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AN INEQUALITY BETWEEN THE VOLUME AND THE CONVEXITY RADIUS OF A RIEMANNIAN MANIFOLD

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1. In [2], Marcel Berger is interested in finding lower bounds on the volume $v(g)$ of a compact n-dimensional Riemannian manifold (M, g) in terms of the injectivity radius $i(g)$ and convexity radius $c(g)$. Recently, Berger [3] proved that

$$v(g) \geq (\alpha(n)/\pi^n)i^n(g)$$

and consequently that

$$v(g) \geq (\alpha(n)/(\pi/2)^n)c^n(g)$$

with equality holding if and only if (M, g) is a sphere of constant curvature. (Here $\alpha(n)$ is the volume of the unit n-sphere.) It is reasonable to expect that stronger inequalities hold when M is not homeomorphic to a sphere.

In this paper we exhibit a constant $\mu'(3) > \alpha(3)/(\pi/2)^3$ such that for any non-simply-connected 3-dimensional manifold (M, g), $v(g) \geq \mu'(3)c^3(g)$. Along the way we refine Loewner's theorem [8], [1] thereby giving a lower bound on the area of a torus or Klein bottle in terms of the two shortest nonhomotopically trivial closed curves.

Throughout this paper, let ρ be the distance function associated to the Riemannian manifold (M, g). The open geodesic ball of radius $r > 0$ centered at the point $p \in M$ is defined by

$$B(p; r) = \{q \in M: \rho(p, q) < r\}.$$

Thus, if $0 < r \leq i(g)$, then the exponential map is a diffeomorphism of the open ball of radius r centered at the origin in the tangent space at p onto $B(p; r)$. Also, if $0 < r \leq c(g)$, then $B(p; r)$ is strongly convex, meaning that any two points of $B(p; r)$ are connected by a unique minimizing geodesic and this geodesic lies in $B(p; r)$. We will frequently use the relation $2c(g) \leq i(g)$ (Remark 1.6 of [2]).

2. Let (M, g) be a compact non-simply-connected Riemannian manifold. Let $l(g)$ denote the length of the shortest nonhomotopically-trivial closed curve. Clearly, $l(g) \geq 2i(g) \geq 4c(g)$. Let γ be a shortest non-

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homotopically-trivial closed curve. Then γ is a closed geodesic having no self-intersections because otherwise we could express γ as the product of two loops both of length less than $l(g)$ and hence homotopically-trivial. Now γ admits a normal tubular neighborhood of radius at least $c(g)$. For if it didn't, the exponential map of the normal bundle of γ into M would fail to be injective on the vectors of length less than $c(g)$, thus implying there is a point p and two geodesic segments τ, τ^* of length less than $c(g)$, emitted from p and meeting γ perpendicularly at points q, q^* respectively. Since the lengths of τ and τ^* are less than $c(g) < i(g)$, neither q nor q^* is a cut point of p. Thus $q \neq q^*$, and thus q, q^* divide γ into two geodesic segments γ_1, γ_2, one of which is the minimizing geodesic joining q to q^*. For if neither is the minimizing geodesic, letting σ denote this geodesic shows that $\gamma \simeq \gamma_1 \cdot \sigma \cdot \sigma^{-1} \cdot \gamma_2$ is homotopically-trivial because the two closed curves $\gamma_1 \cdot \sigma$ and $\sigma^{-1} \cdot \gamma_2$ have length less than $l(g)$. Finally we have the contradiction to convexity (see [6], p. 246) by having both geodesic segments τ, τ^* meeting the minimizing geodesic joining q to q^* perpendicularly in $B(p;\ c(g))$. Denote this normal tubular neighborhood by $B(\gamma; c(g))$.

Remark. If (M, g) is non-orientable, letting $\bar{\gamma}$ be the shortest orientation reversing curve, then $\bar{\gamma}$ is a closed geodesic having no self-intersections and having a normal tubular neighborhood $B(\bar{\gamma};\ c(g))$ of radius $c(g)$. The proof is the same as above, with "non-homotopically-trivial" replaced by "orientation reversing", and "homotopically trivial" by "orientation preserving".

