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CHARACTERIZING 2-DIMENSIONAL MOISHEZON SPACES BY WEAKLY POSITIVE COHERENT ANALYTIC SHEAVES

BY

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1. A considerable amount of effort has been expended in recent years on generalizing Kodaira's Embedding Theorem to a characterization of Moishezon spaces by some form of positive coherent sheaves. There are two possible approaches. One is to consider a relatively weak form of positivity; the problem is then to prove that a complex space carrying such a sheaf is necessarily Moishezon. The other is to give a stronger definition of positivity and then the problem is to show that every Moishezon space carries such a sheaf (see the bibliography in [8]).

One of the first and most natural positivity notions for sheaves was given by Grauert in his fundamental paper "Uber Modifikationen und exceptionelle analytische Mengen" ([4]). Let X be a reduced compact complex space and $\mathscr{S} \to X$ a coherent analytic sheaf. Grauert constructs a linear fibre space $V(\mathscr{S})$ dual to $\mathscr{S}$ such that $\mathscr{S}$ is the sheaf of linear forms on $V(\mathscr{S})$. He then calls $\mathscr{S}$ *weakly positive* if the zero-section of $V(\mathscr{S})$ is exceptional, that is, can be holomorphically contracted to a point. The main theorem of [4] is that a normal compact complex space is projective, and hence by Chow's theorem, projective algebraic, if and only if it carries a weakly positive locally-free sheaf. In light of this result, it seems quite natural to try to characterize Moishezon spaces by weakly positive coherent sheaves. Now it is a simple matter to prove that if X carries a weakly positive coherent sheaf then X is Moishezon (see [8]). The difficulty lies in showing that every Moishezon space carries such a sheaf.

In [8], I gave a slightly weaker definition of positivity than Grauert's. Let X, $\mathscr{S}$, and $V(\mathscr{S})$ be as above. The $V(\mathscr{S})$ is, in general, non-reduced and its reduction is, in general, not irreducible. Let V_R be the reduction of $V(\mathscr{S})$ and let π: $V_R \to X$ be the natural projection. Then there is an analytic set $A \subset X$ such that π: $V \mid (X - A) \to X - A$ is a vector bundle (see [10]). The primary component of $V(\mathscr{S})$, denoted V', is the closure in V_R of $V_R \mid X - A$. If X is irreducible, then V' is the unique irreducible component of V_R which is mapped onto X by π. Although V' is, in general, not a linear fibre space, it does have a well-defined zero-section and $\mathscr{S}$ is called *primary weakly positive* if

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the zero-section of V' is exceptional. The main theorem of [7] (see also [8]) is that a normal irreducible compact complex space is Moishezon if and only if it carries a primary weakly positive coherent sheaf.

The question of whether every Moishezon space carries a weakly positive sheaf remains open and is raised once again in a recent article of Ancona and Vo Van Tan ([1]). My purpose here is to show that every 2-dimensional Moishezon space carries a weakly positive coherent sheaf. Note that this is immediate if X is nonsingular since every nonsingular 2-dimensional Moishezon space is, in fact, projective.

2. We begin with two definitions.

DEFINITION 1 (see [3] for example). Let $\phi\colon Y \to X$ be a proper holomorphic map of reduced complex spaces. We call ϕ a *proper modification mapping* if there is an analytic set A in X such that

(i) $A \subset X$ and $\phi^{-1}(A) \subset Y$ are analytically rare, and
(ii) $\phi\colon Y - \phi^{-1}(A) \to X - A$ is a biholomorphism.

We shall refer to A as the *center* of the proper modification.

