ILLINOIS JOURNAL OF MATHEMATICS
Volume 26, Number 3, Fall 1982

VARIANTS OF BLUMBERG'S THEOREM

BY

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1. Introduction

In [4], J. B. Brown showed that the following statement, which is a variant of Blumberg's theorem [3], holds for any metric space X that is c-typically dense in itself [4, p. 244].

1.1 If f is a real valued function defined on X, then there are subsets D and E of X such that D is contained in E, D is dense in X, $U \cap E$ is of cardinality at least c $(= 2^{\omega})$ for every non-empty open subset U of X, and $f \mid E$ is continuous at every point of D.

Every complete metric space is c-typically dense in itself [4, p. 251], and every topological space for which 1.1 holds is a Baire space [3, p. 667]. In Sections 2 and 3 of this paper, we shall show that an argument of Blumberg's can be used to prove that 1.1 holds for every space in a class $\mathscr{C}$ (the ωBc $\sigma\pi$ spaces) of topological spaces that includes every c-typically dense in itself metric space. If $c = \omega_1$, then any metric space for which 1.1 holds is c-typically dense in itself [4, p. 249]. In Section 5, we shall show that every weakly T_1 $\sigma\pi$ space for which 1.1 holds is in $\mathscr{C}$. In sections 4 and 6, we shall study $\mathscr{C}$ briefly.

The author wishes to thank the referee for noting that the original version of Corollary 4.9 needed an extra hypothesis (that cf $n >$ m where cf n denotes the cofinality of n [7, p. 166]), and making several suggestions that improved the presentation of the results.

2. An argument of Blumberg's

In this section, we shall prove a technical result, Lemma 2.1, that can be used for handling, in the context of metrizable spaces (or $\sigma\pi$ spaces), almost any variant of Blumberg's theorem. The proof of Lemma 2.1 is just Blumberg's proof [1]; however, instead of real valued functions defined on metric spaces, we consider functions defined on spaces of a slightly more general type and taking values in first countable spaces. This allows us to prove corollary 2.2, which is useful in a certain area of topology [17].

The proof of a variant of Blumberg's theorem for a certain class of metric (or near metric) spaces consists of two main parts. Given the function f, part

Received May 18, 1979.

[1] This work was supported, in part, by the Institute for Medicine and Mathematics, Ohio University.

one consists of finding an "appropriately dense" subset C of the domain X of f such that $f \mid C$ has a "weak continuity" property. This first step does not use the fact that X is metric (or near metric). The second part consists of using the metric (or near metric) structure of C, together with the "weak continuity" of f, to find a much "smaller" dense subset D of C and a slightly "larger" set E that is not necessarily between D and C, such that $f \mid E$ is continuous at each point of D. Lemma 2.1 performs the second step.

It is curious that Proposition 1.7 of [16] follows easily from the theorem in [3], while Theorem 3.1, with $m = \omega$, $n = \omega_1$, and Y metrizable, does not seem to follow easily from Theorem 1 of [4], by a similar argument.

The cardinal number of a set S will be denoted by $|S|$. If $\mathscr{F}$ denotes a collection of subsets of S, then: $\mathscr{F}^*$ denotes $\mathscr{F} \sim \{\phi\}$; $\bigcup \mathscr{F}$ (respectively, $\bigcap \mathscr{F}$) denotes

$$\bigcup \{F\colon F \in \mathscr{F}\} \quad \text{(respectively, } \bigcap \{F\colon F \in \mathscr{F}\});$$

and, for any subset A of S, $\mathscr{F} \cap A$ denotes $\{F \cap A\colon F \in \mathscr{F}\}$. A sequence $(A_i, i < \omega)$ will usually be denoted by (A_i); the set $\bigcup \{A_i\colon i < \omega\}$ (respectively, $\bigcap \{A_i\colon i < \omega\}$) will be denoted by $\bigcup_i A_i$ (respectively, $\bigcap_i A_i$).

Suppose $(X, \mathscr{T})$ is a topological space. We shall denote $\mathscr{T}$ by tX and speak of "the topological space X". For any subset A of X, the closure of A is denoted by cl A and the interior of A by int A. We shall denote the collection of all nowhere dense subsets of X by NX. Suppose $\mathscr{K}$ is a subset of NX. We shall say a subset E of X is $\mathscr{K}$ dense in X if, whenever $U \in tX^*$ then $U \cap E$ contains an element of $\mathscr{K}$. Note that a subset K of X is nowhere dense in X if and only if for every U in tX^*, $U \cap X$ is not dense in U. If f is a function defined on X, taking values in the topological space Y, then we shall denote by $\mathscr{B}(f, \mathscr{K})$ the set of all ordered pairs (D, E), where E is a $\mathscr{K}$ dense subset of X, D is a subset of E that is dense in X, and $f \mid E$ is continuous at every point of D.

Now we shall define the class of spaces for which Blumberg's argument is valid. A pseudo-base for a space X is a subset $\mathscr{P}$ of tX^* such that every element of tX^* contains an element of $\mathscr{P}$. A pseudo-base is called σ-disjoint if it is the union of a countable number of disjoint subcollections of tX^*. A space with a σ-disjoint pseudo-base will be called a $\sigma\pi$ space. A pseudo-base $\mathscr{P}$ for a $\sigma\pi$ space X is called a standard pseudo-base for X if $\mathscr{P} = \bigcup \mathscr{P}_i$, where $\mathscr{P}_0 = \{X\}$, and for each i, $\mathscr{P}_{i+1}$ is a disjoint subcollection of tX^* that refines $\mathscr{P}_i$.