Certainly,

$$v(g) \geq \mathrm{Vol}\ (B(\gamma;\ c(g)) = \int_0^{c(g)} \mathrm{Vol}\ (S(\gamma;\ r))\ dr$$

where $S(\gamma;\ r)$, the shell of radius r about γ, is the set of points whose distance from γ is precisely r. Thus, in order to obtain a lower bound on $v(g)$ it suffices to find one on the volume of the shell of radius r.

Example. Let $(M,\ g)$ be an orientable non-simply-connected compact surface. $S(\gamma;\ r)$ consists of two closed curves that are in the same homotopy class as γ. Thus $\mathrm{Vol}\ (S(\gamma;\ r)) \geq 2l(g)$ for $0 < r < c(g)$. Hence $v(g) \geq 2c(g)l(g) \geq 8c^2(g)$.

Remark. This improves a result in [2], [4]. There one finds the relation $v(g) \geq (6/\pi)i^2(g)$ for any non-simply-connected compact surface $(M,\ g)$ which only gives $v(g) \geq (24/\pi)c^2(g)$ when $i(g) \geq 2c(g)$ is taken into account.

If M is 3-dimensional, the shells are homeomorphic to tori or Klein bottles. In the next section we will find lower bounds for the surface areas of these spaces.

3. Let W be either the torus or the Klein bottle. It makes sense to talk about integral multiples of free homotopy classes of closed curves in W. A class is called *basic* if it is not the multiple of another class. In particular, the class of homotopically trivial curves is not basic. Now let W be endowed with a Riemannian metric so that one can define the length $\mathscr{L}(\omega)$ of the free homotopy class ω as the infimum of the lengths of representative curves in ω. Denote the surface area of W by $a(W)$.

LEMMA 3.1. *Suppose that there exists a basic class* ω_0 *and positive numbers* $l_1 \leq l_2$ *such that* (i) $\mathscr{L}(\omega_0) \geq l_1$, *and* (ii) $\mathscr{L}(\omega) \geq l_2$ *for all basic classes* $\omega \neq \pm\omega_0$
Then, (1) *if* W *is a torus,*

$$a(W) \geq \tfrac{1}{2}l_1\sqrt{4l_2^2 - l_1^2}$$

with equality characterizing a flat "isosceles" torus with base l_1 *and sides* l_2, *and* (2) *if* W *is a Klein bottle,*

$$a(W) \geq l_1 l_2$$

with equality characterizing a flat rectangular Klein bottle with sides l_1 *and* l_2.

Proof. The given Riemannian metric on W is conformal to a flat metric. The identity component G of the group of isometries for this flat metric operates transitively on W. Thus, averaging the given metric over G in the manner of Loewner's proof creates a new flat metric such that the area of the new metric is less than or equal to the area of the given metric and such that the length of each free homotopy class in the new metric is greater than or equal to the length of the same free homotopy class in the given metric. (See pp. 303–308 of [1] as well as p. 37 of [2].) Thus, the problem is reduced to considering flat metrics on the torus and the Klein bottle which are classified in [9]. For example, flat Klein bottles are classified by rectangular lattices in the Euclidean plane. The conditions on the lengths of basic homotopy classes become conditions on the lengths of line segments in the plane which connect two lattice points and contain no other lattice points. Thus the length of the shortest such segment must exceed l_1 and the length of any other, which is not parallel to the shortest segment, exceeds l_2. Therefore, the area of the fundamental rectangle must exceed $l_1 l_2$ with equality holding if and only if the sides of the rectangle are l_1 and l_2. The torus case is similar, except in this case the fundamental region is a parallelogram attaining the minimum area of

$$\tfrac{1}{2}l_1\sqrt{4l_2^2 - l_1^2}$$

precisely when the parallelogram may be decomposed, by drawing a diagonal, into two isosceles triangles with base l_1 and sides l_2. Q.E.D.

