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BRANCH SET LINKING FOR NON-ORIENTABLE MANIFOLDS

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1. Introduction

The main theorem generalizes the branch set linking theorems of E. Hemmingsen and W. L. Reddy [5] and the authors [4]. We prove that if $f: M \to N$ is a nice branched covering of (not necessarily orientable) n-manifolds ($n \geq 3$) and $B \subset B_f$ such that $f(B)$ and $f^{-1}f(B) = B$ are nicely embedded $(n-2)$-dimensional manifolds then $f(C)$ links $f(B)$ if C is any path-connected set such that $C \subset M - B$ and $f^{-1}f(C) = C$. This is made precise in the theorem.

This paper is the direct extension of [4] to non-orientable manifolds. The main problem is the necessity to use cohomology with co-efficients in a sheaf. Since the arguments are often the same as those in [4] the reader should have [4] as a ready reference. The geometrical notion which leads to our technique was told to one of us by E. Hemmingsen in 1970.

2. Preliminaries

Cohomology will be sheaf cohomology. If X is an n-manifold then $\mathcal{O}_X$ will denote its orientation sheaf.

DEFINITION. Let $f: M \to N$ be a continuous map between n-manifolds. We say that f is sense preserving if $f^*(\mathcal{O}_N) \cong \mathcal{O}_M$.

Let $f: M \to N$ be a sense-preserving pseudo-covering map [3, Def. 5], between compact connected n-manifolds. There is an integer, deg f, such that if ξ_M is a generator of $H^n(M: \mathcal{O}_M)$ and ξ_N a generator of $H^n(N; \mathcal{O}_N)$ (both are infinite cyclic) then $f^*(\xi_N) = (\deg f)\xi_M$. We shall assume $|\deg f| \geq 1$. Let B_f denote the branch set of f and let $B \subset B_f$ be a closed connected $(n-2)$-dimensional submanifold of M such that:

(a) $B = f^{-1}(f(B))$ and $f(B)$ is an $(n-2)$-dimensional submanifold of N.
(b) $H^{n-2}(B; i^*(\mathcal{O}_M))$ and $H^{n-2}(f(B); i^*(\mathcal{O}_N))$ are infinite cyclic groups.
(c) The homomorphisms

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$$H^{n-2}(M; \mathcal{O}_M) \to H^{n-2}(B; i^*(\mathcal{O}_M)) \quad \text{and} \quad H^{n-2}(N; \mathcal{O}_N) \to H^{n-2}(f(B); i^*(\mathcal{O}_N))$$

induced by the inclusions are both zero.

(d) The groups $H^{n-1}(M; \mathcal{O}_M)$ and $H^{n-1}(N; \mathcal{O}_N)$ are free ($\{0\}$ is a free group).

The assumptions (a)–(d) and that f be a sense-preserving map between compact connected n-manifolds shall be referred to as the standing hypothesis. *We shall assume the standing hypothesis.*

LEMMA 1. (a) *We have Diagram* 1 *in which Φ is the duality isomorphism and δ the co-boundary homomorphism.*

(b) $f_* \circ \Phi \circ f^* = (\deg f)\Phi$

$$\begin{tikzcd}
H_1(M - B) \arrow[r, "f_*"] & H_1(N - f(B)) \\
H_c^{n-1}(M, B; \mathcal{O}_M) \arrow[u, "\Phi"] & H_c^{n-1}(N, f(B); \mathcal{O}_N) \arrow[u, "\Phi"] \arrow[l, "f^*"'] \\
H^{n-2}(B; i^*(\mathcal{O}_M)) \arrow[u, "\delta"'] & H^{n-2}(f(B); i^*(\mathcal{O}_N)) \arrow[u, "\delta"'] \arrow[l, "f^*"']
\end{tikzcd}$$

DIAGRAM 1

Proof (a) The existence of the diagram follows from the Alexander-Lefschetz duality theorem for sheaves [2, V, 9.3], together with the fact that Borel-Moore homology with compact supports is the same as singular homology for manifolds [2, V, II]. The commutativity of the bottom square follows from the fact that f is sense-preserving.