A space X will be called tractable if, whenever $x \in X$ and cl $\{x\} \in NX$, then

(*) there is a countable subcollection $\mathscr{U}$ of tX^* such that $x \in \bigcap \mathscr{U}$ and $\bigcap \{\text{cl } U\colon U \in \mathscr{U}\} \in NX$.

We note that the following types of spaces are tractable:

(1) a first countable Hausdorff space;
(2) a regular space in which every point is a G_δ;

(3) a $\sigma\pi$ space X that has a standard pseudo-base $\mathscr{P}$ such that, for each i, $\bigcup \mathscr{P}_i = X$.

There are spaces (see 6.4) of the third type in which no point is a G_δ.

If f is a function from the space X into the space Y and $x \in X$, then we shall say that f is δ continuous at x if, for every V in tY such that $f(x) \in V$, there is a subset A of X such that $x \in \operatorname{int} \operatorname{cl} A$ and $f[A] \subset V$.

2.1 LEMMA. *Suppose f is a function from the tractable $\sigma\pi$ space X into the first countable space Y, and $\mathscr{K}$ is a subset of NX. If*

(1) *there is a dense subset C of X such that $f \mid C$ is δ continuous at every point of C, and*

(2) *whenever $U \in tX$, $V \in tY$, and $U \cap f^{-1}[V] \neq \phi$, then $U \cap f^{-1}[V]$ contains an element of $\mathscr{K}$,*

then $\mathscr{B}(f, K) \neq \phi$.

If, in addition,

(3) $tX \subset f^{-1}[tY]$,

then there is (D, E) in $\mathscr{B}(f, \mathscr{K})$ such that $tX \cap D$ is metrizable.

Proof. Let W be a function from $Y \times \omega$ into tY such that for each y in Y, $(W(y, i))$ is a non-increasing local base at y. Let

$$I = \{x \in X: \operatorname{int} \operatorname{cl} \{x\} \neq \phi\}, \quad G = \bigcup \{\operatorname{int} \operatorname{cl} \{x\}: x \in I\},$$

and $H = X \sim \operatorname{cl} G$. It suffices to consider two cases: $X = G$; $X = H$.

Case 1. Suppose $X = G$. In this case, the proof is easy. Of course, $D = I$. The set E is constructed as follows. Suppose $x \in I$; because X is a $\sigma\pi$ space, it satisfies the first axiom of countability at x. Let $(S_i(x))$ be a local base at x, each element of which is contained in $\operatorname{int} \operatorname{cl} \{x\}$. Define a sequence $(K_i(x))$ of elements of $\mathscr{K}$ so that for each i,

$$K_i(x) \subset S_i(x) \cap f^{-1}[W(f(x), i)].$$

If

$$E = D \cup \bigcup \{K_i(x): x \in I, i < \omega\},$$

then $(D, E) \in \mathscr{B}(f, \mathscr{K})$.

Case 2. Suppose $X = H$. Because (1) holds, there is a function B from $C \times \omega$ into tX such that: if x is in C, then, for each i,

$$x \in B(x, i + 1) \subset B(x, i)$$

and $B(x, i) \cap C \cap f^{-1}[W(f(x), i)]$ is dense in $B(x, i)$; and

$$\bigcap_i \operatorname{cl} B(x, i) \in NX.$$

Let

$$\mathscr{F} = \{U \cap f^{-1}[V]: U \in tX, V \in tY\},$$

and let γ be a function from $\mathscr{F}^*$ into $\mathscr{K}$ such that $\gamma(F) \subset F$ for every F in $\mathscr{F}^*$. Let $\gamma(\phi) = \phi$, and let $\mathscr{P}$ be a standard pseudo-base for X.

Define, by induction, a sequence $(S_i, D_i, \varepsilon_i)$ where for each i, D_i is a subset of C, ε_i is a function from D_i into ω, and S_i is a function from D_i into tX are such that, for each i, the following hold: $S_i[D_i]$ is a disjoint collection such that $\bigcup S_i[D_i]$ is dense in X; $S_{i+1}[D_{i+1}]$ refines $S_i[D_i]$; $D_i \subset D_{i+1}$; $\varepsilon_{i+1}(x) = \varepsilon_i(x) + 1$ for all x in D_i; $S_{i+1}[D_{i+1} \sim D_i]$ refines $\mathscr{P}_{i+1}$; if $x \in D_i$, then

$$x \in S_i(x) \subset B(x, \varepsilon_i(x));$$

if $x \in D_i$, $x' \in D_{i+1} \sim D_i$, and $S_{i+1}(x') \subset S_i(x)$, then

$$W(f(x'), \varepsilon_{i+1}(x')) \subset W(f(x), \varepsilon_i(x)) \quad \text{and} \quad S_{i+1}(x') \cap K_i(x) = \phi,$$

where

$$K_i(x) = \gamma(f^{-1}[W(f(x), \varepsilon_i(x))] \cap [S_i(x) \sim \text{cl } S_{i+1}(x)]).$$

Let $D = \bigcup_i D_i$ and $E = D \cup \bigcup_i \{K_i(x): x \in D_i\}$. To show that D is dense in X and E is $\mathscr{K}$ dense in X, it suffices to show that $\bigcup_i S_i[D_i]$ is a pseudo-base for X. To do this, suppose that $P \in \mathscr{P}_i$. Then there is an x in D_i such that $P \cap S_i(x) \neq \phi$. Because

$$\bigcap \{\text{cl } S_j(x): j \geq i\} \in NX,$$

there is a k such that $k \geq i$ and

$$U = P \cap [S_k(x) \sim \text{cl } S_{k+1}(x)] \neq \phi.$$

Because $S_{k+1}[D_{k+1} \sim D_k]$ refines $\mathscr{P}_{k+1}$, there is an x' in D_{k+1} such that $S_{k+1}(x') \subset U \subset P$.