Remark. On taking $l_1 = l_2$, (1) reduces to Loewner's theorem and (2) to Proposition 2.5 of [2].

4. Consider γ of Section 2 where (M, g) is now a non-simply connected, compact 3-dimensional Riemannian manifold. The shells $S(\gamma; r)$ for $0 < r \leq c(g)$ are in a natural way S^1-bundles over γ; the projection $\pi\colon S(\gamma; r) \to \gamma$ being the composition of the inverse of the exponential map from the normal bundle of γ into M with the projection of this normal bundle onto γ. Thus the shells are either tori or Klein bottles depending upon whether γ is orientation preserving or orientation reversing. For each basic homotopy class ω we define the winding number $N(\omega)$ to be the integer such that $\pi_* \omega$ is $N(\omega)$ times the generator of the free homotopy classes of closed curves in $\gamma \cong S^1$. Let ω_0 be the class of closed curves which are freely homotopic to the inclusion of a fiber of π. Then ω_0 is basic, $N(\omega_0) = 0$ and $N(\omega) \neq 0$ for any basic $\omega \neq \pm\omega_0$. Furthermore, a representative $\sigma \in \omega$ is in the same free homotopy class in M as the $N(\omega)$-fold multiple of γ, since γ is a deformation retract of $B(\gamma; c(g))$.

LEMMA 4.1. *If ω is basic and $\omega \neq \pm\omega_0$, then $\mathscr{L}(\omega) \geq l(g) \geq 4c(g)$.*

Proof. If $N(\omega) = \pm 1$, then any $\sigma \in \omega$ is in the same free homotopy class as $\pm\gamma$ in M. Thus, by definition of γ and $l(g)$, $\mathscr{L}(\omega) \geq l(g) \geq 4c(g)$.

Now suppose $N(\omega) \neq \pm 1$, thus by taking $-\omega$ if necessary, we can suppose $N(\omega) \geq 2$. Let $\sigma \in \omega$. Since $\pi \circ \sigma$ winds around γ $N(\omega)$ times, σ may be broken up into $N(\omega)$ pieces σ_i such that $\pi \circ \sigma_i$ is the generating class in γ. Now the endpoints of σ_i lie over the same point of γ and are both a distance r in M from that point of γ. If τ_i is the curve consisting of the two geodesic segments connecting the point of γ to the endpoints of σ_i, then $\sigma_i \cup \tau_i$ is a closed curve in the same free homotopy class in M as γ. Thus,

$$L(\sigma_i) + 2r = L(\sigma_i \cup \tau_i) \geq l(g).$$

Summing over i, we obtain

$$L(\sigma) \geq N(\omega)(l(g) - 2r).$$

However, $2r \leq 2c(g) \leq \frac{1}{2}l(g)$. Thus

$$L(\sigma) \geq \tfrac{1}{2}N(\omega)l(g) \geq l(g)$$

since $N(\omega) \geq 2$. Q.E.D.

Remark. The proof of Lemma 4.1 would be simpler if we knew that γ didn't have finite order in M. For in this case, every basic $\omega \neq \pm\omega_0$ would not be trivial in M. Thus giving $\mathscr{L}(\omega) \geq l(g)$.

LEMMA 4.2. $\mathscr{L}(\omega_0) \geq \min(4r, 4c(g) - 2r)$ *for* $0 < r < c(g)$.

Proof. We must show that if $\mathscr{L}(\omega_0) < 4r$, then $\mathscr{L}(\omega_0) \geq 4c(g) - 2r$. Suppose $\sigma \in \omega_0$ and $L(\sigma) < 4r$. Fix $\delta > 0$ and choose a point $p \in M - S(\gamma; r)$ such that p is not a focal point of $S(\gamma; r)$ and the distance between p and σ is less than δ. Then any point on σ is within $\frac{1}{2}L(\sigma) + \delta < 2r + \delta$ of p. Thus if δ is small enough, say $2r + \delta < 2c(g) \leq i(g)$, then $\sigma \subseteq B(p; i(g))$.