DEFINITION 2 (see [9]). Let X be a reduced (irreducible) complex space and $\mathscr{S} \to X$ a coherent analytic sheaf. A monoidal transformation of X with respect to $\mathscr{S}$ is a pair $(\hat{X}, \phi)$ where $\hat{X}$ is a reduced (irreducible) complex space and $\phi\colon \hat{X} \to X$ is a proper modification mapping such that

(i) $\phi^*\mathscr{S}/\mathrm{Tor}\,(\phi^*\mathscr{S})$ is locally-free, and
(ii) if $\psi\colon Y \to X$ is any proper modification mapping such that $\psi^*\mathscr{S}/\mathrm{Tor}\,(\psi^*\mathscr{S})$ is locally-free then there is a unique holomorphic mapping $\eta\colon Y \to \hat{X}$ such that $\psi = \phi \circ \eta$.

The existence and uniqueness of monoidal transformations with respect to coherent sheaves is established by Riemenschneider in [9].

PROPOSITION 1. *Let X be a normal irreducible compact complex space and let A be a finite set of points of X. Suppose there exists a proper modification $\pi\colon \hat{X} \to X$ with center A such that $\hat{X}$ is a projective algebraic manifold. Then X carries a weakly positive coherent analytic sheaf.*

Proof. Let $H \to \hat{X}$ be a very ample line bundle. Then there is an exact sequence

$$\mathscr{O}_{\hat{X}}^{q} \xrightarrow{\lambda} \mathscr{H} \to 0 \tag{1}$$

where $\mathscr{H} = \mathscr{O}_{\hat{X}}(H)$ and $q = \dim_{\mathbf{C}}(\Gamma(\hat{X}, \mathscr{H}))$. Let V be the contravariant functor assigning to every coherent sheaf its associated linear fibre space (see

[2]). Then $V(\mathscr{H}) = H^* = H^{-1}$ and $V(\mathcal{O}^q_{\hat{X}}) = \hat{X} \times \mathbf{C}^q$ so that (1) induces an exact sequence

$$(2) \qquad 0 \to H^* \to \hat{X} \times \mathbf{C}^q.$$

Since H is very ample, the mapping $\mu: \hat{X} \to P^{q-1}$ defined by $\mu(x) = H^*_x \in P(\mathbf{C}^q) = P^{q-1}$ is a holomorphic embedding of $\hat{X}$ into P^{q-1}. Moreover, $H = \mu^* G$ where $G \to P^{q-1}$ is the (positive) hyperplane bundle.

The exact sequence (1) induces a morphism

$$(3) \qquad \pi_*(\mathcal{O}^q_{\hat{X}}) = \mathcal{O}^q_X \xrightarrow{\pi_*\lambda} \pi_*(\mathscr{H}).$$

Let $\mathscr{S} = \operatorname{Im}(\pi_* \lambda)$. Then $\mathscr{S}$ is a torsion-free coherent sheaf over X and there is an exact sequence

$$(4) \qquad \mathcal{O}^q_X \to \mathscr{S} \to 0.$$

We shall prove that $\mathscr{S}$ is weakly positive. A key element in our argument will be the fact that $(\hat{X}, \pi)$ is the monoidal transformation of X with respect to $\mathscr{S}$. For a proof of this, see [7].

The exact sequence (4) induces an exact sequence

$$(5) \qquad 0 \to V(\mathscr{S}) \to X \times \mathbf{C}^q.$$

Let $X \times P^{q-1}$ be the projectivization of $X \times \mathbf{C}^q$; that is, $X \times P^{q-1}$ is the quotient of $X \times (\mathbf{C}^q - \{0\})$ by the action of $\mathbf{C}^*$ on the second factor. Let $\rho: X \times P^{q-1} \to X$ and $\phi: X \times P^{q-1} \to P^{q-1}$ be the natural projections and let $F \to X \times P^{q-1}$ be defined by $F = \phi^* G$ where $G \to P^{q-1}$ is the (positive) hyperplane bundle.