If $x \in D$, then $f \mid E$ is continuous at x, because for each i,

$$f[E \cap S_i(x)] \subset W(f(x), \varepsilon_i(x)).$$

Finally, if (3) holds, then $[\bigcup_i S_i[D_i]] \cap D$ is a σ-discrete base for $f^{-1}[tY] \cap D = tX \cap D$. ▮

The main use of Lemma 2.1 is in proving Theorem 3.1. However, the following corollaries are also of some interest.

2.2 COROLLARY. *Every first countable Hausdorff $\sigma\pi$ space has a dense metrizable subspace.*

Proof. If X is a first countable Hausdorff $\sigma\pi$ space, let f be the identity mapping of X into X and let $\mathscr{K} = \phi$. Because $tX \subset f^{-1}[tX]$, Lemma 2.1 implies that X has a dense metrizable subspace. ▮

If X is regular and has a dense metrizable subspace, then it is a $\sigma\pi$ space. As Example 6.4 shows, the converse of this statement is false. The following result indicates, however, that many $\sigma\pi$ spaces have metrizable spaces associated with them, in a rather simple fashion.

2.3 COROLLARY. *If the $\sigma\pi$ space X has a dense tractable subspace, then there is a dense subset D of X and a topology $\mathscr{W}$ such that $(D, \mathscr{W})$ is metrizable and $\mathscr{W}$ is a pseudo-base for $(tX) \cap D$.*

Proof. We may assume that X is tractable. Referring to the proof of Lemma 2.1, let $\mathscr{W}$ be the topology on D generated by

$$(\bigcup_i S_i[D_i]) \cap D. \quad \blacksquare$$

Corollary 2.3 is trivial if X is a Baire space. For let $\mathscr{P}$ be a standard pseudo-base for X and let D be a subset of $\bigcap_i [\bigcup \mathscr{P}_i]$ such that if (P_i) is a sequence for which $P_i \in \mathscr{P}_i$ for each i and $\bigcap_i P_i \neq \phi$, then $|D \cap \bigcap_i P_i| = 1$. Let $\mathscr{W}$ be the topology generated by $\mathscr{P} \cap D$; then $(D, \mathscr{W})$ is a metrizable Baire space. If, in addition, X is α-favorable (see Section 6), then D may be chosen so that $(D, \mathscr{W})$ is completely metrizable.

Suppose f is a function from the space X into the space Y and $x \in X$. We shall say f is Δ continuous at x if there is a subset A of X such that $x \in \operatorname{int} \operatorname{cl} A$ and $f \mid A \cup \{x\}$ is continuous at x. It is clear that if f is Δ continuous at x, then f is δ continuous at x.

2.4 COROLLARY. *If f is a function from a regular $\sigma\pi$ space X into a first countable T_1 space Y that is Δ continuous at every x in X, then there is a dense subset D of X such that $f \mid D$ is continuous.*

Proof. Without loss of generality, we may, and do, assume that, for each x in X, $\{x\}$ is nowhere dense in X. Let X^+ be the set of all points x of X for which $(*)$ holds. Let $G = \operatorname{int} X^+$ and $H = X \sim \operatorname{cl} G$. It suffices to consider two cases, when $X = G$ and when $X = H$. If $X = G$, then Lemma 2.1, with $\mathscr{K} = \phi$ and $C = X$, implies the existence of the required set D. Suppose $X = H$. It suffices to show that if $U \in tX^*$, then f is constant on some subset of U that is not nowhere dense in X. Given U, choose x in $U \sim X^+$ and a subset A of X such that $x \in \operatorname{int} \operatorname{cl} A$ and $f \mid A \cup \{x\}$ is continuous at x. Let (V_i) be a local base at $f(x)$, and define a sequence (U_i) of open subsets of int cl A that contain x such that for each i, $\operatorname{cl} U_{i+1} \subset U_i$ and $f[A \cap U_i] \subset V_i$. If $K = A \cap [\operatorname{int} \bigcap_i U_i]$, then $K \notin NX$ and $f(y) = f(x)$ for every y in A. $\blacksquare$

We shall denote the real line by R, the Euclidean topology on R by tR, and Lebesgue outer measure on R by μ^*.

2.5 COROLLARY. *If f is a real-valued function defined on R for which*

$$\mu^*(U \cap f^{-1}[V]) > 0$$

whenever $U, V \in tR$ and $U \cap f^{-1}[V] \neq \phi$, then there are subsets D and E of R such that D is dense in R, $D \subset E$, $\mu^*(E \cap U) > 0$ for all U in tR^*, and $f \mid E$ is continuous at every point of D.