Consider the function $f\colon S(\gamma; r) \cap B(p; i(g)) \to \mathbf{R}$ defined by $f(x) = \rho(x, p)$. This is a Morse function having the property that critical points occur at points of $S(\gamma; r) \cap B(p; i(g))$ where the unique minimizing geodesic segment to p is perpendicular to $S(\gamma\colon r)$ and the index of such a critical point is the number of focal points, counted with multiplicity, on this geodesic segment. (This is essentially the Morse index theorem in the one-fixed endpoint case. One should note that by choosing p to be non-focal, the critical points of f are non-degenerate.)

Now, f must have a critical value a between the minimum value of f and $L(\sigma) + \delta$, for otherwise, by using the deformation retraction defined from the gradient flow of f, we could show that σ is contractible in $S(\gamma; r) \cap B(p; i(g))$.

More precisely, let $\mathscr{H}$ be the family of closed curves

$$\psi\colon [0, 1] \to S(\gamma; r) \cap B(p; i(g)), \quad \psi(0) = \psi(1),$$

that are not homotopically trivial in $S(\gamma; r)$. $\mathscr{H}$ is invariant under ambient isotopies of $S(\gamma; r) \cap B(p; i(g))$. Thus by the mini-max principle [7],

$$a = \inf_{\psi \in \mathscr{H}} \sup_{t \in [0, 1]} \{f(\psi(t))\}$$

is a critical value of f. Clearly, since $\sigma \in \mathscr{H}$ and

$$\sup_{t \in [0, 1]} f(\sigma(t)) \leq \tfrac{1}{2}L(\sigma) + \delta,$$

we have $0 < a \leq \frac{1}{2}L(\sigma) + \delta$.

Hence there are critical points of f on the a-level, and some of these critical points must be of index one. (We can always slide a curve off an index two critical point, and a curve caught in a neighborhood of an index zero critical point would be contractible.)

Now we want to show that $a \geq 2c(g) - r$. For once we show this, it follows that $L(\sigma) + 2\delta \geq 2a \geq 4c(g) - 2r$. Thus, since δ is arbitrary, $L(\sigma) \geq 4c(g) - 2r$.

There are finitely many critical points, $x_1, \ldots, x_k$, of index one on the a-level of f. Let τ_i be the unique minimizing geodesic segment connecting x_i to p. Observe, $L(\tau_i) = f(x_i) = a$ and τ_i is perpendicular to $S(\gamma; r)$ at x_i for all i. Furthermore, there is a focal point of $S(\gamma; r)$ on τ_i between x_i and p. Now, τ_i is either pointed "inward" or "outward" with respect to the manifold with boundary $B(\gamma; r)$. If inward, the focal point on τ_i occurs when it meets γ at distance r. τ_i will not meet $S(\gamma; r)$ again until distance $2r$. Now since p is within distance δ of $S(\gamma; r)$, we see the length of τ_i must be at least $2r - \delta$, i.e., $2r - \delta \leq a$. Thus,

$$2r - \delta \leq a \leq \tfrac{1}{2}L(\sigma) + \delta,$$

which implies $L(\sigma) + 3\delta > 4r$. However, since $L(\sigma) < 4r$, this doesn't occur if δ is small enough. Therefore, we may assume that τ_i is outward pointing for all i.

Extend τ_i inward until it meets γ to obtain a geodesic $\hat{\tau}_i$ which connects some point $y_i \in \gamma$ to p. Thus $L(\hat{\tau}_i) = r + L(\tau_i) = r + a$.

Recall that we are trying to prove $a \geq 2c(g) - r$. We argue by contradiction. So suppose $a + r < 2c(g)$. Since, $2c(g) \leq i(g)$, it follows that $\hat{\tau}_i$ is the unique minimizing geodesic between y_i and p.