(b) We note that middle groups in the diagram are given by

$$H_c^{n-1}(M - B; i^*(\mathcal{O}_M)) \quad \text{and} \quad H_c^{n-1}(N - f(B); i^*(\mathcal{O}_N)).$$

Since $M - B$ and $N - f(B)$ are open, the inclusion maps into M and N are sense-preserving. Thus Φ is the respective duality mapping for $M - B$ or $N - f(B)$ and is given by cap product with the fundamental class [2, V, 10.2]. The standard formula $f_*(\alpha \cap f^*(\beta)) = f_*(\alpha) \cap \beta$ completes the proof.

DEFINITION. The images of $\Phi \circ \delta$ in $H_1(M - B)$ and $H_1(N - f(B))$ in Diagram 1 are called the subgroups transverse to B and $f(B)$ respectively.

LEMMA 2. (a) *In Diagram* 1, *the homomorphisms denoted by δ are monomorphisms onto a direct summand.*

(b) *The homomorphism f_* (in Diagram* 1*) maps the subgroup transverse to B into the subgroup transverse to $f(B)$.*

The proof is similar to those of Lemmas 1 and 2 of [4].

3. Localization

In this section, the action of f_* on a transverse subgroup is reduced to studying the action of f_* on a Church-Hemmingsen neighborhood whose definition we now recall.

DEFINITION. An open set U in M is called a Church-Hemmingsen neighborhood if there is a commutative diagram of maps of pairs.

$$\begin{tikzcd}
(\mathrm{cl}\, U, \mathrm{cl}\,(U \cap B)) \arrow[r, "g"] \arrow[d, "f|_{\mathrm{cl}\, U}"] & (D^{n-2} \times D^2, D^{n-2} \times \{0\}) \arrow[d, "\mathrm{id} \times z^d"] \\
(f(\mathrm{cl}\, U), f(\mathrm{cl}\,(U \cap B))) \arrow[r, "h"] & (D^{n-2} \times D^2, D^{n-2} \times \{0\})
\end{tikzcd}$$

in which g and h are homeomorphisms and $|d| > 1$. Here cl denotes closure and D^k is the closed k-disc. d is called the local degree on $U \cap B$. We *assume* B has a Church-Hemmingsen neighborhood.

LEMMA 3. *Let U be a Church-Hemmingsen neighborhood and let $V = U \cap B$. Then Diagram 2 is a commutative diagram where Φ is the duality isomorphism and δ the co-boundary.*

$$\begin{tikzcd}
H_1(U - V) \arrow[r, "i_{0*}"] & H_1(M - B) \\
H_c^{n-1}(M, M - (U - V); \mathcal{O}_M) \arrow[u, "\Phi"] \arrow[r, "i_1^*"] & H_c^{n-1}(M, B; \mathcal{O}_M) \arrow[u, "\Phi"] \\
H^{n-2}(M - (U - V); i^*(\mathcal{O}_M)) \arrow[u, "\delta"] \arrow[r, "i_2^*"] & H^{n-2}(B; i^*(\mathcal{O}_M)) \arrow[u, "\Phi"]
\end{tikzcd}$$

DIAGRAM 2

Proof. The bottom square is part of the cohomology ladder for

$$(M, B) \to (M, M - (U - V)).$$

The top square is commutative because of the naturality of Φ with respect to sheaves and hence inclusions [2, Theorem V.9.2].

The analogous statement about the range is true.

LEMMA 4. *Diagram 3 is a commutative diagram which is exact at the middle of the top row. Moreover, $i_2^* \circ \delta^*$ is an isomorphism. Here δ^* is the Mayer-Vietoris coboundary,* cl *denotes closure, and $H^p(A)$ denotes $H^p(A; i^*(\mathcal{O}_M))$ for any p and any $A \subset M$.*

$$\begin{tikzcd}
H^{n-3}((B - U) \cap \mathrm{cl}\, V) \arrow[r, "\delta^*"] & H^{n-2}(B) \arrow[r] & H^{n-2}(B - U) \otimes H^{n-2}(\mathrm{cl}\, V) \\
H^{n-3}((M - U) \cap \mathrm{cl}\, V) \arrow[u, "i^* = \mathrm{id}"] \arrow[r, "\delta^*"] & H^{n-2}((M - U) \cup V) \arrow[u, "i_2^*"] &
\end{tikzcd}$$

