + 1.$$

Condition (d) is Lemma 4. ∎

COROLLARY 2. *Let* $Q_k = (2^k n, \ldots, n)$. *Let* $f\colon X \to K(Z_2, n)$ *classify* $x \in H^n(X; Z_2)$. *If* x *is annihilated by* A_2 *and for every* $k \geq 1$,

$$Sq^{Q_k}\beta_{(2)}(x) \neq 0 \quad \text{mod } (\text{im } Sq^{2^{k+1}} + \text{im } Sq^{Q_k}Sq^1),$$

then f *is incompressible.*

Proof. Conditions (a), (b), and (c) of the theorem are exactly as in Corollary 1. ∎

REFERENCES

1. J. GOLDFEATHER, *Incompressibility and fibrations*, Illinois J. Math., vol. 21 (1977), pp. 688–702.
2. L. KRISTENSON, *On the cohomology of two stage Postnikov systems*, Acta Math., vol. 107 (1962), pp. 73–123.
3. J. P. SERRE, *Homologie singuliere des espaces fibres*, Ann. of Math., vol. 54 (1951), pp. 425–505.

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