Let σ_i be the fiber of $S(\gamma; r)$ over the point y_i. Certainly $x_i \in \sigma_i$ and $\rho(x, y_i) = r$ for any $x \in \sigma_i$. Thus, since $\hat{\tau}_i$ is the unique minimizing geodesic between y_i and p,

$$\rho(y_i, x) + \rho(x, p) \geq r + a$$

for all $x \in \sigma_i$, with equality holding if and only if $x = x_i$. Thus

$$(*) \qquad f(x) \geq a$$

for all $x \in \sigma_i \cap B(p; i(g))$ with equality holding if and only if $x = x_i$. (Consequently, the σ_i are distinct for two distinct critical points cannot be on the same fiber.)

Now there exists a curve $\psi \in \mathscr{H}$ such that $\sup_{t \in [0,1]} f(\psi(t)) = a$ and ψ meets each σ_i transversely. The construction is as follows. Since $f^{-1}((a, a + \varepsilon])$ is free of critical points for small enough ε, any $\psi \in \mathscr{H}$ with $\sup_{t \in [0,1]} f(\psi(t)) < a + \varepsilon$ may be deformed into $f^{-1}([0, a])$ via the downward gradient flow of f. Thus we have ψ such that $\sup_{t \in [0,1]} f(\psi(t)) = a$. Now, by $(*)$, the only possible points of intersection of ψ and σ_i are x_i. We want to make ψ transverse to σ_i. There exist disjoint neighborhoods U_i of x_i in $S(\gamma; r)$ and Morse coordinates (u, v) in U_i such that

$$f(u, v) = a - u^2 + v^2 \quad \text{in } U_i.$$

Let

$$A_i^+ = \{(u, v) \in U_i \colon u^2 - v^2 \leq 0 \text{ and } u \geq 0\}$$

and

$$A_i^- = \{(u, v) \in U_i \colon u^2 - v^2 \leq 0 \text{ and } u \leq 0\}$$

be the right and left descending cones in U_i. Also let

$$B_i(\varepsilon) = \{(u, v) \in U_i \colon u^2 + v^2 < \varepsilon\}$$

and similarly for $B_i(2\varepsilon)$. We may choose ε small enough so that $B_i(2\varepsilon) \subset U_i$ for all i. Thus the sets $B_i(2\varepsilon)$ and

$$\hat{S} = S(\gamma; r) \cap B(p; i(g)) - \bigcup_{i=1}^{k} \overline{B_i(\varepsilon)}$$

form an open cover of

$$S(\gamma; r) \cap B(p; i(g)).$$

Thus the $\psi^{-1}(B_i(2\varepsilon))$ and $\psi^{-1}(\hat{S})$ gives an open cover of $[0, 1]$. Using the Lebesgue number, we can find a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of $[0, 1]$

such that for all $j = 1, \ldots, m$, $\psi([t_{j-1}, t_j])$ is contained in either S or $B_i(2\varepsilon)$ for some i. Furthermore, by deleting unnecessary points of the partition, we can assume that if $\psi([t_{j-1}, t_j]) \not\subset \hat{S}$ then

$$\psi(t_{j-1}),\ \psi(t_j) \in \hat{S} \cap B_i(2\varepsilon)$$

for that i such that $\psi([t_{j-1}, t_j]) \subset B_i(2\varepsilon)$. This insures that $\psi(t_j)$ is not one of the critical points x_i. Now, we will change ψ to make it transverse. We define a new curve $\tilde{\psi}$ by

$$\tilde{\psi} \mid [t_{j-1}, t_j] = \psi \mid [t_{j-1}, t_j]$$

if $\psi([t_{j-1}, t_j]) \subset \hat{S}$. If $\psi([t_{j-1}, t_j]) \subset B_i(2\varepsilon)$, then

$$\psi([t_{j-1}, t_j]) \subset A_i^+ \cup A_i^- .$$

Case 1. If $\psi(t_{j-1})$ and $\psi(t_j)$ are both in A_i^+, these points may be connected by a curve in A_i^+ that misses x_i. Replace $\psi \mid [t_{j-1}, t_j]$ by this curve. Similarly, if both $\psi(t_{j-1})$ and $\psi(t_j)$ are in A_i^-.