Let V' be the primary component of $V(\mathscr{S})$. Then V' is, in general, not a linear fibre space. It is, however, invariant under the action of $\mathbf{C}^*$ so there is a complex space $P(V')$,

$$P(V'(\mathscr{S})) = (V' - \{0\})/\mathbf{C}^*,$$

associated to V'. Moreover, $P(V')$ is a reduced complex subspace of $X \times P^{q-1}$. Now, by its construction, $P(V')$ is precisely the monoidal transformation of X with respect to $\mathscr{S}$ (see [10], [9]). Thus $\hat{X} = P(V')$. We claim that $\phi \mid \hat{X} = \mu$. This follows from the fact that $\hat{X} \cap [\rho^{-1}(X - A)]$ is dense in $\hat{X}$ (since $\rho^{-1}(A)$ is a proper analytic set in the manifold $\hat{X}$) and ϕ certainly agrees with μ on this set. Thus $F \mid \hat{X} = \phi^* G \mid \hat{X} = \mu^* G = H$ so that $F \mid \hat{X}$ is a positive line bundle in the sense of Kodaira (see [4], for example).

Let S be the analytic set in X over which $\mathscr{S}$ is not locally-free (see [10]). Since $\pi: \hat{X} - \pi^{-1}(A) \to X - A$ is a biholomorphism, $V(\mathscr{S}) \mid X - A$ is a line bundle. Clearly, then, $S \subset A$ so that S consists of a finite set of points. Now let $P(V_R) = V_R - \{0\}/\mathbf{C}^*$ be the projectivization of the reduction of $V(\mathscr{S})$. We claim that $F \mid P(V_R)$ is a positive line bundle. To prove this, we make use of a lemma of Grauert to the effect that a line bundle over a reduced compact complex space is positive if its restriction to each irreducible component of the space is positive (see [4], p. 349).

For $x \in X$, let $V_x = V(\mathscr{S})_x$ denote the fibre of $V(\mathscr{S})$ over x. Then (see [6]) V_x is a reduced complex vector space (even if $V(\mathscr{S})$ is non-reduced) so that V_x is also the fibre of V_R over x. Let V'_x denote the fibre of V' over x. Finally, let $x_1, \ldots, x_k$ be those points of S for which V'_x is properly contained in V_x. (Clearly, $V_x = V'_x$ for all $x \notin S$.) Then the irreducible components of $P(V_R)$ are $P(V') = \hat{X}$ and $P(V_{x_i})$, $i = 1, \ldots, k$. Now for each i, ϕ: $P(V_{x_i}) \to P^{q-1}$ is an embedding; it follows that $F \mid P(V_{x_i})$ is positive. Since $F \mid \hat{X} = H \mid \hat{X}$ is also positive, it follows from Grauert's lemma that $F \to P(V_R)$ is a positive line bundle.

Let $L \to X \times P^{q-1}$ be the line bundle corresponding to the principal $\mathbf{C}^*$-bundle $X \times (\mathbf{C}^q - \{0\}) \to X \times P^{q-1}$. Then (see [7], [8]) $L^* = F$ so $L \mid P(V_R)$ is a negative line bundle. Since every negative line bundle is weakly negative (see [4]), it follows that the zero-section of $L \mid P(V_R)$ is exceptional. Since $(L - \{0\}) \mid P(V_R)$ is biholomorphic to $V(\mathscr{S})_R - \{0\}$, it then follows that the zero-section of V_R is exceptional. Finally, $V(\mathscr{S})_R$ has exceptional zero-section if and only if $V(\mathscr{S})$ has exceptional zero-section (see [11], for example), so $\mathscr{S}$ is weakly positive, Q.E.D.

It is now a simple matter to prove our principal result.

PROPOSITION 2. *Let X be a 2-dimensional normal irreducible compact complex space. Then X is Moishezon if and only if it carries a weakly positive coherent analytic sheaf*, (cf. [9]).

Proof. If X carries a weakly positive sheaf, then, a fortiori, it carries a primary weakly positive sheaf. By Theorem 2 of [8], it follows that X is Moishezon.

For the converse, let X be a Moishezon space. Then (see [5]) there is a proper modification π: $\hat{X} \to X$ with center A such that $\hat{X}$ is a projective manifold and A is an analytic set in X of codimension greater than or equal to 2. If, in particular, X has dimension 2, it follows that A consists of a finite number of points. Proposition 1 then implies that X carries a weakly positive sheaf, Q.E.D.