Proof. Suppose f is as hypothesized. Because R is a metric Baire, there is a dense subset C of R such that $f \mid C$ is continuous. Lemma 2.1, with $\mathscr{K} = \{K \in NR: \mu^*(K) > 0\}$ implies the existence of the required sets D and E. ▮

In [5], J. B. Brown gives an example that shows the conclusion of Corollary 2.5 is false for some real valued functions defined of R.

3. A generalization of two theorems of J. B. Brown

In this section, we shall prove a statement, Theorem 3.1, that generalizes Theorems 1 and 1′ of [4].

We shall use m and n to denote cardinal numbers. If X is a topological space and $n \geq \omega_1$, then we denote by nNX the collection of all nowhere dense subsets of X of cardinality at least n. We denote by $nGNX$ the collection of all subsets A of X such that for every U in tX^*, $U \cap A$ is not nNX dense in U. The elements of $nGNX$ can be described as "generalized nowhere dense" subsets of X. It is convenient to let $0GNX = NX$. Note that a subset of a T_1 space that has no isolated points is dense in X if and only if it is NX^* dense in X. If $m \geq \omega$ and either $n = 0$ or $n \geq \omega_1$, we shall denote by $mMnX$ the collection of all subsets of X that are unions of subcollections of $nGNX$ of cardinality at most m. Elements of $mMnX$ can be described as "generalized meager" subsets of X. In fact, $\omega M0X$ consists precisely of the meager subsets of X.

Suppose X is metrizable. It follows from 4.3 that $\omega M\omega_1 X$ consists of what J. B. Brown calls the nowhere typically dense [4, p. 244] subsets of X. And it is easily verified that every element of ωMcX is what J. B. Brown calls nowhere c-typically dense [4, p. 250].

We shall say X is an mBn space if $(tX^*) \cap mMnX = \phi$. It follows that: a space is a Baire space if and only if it is an $\omega B0$ space; a metric space is an $\omega B\omega_1$ space if and only if it is typically dense in itself [4, p. 244]; any c-typically dense in itself [4, p. 250] metric space is an ωBc space.

3.1 THEOREM. *If f is a function from a $\sigma\pi$ mBn space X into a first countable space Y of weight* [7, p. 164] *at most m, then $\mathscr{B}(f, nNX) \neq \phi$.*

Remark. Theorem 3.1, with $n = 0$, is a generalization of the "if" part of the theorem in [2]. If $m = \omega$ and $n = 0$, Theorem 3.1 is essentially Proposition 1.7 of [16]. If $m = \omega$ and $n = \omega_1$, then Theorem 3.1 generalizes Theorem 1 of [4]; if $m = \omega$ and $n = c$, then it generalizes Theorem 1′ of [4].

As Professor B. J. Pettis observed, Theorem 3.1 implies the following generalization of itself.

3.2 COROLLARY. *Suppose X is a $\sigma\pi$ mBn space. If for each i, f_i is a function from X into a first countable space Y_i of weight at most m, then $\bigcap_i \mathscr{B}(f_i, nNX) \neq \phi$.*

Proof. Apply Theorem 3.1 to f, where Y is the product of the Y_i and $f(x) = (f_i(x))$ for all x in X. ▌

We shall now prove Theorem 3.1. The proof is essentially the same as the proof of the theorem in [3]. In particular, the next few lines are similar to Lemmas 1, 2, and 3 of [3]. Therefore, the presentation is kept brief.

3.3 LEMMA. *If $\mathscr{F}$ is a subset of $mMnX$ of cardinality at most m, then $\bigcup \mathscr{F} \in mMnX$.*

3.4 LEMMA. *Suppose A is a subset of the space X such that every element U of tX^* contains an element V of tX^* such that $V \cap A \in mMnX$. Then $A \in mMnX$.*

The proof of Lemma 3.4 is a straightforward generalization of the proof of the Banach category theorem (see [13] and pages 201, 202 of [11]).

Now, for any subset A of the space X, let $M(A, m, n)$ denote the set of all x in A such that every open subset U of X that contains x contains an element V of tX^* such that $V \cap A \in mMnX$. It follows from Lemma 3.4 that $M(A, m, n) \in mMnX$.

Proof of 3.1. Suppose f, X, and Y are as hypothesized, and let $\mathscr{P}$ be a standard pseudo-base for X. If

$$d(f) = \{M(f^{-1}[V], m, n): V \in tY\},$$

then, by Lemma 3.3, $d(f) \in mMnX$. So, if

$$X_f = [X \sim d(f)] \cap [\textstyle\bigcap_i \cup \mathscr{P}_i],$$

then X_f is a tractable $\sigma\pi$ space that is dense in X.

It is easy to verify that: $f \mid X_f$ is δ continuous at every point of X_f, and if $U \in tX_f$, $V \in tY$, and $U \cap f^{-1}[V] \neq \phi$, then $U \cup f^{-1}[V]$ contains an element of nMX_f. So Lemma 2.1 implies that $\mathscr{B}(f \mid X_f, nNX_f) \neq \phi$. Hence $\mathscr{B}(f, nNX) \neq \phi$. ▌

We conclude this section with a result that characterizes ωBn $\sigma\pi$ spaces.