Case 2. If $\psi(t_{j-1}) \in A_i^+$ and $\psi(t_j) \in A_i^-$, or vice versa, replace $\psi \mid [t_{j-1}, t_j]$ by the curve made up of the u = constant segment connecting $\psi(t_{j-1})$ to the u-axis, the u-axis and the u = constant segment connecting the u-axis to $\psi(t_j)$.

Thus $\tilde{\psi}$ is homotopic to ψ, since we made changes only in coordinate neighborhoods and thus $\tilde{\psi} \in \mathscr{H}$. Furthermore,

$$\sup_{t \in [0,1]} f(\tilde{\psi}(t)) \leq a$$

since the replacement curves had this property. Finally $\tilde{\psi}$ meets the σ_i transversely since the u-axis meets σ_i transversely by (*). Thus there are only a finite number of intersections between $\tilde{\psi}$ and all the σ_i.

Now, take $\psi \in \mathscr{H}$ meeting the σ_i transversely in the least possible number of points and satisfying $\sup_{t \in [0,1]} f(\psi(t)) \leq a$. There must be at least one point of intersection; otherwise we could deform ψ below the a-level. Suppose $\psi \cap \sigma_{i_0} \neq \phi$. Now, $\psi \subset B(p;\, i(g))$ and hence is contractible in M. In particular, the winding number of ψ is not ± 1, for otherwise it would be freely homotopic to γ in M which is not contractible. Thus ψ must intersect σ_{i_0} at least two times. Consider the Morse coordinate system about x_{i_0}. As we traverse the closed curve ψ we must pass through x_{i_0} in one of two directions, either first through $A_{i_0}^+$ and then $A_{i_0}^-$ or first through $A_{i_0}^-$ and then $A_{i_0}^+$. If both times ψ passes through x_{i_0} in the same direction, we can decompose ψ into the product of two closed curves. One of these must be noncontractible since ψ is. Thus we obtain a curve in $\mathscr{H}$ with one less intersection with the σ_i. If ψ passes through x_{i_0} in different directions, again we decompose ψ into the product of two closed curves, one of which is non-contractible. We can pull this curve off of x_{i_0} to make all intersections transverse. Hence we obtain a curve in $\mathscr{H}$ with two less intersections with the σ_i. Either case is a contradiction to our choice of ψ.

Q.E.D.

5. We are now in a position to prove the following result.

THEOREM 5.1. *There exists a constant $\mu'(3) > \alpha(3)/(\pi/2)^3$ such that for every non-simply-connected compact 3-dimensional Riemannian manifold (M, g), $v(g) \geq \mu'(3)c^3(g)$.*

Proof. Using the estimates of Sections 2–4, we have either (1) if γ preserves orientation

$$\begin{aligned} v(g) &\geq \int_0^{c(g)} \tfrac{1}{2} l_1(r)\sqrt{4l_2^2 - l_1^2(r)}\, dr \\ &= \frac{1}{3}\left(64 + 15\sqrt{60} - \frac{1024\sqrt{2}}{9}\right)c^3(g) \\ &\geq \tfrac{59}{10} \cdot c^3(g), \end{aligned}$$

or (2) if γ reverses orientation

$$v(g) \geq \int_0^{c(g)} l_1(r) l_2\, dr = \tfrac{20}{3} \cdot c^3(g)$$

where $l_1(r) = \min(4r, 4c(g) - 2r)$ and $l_2 = 4c(g)$. One can easily check that both constants are greater than $\alpha(3)/(\pi/2)^3 = 16/\pi$. Q.E.D.

Remark. If M is non-orientable, one can take the shortest orientation reversing curve $\bar{\gamma}$ and insure the better constant 20/3 in Theorem 5.1.

It is likely that the bound in Theorem 5.1 is very weak. For one might suspect that the ratio $v(g)/c^3(g)$ would be least on manifolds with relatively simple topology. For example, this ratio is $64/\pi$ when calculated for the canonical metrics on $\mathbf{RP}^3$ and $S^1 \times S^2$, which is over three times the value of the constant determined in Theorem 5.1.

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