DIAGRAM 3

Proof. The Mayer-Vietoris sequence follows from [2, page 68]. It is easily seen that $(M - U) \cap \text{cl } V = (B - U) \cap \text{cl } V$, hence i^* is the identity. The square commutes because all maps are induced by inclusions. Exactness at $H^{n-2}(B)$ follows from the Mayer-Vietoris sequence. We next note that cl V is homeomorphic to a disc and that $(B - U) \cap \text{cl } V$ is homeomorphic to $\partial D^{n-2} = S^{n-3}$. This implies that $H^{n-2}(\text{cl } V) = \{0\}$ and that the inclusion $S^{n-3} \cong (B - U) \cap \text{cl } V \to \text{cl } V \to M$ is null-homotopic. Since $\mathcal{O}_M$ is a locally-constant sheaf this implies that $\mathcal{O}_M(B - U) \cap \text{cl } V$ is the constant sheaf. Thus $H^{n-3}((B - U) \cap \text{cl } V)$ is an infinite cyclic group. Thus, since $H^{n-2}(B)$ is an infinite cyclic group, to show $i_2^* \circ \delta^*$ is an isomorphism we need only show that the top δ^* is surjective. This, by exactness, will follow from $H^{n-2}(B - U) = \{0\}$ which we now show. Since $B - U = B - V$ we may apply duality and obtain

$$H^{n-2}(B - U; i^*(\mathcal{O}_M)) \cong H_0(B, V; \mathcal{O}_B^{-1} \otimes i^*(\mathcal{O}_M)).$$

The sheaf $\mathcal{O}_B^{-1} \otimes i^*(\mathcal{O}_M)$ is a locally constant sheaf induced by a homomorphism from the fundamental group of B into $\mathbf{Z}_2$. This homomorphism is either trivial or surjective. In the first case

$$H_0(B, V; \mathcal{O}_B^{-1} \otimes i^*(\mathcal{O}_M) = \{0\}$$

because B is path connected and the sheaf is constant. In the second case one checks the definition to see that there are no non-trivial chains in dimension 0.

The analogous statement is true about the range. We denote by $T(B)$ (resp. $T(f(B))$) the subgroup transverse to B (resp. $f(B)$).

PROPOSITION 5. *Diagram 4 is a commutative diagram in which i_{0*} is an isomorphism of $H_1(U - V)$ onto $T(B)$. Here $H^p(A) = H^p(A; i^*(\mathcal{O}_M))$ for any $A \subseteq M$.*

$$\begin{array}{ccc}
H_1(U - V) & \xrightarrow{i_0^*} & H_1(M - B) \\
\uparrow{\scriptstyle \Phi} & & {\scriptstyle \Phi} \\
H_c^{n-1}(M, M - (U - V)) & \xrightarrow{i_1^*} & H_c^{n-1}(B) \\
\uparrow{\scriptstyle \delta \circ \delta^*} & & {\scriptstyle \delta} \\
H^{n-3}((M - U) \cap \text{cl } V) & \xrightarrow{i_2^* \circ \delta^*} & H^{n-2}(B)
\end{array}$$

DIAGRAM 4

Proof. Diagram 4 is formed by hooking together Diagram 2 together with Diagram 3 and noting that $(M - U) \cup V = M - (U - V)$. Since $i_2^* \circ \delta^*$ is an isomorphism,

$$(\Phi \circ \delta \circ i_2^* \circ \delta^*)(H^{n-3}((M - U) \cap \text{cl } V))$$

is $T(B)$; thus i_0^* is surjective. Since $H_1(U - V)$ and $T(B)$ are infinite cyclic groups i_0^* is an isomorphism.

4. A linking theorem

DEFINITION. Under the standing hypotheses f will be said to be transverse to B if there are complements to $T(B)$ and $T(f(B))$ denoted by $T(B)^{\perp}$ and $T(f(B))^{\perp}$ such that $f_*(T(B)^{\perp}) \subset T(f(B))^{\perp}$.