3.5 THEOREM. *For any $\sigma\pi$ space, the following statements are equivalent.*

(1) *X is an ωBn space.*
(2) *If f is a real valued function defined on X, then $B(f, nNX) \neq \phi$.*
(3) *If f is a function from X into ω, then $\mathscr{B}(f, nNX) \neq \phi$.*

Proof. All that remains to be shown is that (3) implies (1). So suppose (3) holds and $X = \bigcup_i B_i$, where: $B_i \cap B_j = \phi$ if $i \neq j$; and if $i \geq 1$, then $B_i \in nGNX$. Define f by letting $f(x) = i$ if $x \in B_i$, and let (D, E) be an element of $\mathscr{B}(f, nNX)$. Then there is a k, and a U in tX^*, such that $B_k \cap U \supset U \cap U$; therefore, $B_k \cap U$ is nNX dense in U. So $k = 0$ and $X \neq \bigcup\{B_i\colon 1 \leq i < \omega\}$. ▮

4. Some useful properties of $\sigma\pi$ spaces

Part of the material in this section will be used in Sections 5 and 6.

We shall denote the cellular number [7, p. 164] of a space X by oX. A space X satisfies the countable chain condition if $oX \leq \omega$. If $oX > \omega$ for every U in tX^*, then X is called nowhere CCC. The space X is called weakly T_1 if for each x in X, either cl $\{x\} = \{x\}$ or cl $\{x\}$ is nowhere dense in X. Note that any space that is nowhere CCC is weakly T_1.

4.1 THEOREM. *Suppose X is a $\sigma\pi$ space.*

(1) *There is a disjoint subcollection of tX^* of cardinality oX (i.e., oX is assumed).*

(2) *The density character* [7, p. 164] *of X equals oX.*

(3) *If X is nowhere CCC, then there is a family $(F_\alpha, \alpha < \omega_1)$ of closed, nowhere dense subsets of X such that*

$$X = \bigcup \{F_\alpha\colon \alpha < \omega_1\}$$

and if $\alpha < \beta < \omega_1$, then $F_\alpha \subset F_\beta$.

(4) *If X is weakly T_1 and has no isolated points, then it is the union of a subcollection of NX that is of cardinality at most c.*

The proofs of all four statements in Theorem 4.1 are the same for $\sigma\pi$ spaces as they are for metrizable spaces. See pages 167, 168 of [7] for proofs of (1) and (2), when X is metrizable. We shall include a sketch of a proof of (3), which was given for metrizable spaces in [15].

Proof of (3). Let $\mathscr{P}$ be a standard pseudo-base for X. Because no element of $\mathscr{P}$ satisfies the countable chain condition, for each i, there is a family $(V(i, \alpha), \alpha < \omega_1)$ of non-empty open subsets of X such that: if $\alpha < \beta < \omega_1$, then $V(i, \alpha) \cap V(i, \beta) = \phi$; if $\alpha < \omega_1$ and $P \in \mathscr{P}_i$, then $P \cap V(i, \alpha) \neq \phi$. For each α less than ω_1, let

$$F_\alpha = X \sim \bigcup \{V(i, \beta)\colon i < \omega, \alpha < \beta < \omega_1\}.$$

It is easy to verify that $(F_\alpha, \alpha < \omega_1)$ has the required properties. ▮

We shall now give some applications of Theorem 4.1. The first is a simplification of the definition of mBn space when $n = \text{cf } n > m$. We shall denote by HX the collection of all U in tX^* for which $oV = oU$ for all V in tU^*. The collection HX is a pseudo-base for X.

4.2 PROPOSITION. *Suppose X is a $\sigma\pi$ space.*

(1) *If $X \in HX$ and $oX \geq n \geq \omega_1$, then $mMnX = mM0X$.*

(2) *If* $\operatorname{cf} n > \max(oX, m)$, *then a subset A of X is in $mMnX$ if and only if there is B in $mM0X$ such that $A \sim B$ contains no element of nNX.*

Proof. (1) Under the hypothesis of (1), every dense subset of X is nNX dense. To see this, suppose B is dense in X and $W \in tX^*$. By 4.1(1), there is a disjoint subcollection $\mathscr{U}$ of tW^* of cardinality at least n. Let A be a subset of $(\bigcup \mathscr{U}) \cap B$ such that $|A \cap B \cap U| = 1$ for every U in $\mathscr{U}$; then $A \in nNX$, and B is nNX dense in X. Part (1) follows easily from this.

(2) Suppose $\mathscr{F}$ is a subset of $nGNX$ of cardinality at most m, and let $A = \bigcup \mathscr{F}$. If $F \in \mathscr{F}$, then because $\operatorname{cf} n > oX$, there is a disjoint subcollection $\mathscr{U}_F$ of tX^* such that $\bigcup \mathscr{U}_F$ is dense in X and $\bigcup \mathscr{U}_F$ contains no element of nNX. If

$$B = \bigcup \{A \sim \bigcup \mathscr{U}_F \colon F \in \mathscr{F}\},$$

then B has the required properties. ▮

4.3 COROLLARY. *If* $\operatorname{cf} n = n > m \geq \omega$, *then $mMnX$ consists of all subsets A of X for which there is B in $mM0X$ and C in $nGNX$ such that $A = B \cup C$.*

We note that 4.1(3) and 4.1(4) restrict the cardinals for which a $\sigma\pi$ space can be an mBn space.

