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APPROXIMATION BY RATIONAL MODULES IN Lip α NORMS

BY

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1. Introduction

Let X be a compact subset of the complex plane $\mathbb{C}$. We denote by $\mathscr{R}(X)\bar{p}_m$ the rational module

$$\{r_0 + r_1\bar{z} + \cdots + r_m\bar{z}^m\},$$

where each r_j denotes a rational function with poles off X. Not long ago, a joint work by Trent and the author [8] proved that $\mathscr{R}(X)\bar{p}_1$ is always uniformly dense in $C(X)$ if X has empty interior. On the other hand, it is easy to see that $\mathscr{R}(X)\bar{p}_1$ is not always dense in $D^1(X)$, the Lip 1 closure of smooth functions on X, even if X has empty interior. Actually every Swiss cheese X will do the job. Thus it is interesting to ask where the cutoff point is. To be precisely, one may ask whether $\mathscr{R}(X)\bar{p}_1$ is always dense in lip (α, X) for each α, $0 < \alpha < 1$, if X has no interior. In this paper, we answer this question negatively. Furthermore, we prove that there is a close relation between the L^p density of $\mathscr{R}(X)$ and the Lip α density of $\mathscr{R}(X)\bar{p}_1$.

THEOREM 1. (i) *Let $2 \le p < 2/(1-\alpha)$. If $\mathscr{R}(X)$ is not dense in $L^p(X)$ then $\mathscr{R}(X)\bar{p}_1$ is not dense in* lip (α, X).

(ii) *Let $2/(1-\alpha) < p < \infty$. Then there exists a compact set X such that $\mathscr{R}(X)$ is not dense in $L^p(X)$ but $\mathscr{R}(X)\bar{p}_1$ is dense in* lip (α, X).

The fact that the compact set X in (ii) of this theorem must have empty interior is clear, since $\mathscr{R}(X)\bar{p}_1$ would not be dense in lip (α, X) otherwise. Similarly, in part (i) only the case X having empty interior is interesting, although the statement is true for general compact set.

For each fixed p, $2 \le p < \infty$, there are necessary and sufficient condition in terms of capacity for $\mathscr{R}(X)$ to be dense in $L^p(X)$ [2], [3]. Therefore there is a way, though generally hard, to verify the hypothesis in Theorem 1. Many examples of nowhere dense sets X so that $\mathscr{R}(X)$ is not dense in $L^2(X)$ are known (e.g., see [2]). Thus for any of such X, $\mathscr{R}(X)\bar{p}_1$ is not dense in lip (α, X) for any $0 < \alpha < 1$.

The case $p = 2/(1-\alpha)$, however, remains open.

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THEOREM 2. (i) *Let $2 \leq p \leq 2/(1-\alpha)$. Then there exists a compact set X such that $\mathscr{R}(X)$ is dense in $L^p(X)$ but $\mathscr{R}(X)\bar{p}_1$ is not dense in* lip (α, X).

(ii) *Let $2/(1-\alpha) \leq p < \infty$. If $\mathscr{R}(X)$ is dense in $L^p(X)$ then $\mathscr{R}(X)\bar{p}_1$ is dense in* lip (α, X).

Again the compact set X in part (i) of this theorem must have no interior. Part (ii) of Theorem 2 is proved by O'Farrell in [4]. The reason is simple: $\| \hat{\phi} \|_\alpha \leq K \| \phi \|_p$ for all compactly supported smooth function ϕ, where $\hat{\phi}$ denotes the Cauchy transform of ϕ.

In Section 2, we will prove part (i) of Theorem 1. In Section 3 we will discuss the construction of a certain type nowhere dense sets and prove part (ii) of Theorem 1 and part (i) of Theorem 2. In Section 4 we extend these results to a more general type of rational modules.