THEOREM 7. *Let f: $M \to N$ be transverse to B and suppose B has a Church-Hemmingsen neighborhood with local degree d. Let $C \subset M - B$ be path connected and such that $f^{-1}(f(C)) = C$. Then the image of $H_1(f(C))$ in $H_1(N - f(B))$ under the inclusion homomorphism is* **not** *contained in $T^{\perp}(f(B))$.*

The proof in [4, page 332] is valid here.

COROLLARY 8. *Under the hypothesis of the theorem, the image of $H_1(C)$ in $H_1(M - B)$ under the inclusion homomorphism is not contained in $T^{\perp}(B)$.*

Proof. By the theorem there is a $c \in H_1(f(C))$ such that

$$i_*(c) \notin T^{\perp}(f(B)).$$

Then $i_*(c) = a \oplus b$ with, $a \neq 0$, $a \in T(f(B))$, and $b \in T^{\perp}(f(B))$. Let γ be a closed curve in $f(C)$ representing c. By repeated application of the path-lifting theorem [6, Theorem $X(2.1)$, page 186] there is a closed path $\tilde{\gamma}$ such that $f \circ \tilde{\gamma} = \gamma^q$ for some $q \geq 1$. Let $\tilde{c}$ denote the homology class of $\tilde{\gamma}$ in $H_1(C)$. Then,

$$i_*(f_*(\tilde{c})) = qc = qa \oplus qb.$$

Since $T(B)$ is torsion free, $qa \neq 0$. Thus,

$$i_*(f_*(c)) \notin T^{\perp}(f(B)).$$

But

$$f_*(T^{\perp}(B)) \subset T^{\perp}(f(B))$$

by hypothesis, hence $i_*(c) \notin T^{\perp}(B)$.

5. Applications

In this section we consider when the theorem may be applicable and make some remarks concerning some of our hypothesis.

Example. Consider g: $S^n \to S^n$ obtained by suspending (an odd number of times) the $p - 1$ branched covering of S^3 to itself in which B_f and $f(B_f)$ are linked circles. Take the connected sum of $\mathbf{R}P^n$ (non-orientable since n is even) with the image and the equivalent connected sum of p disjoint copies of $\mathbf{R}P^n$ with the domain. The result is a branched covering $\tilde{g}$: $M \to N$ (of non-orientable manifolds) to which the theorem can be applied.

The theorem in fact states that the linking behavior of the cases governed by the hypothesis are similar to those of the example.

We next consider the hypothesis that f be sense-preserving, i.e., $f^*(\mathcal{O}_N) \cong \mathcal{O}_M$. We have the following:

PROPOSITION 9. *Let $f: M \to N$ be a pseudo-covering between closed n-manifolds such that B_f is a co-dimension 2 submanifold, $f^{-1}(f(B_f)) = B_f$, and every point of B_f has a Church-Hemmingsen neighborhood. Then $f^*(\mathcal{O}_N) \cong \mathcal{O}_M$.*

Proof. Let

$$W^M = 1 + W_1^M + \cdots \in H^1(M; \mathbf{Z}_2) \quad \text{and} \quad W^N = 1 + W_1^N + \cdots \in H^1(N; \mathbf{Z}_2)$$

be the respective total Stiefel-Whitney class. As is well known $\mathcal{O}_M$ and $\mathcal{O}_N$ are classified by W_1^M and W_1^N respectively. Thus we must show $f^*(W_1^N) = W_1^M$. However, under the hypothesis of the theorem, Brand [1, Corollary 3] has shown that

$$W(\tau(M) - f^*(\tau(N))) = 1 + x_1 + x_2 + \cdots \quad \text{with } x_1 = 0.$$

Here, τ denotes the tangent bundle. Thus $W^M = f^*(W^N)(1 + 0 + X_2 + \cdots)$ and $W_1^M = f^*(W_1^N)$.

This shows that the strongest hypotheses of the theorem are those needed to define $T(B)$, and the condition that f be transverse.

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