4.4 PROPOSITION. *Suppose X is a $\sigma\pi$ mBn space.*

(1) *If X is weakly T_1 and has no isolated points, then $m < c$.*
(2) *If X is nowhere CCC, then $m = \omega$.*
(3) *If $m > \omega$ and X is Hausdorff, then $n \leq 2^c$.*

Proof. Only (3) requires proof. In this case, we may, because of (2), assume that X satisfies the countable chain condition. By 4.1(2), X is separable. It follows from Lemma 15 of [7] that $|X| \leq 2^c$. ▮

The next application will be used in Section 5. Suppose $n \geq \omega_1$. A subset A of a space is called n dense in X if $|U \cap A| \geq n$ for every U in tX^*. Clearly, an nNX dense subset of X is n dense. Example 4.10 shows that the converse of this statement is false, even for $\sigma\pi$ spaces. The following statements indicate that in some situations the converse is true.

4.5 THEOREM. *If X is nowhere CCC $\sigma\pi$ space, then every n dense subset of X is nNX dense.*

Proof. It suffices to show that if the hypothesis holds and X is n dense in X, then $nNX \neq \phi$. To show this, suppose that $(F_\alpha, \alpha < \omega_1)$ satisfies the conclusion of Theorem 4.1(3).

Case 1. Suppose cf $n > \omega_1$. Because

$$n \leq |X| = \sup\{|F_\alpha| : \alpha < \omega_1\},$$

one of the $F_\alpha s$ is in nNX.

Case 2. Suppose cf $n = \omega$. Let (n_i) be a sequence of regular cardinals such that $n = \sup\{n_i : i < \omega\}$. For each i, there is an α_i less than ω_1 such that $|F_{\alpha_i}| \geq n_i$. Let

$$\gamma = \sup\{\alpha_i : i < \omega\} + 1.$$

Then $F_\gamma \in nNX$, because $(|F_\alpha|, \alpha < \omega_1)$ is non-decreasing.

Case 3. Suppose $n >$ cf $n = \omega_1$. Let $(n_\alpha, \alpha < \omega_1)$ be a family of regular cardinals, each of which is greater than ω, such that

$$n = \sup\{n_\alpha : \alpha < \omega_1\},$$

and let $(U_\alpha, \alpha < \omega_1)$ be a disjoint family of elements of tX^*. If $\alpha < \omega_1$, then by Case 1, there is a K_α in $n_\alpha NU_\alpha$. Then

$$\bigcup\{K_\alpha : \alpha < \omega_1\} \in nNX.$$

Case 4. Suppose $n = \omega_1$. As in the proof of 4.2(1), every dense subset of X is nNX dense. ▮

4.6 PROPOSITION. *Suppose X is a weakly T_1 $\sigma\pi$ space. If either* (1) cf $n > c$, *or* (2) $n > c$ *and* cf $n = \omega$, *then every n dense subset of X is nNX dense.*

Proof. Because of 4.5, we may assume that $oX = \omega$. In both cases, it suffices to show that if X is n dense in X, then $nNX \neq \phi$. If (1) holds, then this follows from 4.1(4). If (2) holds, then the proof is similar to Case 3 of the proof of 4.5, using 4.1(4) instead of 4.1(3). ▮

The following statement is an often useful alternative to the continuum hypothesis.

4.7 MARTIN'S AXIOM (topological form). *If $\omega < m < c$, then every compact Hausdorff space that satisfies the countable chain condition is an $mB0$ space.*

If $\omega_1 = c$, then 4.7 definitely holds. In [14], it is proven that it is consistent with ZFC that 4.7 holds and $\omega_1 < c$. It is shown in [12] that if 4.7 holds, then c is regular; in fact, it is shown that in this case, $2^m = c$ whenever $\omega < m < c$. In any statement in this paper, "[MA]" indicates that 4.7 is part of the hypothesis of that statement.

4.8 THEOREM [MA]. *If X has a countable pseudo-base and $\omega < m < c$, then $mN0X = \omega M0X$.*

The proof of 4.8 is the same as the proof of the theorem on page 170 of [12].

4.9 COROLLARY [MA]. *Suppose X and m satisfy the hypothesis of* 4.8. *If* cf $n > m$ *and X is an ωBn space, then it is an mBn space.*

Proof. It follows from 4.8 and 4.2(2) that under the hypothesis of 4.9, we have $\omega MnX = mMnX$. ▮

4.10 *Example* [MA]. There are subsets of R that are c dense, but not cNR dense, in R.

Using 4.8 and a simple modification of the argument on pages 146, 147 of [9], we can construct a c dense subset A of R such that $|A \cap F| < c$ for every F in cNR. ▮

4.11 PROPOSITION [MA]. *Suppose X is a weakly T_1 $\sigma\pi$ space. If $n \geq \omega_1$ and* cf $n \neq c$, *then every n dense subset of X is nNX dense.*

Proof. Because of 4.5 and 4.6, we may assume that cf $n < c$ and $oX = \omega$. Suppose X is n dense in X. We first show that there is a set A in $\omega M0X$ of cardinality n. If $n < c$, let A be any subset of X of cardinality n; it follows from 4.8 that $A \in \omega M0X$. If $n > c$, then by 4.1(4), there is a subcollection $\mathscr{F}$ of NX of cardinality at most c such that $|\bigcup \mathscr{F}| \geq n$. Let $(n_\alpha, \alpha < \text{cf } n)$ be a family of regular cardinals such that for each α, $c < n_\alpha < n$, and $n = \sup \{n_\alpha : \alpha < \text{cf } n\}$. For each α less than ω_1, there is an F_α in $\mathscr{F}$ such that $|F_\alpha| \geq n_\alpha$. If

$$A = \bigcup \{F_\alpha : \alpha < \text{cf } n\},$$

then $A \in (\text{cf } n)M0X = \omega M0X$ and $|A| \geq n$.