3. We recall the following definition (see [8]):

DEFINITION 3. Let X be a reduced compact complex space and $\mathscr{S} \to X$ a coherent analytic sheaf. Then $\mathscr{S}$ is (*primary*) *cohomologically positive* if for any coherent sheaf $\mathscr{J} \to X$ there is an integer $\mu = \mu(\mathscr{J})$ such that $H^k(X, \mathscr{S}^{(v)} \otimes \mathscr{J}) = 0(H^k(X, \mathscr{S}^{(v)} \otimes \mathscr{J}/\mathrm{Tor}\,(\mathscr{S}^{(v)} \otimes \mathscr{J})) = 0)$ for all $v \geq \mu$ and all $k \geq 1$.

It is proven in [8] that (primary) weak positivity is equivalent to (primary) cohomological positivity.

DEFINITION 4. A Moishezon space is *proper* if it carries a weakly positive coherent sheaf.

Thus, whereas every Moishezon space carries a primary cohomologically positive sheaf, every proper Moishezon space carries a cohomologically positive sheaf. The following proposition follows immediately from Propositions 1, 2, and Grauert's characterization of projective varieties.

PROPOSITION 3. (i) *Every projective variety is a proper Moishezon space.*

(ii) *Every Moishezon space of dimension 2 is proper.*

(iii) *If $\pi\colon \hat{X} \to X$ is a proper modification with center A such that $\hat{X}$ is projective and A is finite, then X is a proper Moishezon space.*

Since there exist 2-dimensional Moishezon spaces (necessarily singular) which are not projective, the class of proper Moishezon spaces is strictly larger than the class of projective varieties.

The question remains whether there exist improper Moishezon spaces, that is, whether every Moishezon space carries a weakly positive sheaf.

REFERENCES

1. V. ANCONA and V. VAN TAN, *Embedding Moishezon spaces into 1-convex spaces*, Math. Ann., vol. 247 (1980), pp. 143–147.

2. G. FISCHER, *Lineare Fasserraume und koharente Modulgarben uber komplexen Raumen*, Arch. Math., vol. 18 (1967), pp. 609–617.

3. ———, *Complex Analytic Geometry*, Lecture Notes in Math., vol. 538, Springer-Verlag, Berlin and New York, 1976.

4. H. GRAUERT, *Uber Modifikationen und exceptionelle analytische Mengen*, Math, Ann., vol. 146 (1962), pp. 331–368.

5. B. G. MOISHEZON, *On n-dimensional compact varieties with n algebraically independent meromorphic functions*, I, II, III, Izv. Akad. Nauk SSSR Ser. Mat., vol. 30 (1966), pp. 133–174; pp. 345–386; pp. 621–656; English transl., Amer. Math. Soc. Trans. (2), vol. 63 (1967), pp. 51–177.

6. F. OORT, *Algebraic group schemes in characteristic zero are reduced*, Invent. Math., vol. 2 (1966/67), pp. 79–80.

7. J. H. RABINOWITZ, *Moisezon spaces and positive coherent sheaves*, Proc. Amer. Math. Soc., vol. 71 (1978), pp. 237–240.

8. ———, *Positivity notions for coherent sheaves over compact complex spaces*, Invent. Math., vol. 62 (1980), pp. 79–87.

9. O. RIEMENSCHNEIDER, *Characterizing Moisezon spaces by almost positive coherent analytic sheaves*, Math. Zeitschr., vol. 123 (1971), pp. 263–284.

10. H. ROSSI, *Picard variety of an isolated singular point*, Rice Univ. Studies, vol. 54 (1968), pp. 63–73.

11. M. SCHNEIDER, *Familien negativer Vektorraumbundel und 1-convexe Abbildungen*, Abh. Math. Seminar Univ. Hamburg, vol. 47 (1978), pp. 150–170.

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