Now, if cf $n > \omega$, then it is clear that the existence of an element of $\omega M0X$ of cardinality n implies that $nNX \neq \phi$. So suppose cf $n = \omega$. Let (n_i) be a sequence of uncountable regular cardinals, each of which is less than n and different from c, such that $n = \sup \{n_i : i < \omega\}$, and (U_i) be a disjoint sequence of elements of tX^*. For each i, there is K_i in n_iNU_i; then $\bigcup_i K_i \in nNX$. ▮

5. Converses

In this section we shall prove some converses of Theorem 3.1. One of them, Theorem 5.2(1), generalizes Theorems 2 and 2′ of [4]. And Theorems 5.1 (with $n = c$) and 5.2(1) characterize the weakly T_1 $\sigma\pi$ spaces for which Proposition C of [4] holds (Proposition C of [4] is just 1.1): they are just the ωBc spaces. This naturally leads to another question. Are the weakly T_1 $\sigma\pi$ spaces for which Proposition B of [4] (which is 1.1 with c replaced by ω_1) holds just the $\omega B\omega_1$ spaces? Theorem 5.2(4) implies this is true if Martin's axiom holds. And, if X is a nowhere CCC $\sigma\pi$ space, then it follows from Theorems 3.5 and 4.5 that Proposition B of [4] holds for X if and only if X is an $\omega B\omega_1$ space. So the question reduces to the following. Must a weakly T_1 space with a countable pseudo-base for which Proposition B of [4] holds be an $\omega B\omega_1$ space?

If X and Y are topological spaces, f is a function from X into Y, and $n \geq \omega_1$, then we shall denote by $\mathscr{B}'(f, n)$ the set of all ordered pairs (D, E) such that E is an n dense subset of X, D is a dense subset of E, and $f \mid E$ is continuous at every point of D. A space X will be called an n Brown space if $\mathscr{B}'(f, n)$ is non-empty for every real-valued function f defined on X.

The following statement follows from Theorem 3.1.

5.1 THEOREM. *If $n \geq \omega_1$, then every $\sigma\pi$ ωBn space is an n Brown space.*

We shall prove the following converses.

5.2 THEOREM. *If n satisfies any of the following conditions, then every weakly T_1, $\sigma\pi$ n Brown space is an ωBn space:*

(1) $n = c$;
(2) $\mathrm{cf}\, n > c$;
(3) $n > c$ and $\mathrm{cf}\, n = \omega$;
(4) [MA] $\mathrm{cf}\, n \neq c$.

Parts (2), (3), and (4) of Theorem 5.2 follow from 3.5, 4.6, and 4.11. We shall now prove (1), starting with a lemma whose proof is omitted.

5.3 LEMMA. *If X is a c Brown space and Y is a subset of X such that $X \sim Y$ is either closed or meager in X, then Y is a c Brown space.*

Now suppose that X is a weakly T_1, $\sigma\pi$ c Brown space. Because of 4.5, we may assume that X satisfies the countable chain condition. Because of 4.2(2) and 5.3, it suffices to show that $cNX \neq \phi$. So suppose, to the contrary, that every nowhere dense subset of X has cardinality less than c. It follows from 4.1(4) that X has cardinality at most c; hence $|X| = c$. Let $\mathscr{P}$ be a standard pseudo-base for X, let $\mathscr{S}$ denote the σ-algebra generated by $\mathscr{P}$, and let $\mathscr{M}$ denote the set of all real valued functions defined on X that are measurable $(\mathscr{S})$. By Exercise 9 on page 26 of [10], $\mathscr{S}$ is of cardinality at most c. Because each element of $\mathscr{M}$ is the limit of a sequence of elements of $\mathscr{M}$, each of which has finite range, it follows that $|\mathscr{M}| = c$.

The argument on page 148 of [9] shows that there is a function h from X into R such that

$$|\{x: h(x) = g(x)\}| < c$$

for every g in $\mathscr{M}$. We shall obtain a contradiction by showing that the hypothesis on X implies that there is an f in $\mathscr{M}$ such that $\{x: h(x) = f(x)\}$ has cardinality c. To show this, first pick (D, E) in $\mathscr{B}'(h, c)$. Define, by induction, a sequence $(\mathscr{Q}_i)$ of disjoint subcollections of $\mathscr{P}$ such that for each i, $\bigcup \mathscr{Q}_i$ is dense in X, $\mathscr{Q}_{i+1}$ refines $\mathscr{Q}_i$, and if $Q \in \mathscr{Q}_i$ and $x, y \in E \cap Q$, then $|h(x) - h(y)| < 2^{-i}$. Let $Y = \bigcap_i \cup Q_i$, and for each i, let

$$f_i(x) = \sup\,\{h(y): y \in Q \cap E\} \quad \text{if } x \in Q \in \mathscr{Q}_i,$$

and

$$f_i(x) = 1 \quad \text{if } x \in X \sim \bigcup \mathcal{Q}_i.$$

Then $f = \lim_i f_i$ exists, f is measurable ($\mathcal{S}$), and

$$E \cap Y \subset \{x: h(x) = f(x)\}.$$

But, because $X \sim Y$ is meager in X, it has cardinality less than c. Hence $|\{x: h(x) = f(x)\}| = c$. ▮

6. Existence of *mBn* $\sigma\pi$ spaces

Suppose X is a $\sigma\pi$ space. If X satisfies the countable chain condition and is an ωBn space, then by Lemma 15 of [7], $n \leq 2^c$. (If, in addition, X is metrizable, then $n \leq c$.) In this section, we shall show (Example 6.4) that there is a compact Hausdorff space with a countable pseudo-base that is an $\omega B(2^c)$ space. First, however, we shall identify some ωBc spaces.

A space X is called α-favorable [6, p. 116], if there is a function $\theta: tX^* \to tX^*$ such that: $\theta(U) \subset U$ for all U in tX^*; if (U_i) is a sequence of elements of tX^* such that for each i, $U_{i+1} \subset \theta(U_i)$, then $\bigcap_i U_i \neq \phi$. In [6], it is shown that every locally compact Hausdorff space and every completely metrizable space is α-favorable.

6.1 PROPOSITION. *Every α-favorable, weakly T_1 $\sigma\pi$ space without isolated points is an ωBc space.*

Proof. This proof is similar to the proof of the corollary on page 251 of [4]. Suppose X satisfies the hypothesis of 6.1. Let θ denote the function that exists because X is α-favorable, and let $\mathcal{P}$ be a standard pseudo-base for X. Suppose (B_i) is a sequence of elements of $cGNX$; it suffices to show that $X \neq \bigcup_i B_i$. Define, by induction, a sequence $(\mathcal{C}_i)$ of disjoint subcollections of $\mathcal{P}$ such that the following hold for each i: $\mathcal{C}_i$ is a subset of

$$\bigcup \{\mathcal{P}_j: i \leq j < \omega\}$$

of cardinality 2^i; $\mathcal{C}_{i+1}$ refines $\mathcal{C}_i$; if $C \in \mathcal{C}_i$ and

$$\mathcal{A}(C) = \{D \in \mathcal{C}_{i+1}: D \subset C\},$$

then $|\mathcal{A}(C)| = 2$ and $\bigcup \mathcal{A}(C) \neq C$; if $C \in \mathcal{C}_i$, then $C \cap B_i$ contains no element of cNX; and, if $C \in \mathcal{C}_i$ and $D \in \mathcal{A}(C)$, then there is U in tX^* such that $D \subset \theta(U) \subset U \subset C$. Let $A = \bigcap_i \cup \mathcal{C}_i$. An adaptation of the argument on page 251 of [4] shows that $A \in cNX$ and for each i, $A \cap B_i \notin cNX$. Hence $A \sim \bigcup_i B_i \neq \phi$ and $\bigcup_i B_i \neq X$. ▮

For any topological space X, we shall denote the smallest cardinal number of a non-empty G_δ subset of X by $\# X$.

6.2 PROPOSITION. *Suppose X is a weakly T_1, $mB0$ $\sigma\pi$ space. If $\# X \geq n$ and* cf $n > m$, *then X is an mBn space.*

Proof. Suppose $\mathscr{P}$ is a standard pseudo-base for X, and $\mathscr{F}$ is a subset of $nGNX$ of cardinality at most m; we shall show that $\bigcup \mathscr{F} \neq X$. For each F in $\mathscr{F}$, there is a disjoint subcollection $\mathscr{U}_F$ of $\mathscr{P}$ such $\bigcup \mathscr{U}_F$ is dense in X and if $U \in \mathscr{U}_F$, then $U \cap F$ contains no element of nNX. By hypothesis, there is an x is

$$\bigcap \{\bigcup \mathscr{U}_F: F \in \mathscr{F}\} \cap [\bigcap_i \cup \mathscr{P}_i].$$

If $A = \bigcap \{P \in \mathscr{P}: x \in P\}$, then A is a nowhere dense G_δ set. And, because $|F| < \mathrm{cf}\, n$,

$$|A \cap [\bigcup \mathscr{F}]| \leq \sum \{|A \cap F| : F \in \mathscr{F}\} < n.$$

Hence A is not contained in $\bigcup \mathscr{F}$. ▮

6.9 COROLLARY. *Suppose X is a $\sigma\pi$ space.*

(1) *If X is a T_1 Baire space and the set*

$$\{x \in X: \{x\} \text{ is a } G_\delta\}$$

is meager in X, then X is an $\omega B \omega_1$ space.

(2) *Suppose X is a completely regular, Hausdorff meager space. If βX, the Stone-Čech compactification of X, is an $mB0$ space, then βX and $\gamma X = \beta(\beta X \sim X)$ are $mB(2^c)$ spaces.*

Proof. We shall prove (2); the proof of (1) is similar. Let Y be a dense G_δ subset of βX that is contained in $\beta X \sim X$. It suffices to show that Y is an $mB(2^c)$ space; to do this, we shall show that $\# Y \geq 2^c$. So suppose K is a non-empty G_δ subset of Y, and choose x in K. Because K is a G_δ in βX, by 3.11(b) of [8], there is a closed G_δ subset C of βX such that $x \in C \subset K$. Because $C \subset \beta X \sim X$, by Theorem 9.5 of [8], the cardinality of C is at least 2^c. ▮

6.4 *Example.* Let Q denote the space of rational numbers. By 6.3(2), γQ is an $\omega B(2^c)$ space. Because $\# \gamma Q = 2^c$, any metrizable subspace of γX is nowhere dense. And, if Martin's axiom holds, then γQ is an $mB(2^c)$ space for any cardinal m such that $\omega \leq m < c$. ▮

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