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2. Proof of (i) of Theorem 1

Let $2 \leq p < 2/(1-\alpha)$ and suppose $\mathscr{R}(X)$ is not dense in $L^p(X)$. Then there exists a function $g \in L^p(X)$, $p^{-1} + q^{-1} = 1$, such that $g \neq 0$, $\int gf\, dm = 0$ for all $f \in \mathscr{R}(X)$, where dm denotes the two-dimensional Lebesgue measure. Let $\bar{\partial} g$ be the partial derivative of g with respect to $\bar{z}$ in the distribution sense. Then $\bar{\partial} g \neq 0$ and $\bar{\partial} g \perp \mathscr{R}(X)\bar{p}_1$. On the other hand, given any smooth function ϕ with a directional derivative $D_u \phi$, we have

$$\begin{aligned}
\left| \int g \cdot D_u \phi\, dm \right| &= \left| \int g \cdot \lim_{h \to 0} \frac{\phi(z+hu) - \phi(z)}{h}\, dm \right| \\
&\leq \lim_{h \to 0} \int |g| \cdot \frac{|\phi(z+hu) - \phi(z)|}{|h|}\, dm \\
&\leq \| \phi \|_\alpha \sup_\xi \int \frac{|g|}{|\xi - z|^{1-\alpha}}\, dm \\
&\leq K' \| \phi \|_\alpha,
\end{aligned}$$

since the last integral gives a bounded continuous function. Therefore

$$|\bar{\partial} g(\phi)| \leq K \| \phi \|_\alpha$$

for all smooth function ϕ and $\bar{\partial} g$ is a continuous linear functional on lip (α, X), hence $\mathscr{R}(X)\bar{p}_1$ cannot be dense in lip (α, X).

3. Examples

Sinanjan [7] has constructed a nowhere dense compact set X such that $\mathscr{R}(X)$ is not dense in $C(X)$ but is dense in every $L^p(X)$, $p \geq 1$. The following Lemma which extends this result should surprise nobody. We are grateful to J. Brennan for a discussion of the Lemma.

LEMMA. *Let $2 \le p_0 < \infty$ be fixed. Then* (i) *there exists a compact X_1 such that $\mathscr{R}(X_1)$ is not dense in $L^{p_0}(X_1)$ but is dense in every $L^p(X_1)$, $p < p_0$,* (ii) *there exists a compact X_2 such that $\mathscr{R}(X_2)$ is dense in $L^{p_0}(X_2)$ but is not dense in any $L^p(X_2)$, $p > p_0$.*

Proof. We consider only the case $2 < p_0 < \infty$. When $p_0 = 2$ the construction is similar. We will use Hedberg's capacity theorems [3], [4]. Let C_q be the q-capacity, $1 < q < 2$, $p^{-1} + q^{-1} = 1$. Then by Lemma 1, 3 in [3] there exists constants F_1 and F_2 so that

$$F_1 \delta^{2-q} \le C_q(B_x(\delta)) \le F_2 \delta^{2-q}$$

for any ball with radius δ.

(A) The construction of X_1. Choose n_0 such that

$$\sum_{n_0}^{\infty} F_2 n^{-2} < C_{q_0}(B_0(\tfrac{1}{2})).$$

Let X_0 be the closed unit square with center at the origin. Cover X_0 with 4^n squares with side 2^{-n}. Call the squares $A_n^{(i)}$. In every $A_n^{(i)}$ put an open disk $B_n^{(i)}$, such that $B_n^{(i)}$ and $A_n^{(i)}$ have the same center and such that the radius of $B_n^{(i)}$ is

$$\delta_n = 2^{-2n/(2-q_0)} n^{-2/(2-q_0)}.$$

Let $X_1 = X_0 - \bigcup_{n \ge n_0} (\bigcup_i B_n^{(i)})$. Since

$$\begin{aligned} C_{q_0}(B_0(\tfrac{1}{2}) \backslash X_1) &\le \sum_{n_0}^{\infty} 4^n C_{q_0}(B_n^{(i)}) \\ &\le F_2 \sum_{n_0}^{\infty} n^{-2} \\ &< C_{q_0}(B_0(\tfrac{1}{2})), \end{aligned}$$

$\mathscr{R}(X_1)$ is not dense in $L^{p_0}(X_1)$. Within any disk centered at x and having radius 2^{-n}, there is a disk in $\mathbb{C} \backslash X_1$ having radius at least $4^{-1}\delta_n$. Hence

$$\lim_{n \to \infty} 2^{2n} C_q(B_x(2^{-n}) \backslash X_1) \ge \lim_{n \to \infty} 4^{-1} F_1 2^{2n} 2^{-2n(2-q)/(2-q_0)} n^{-2(2-q)/(2-q_0)} \to \infty$$

when $q > q_0$. Thus $\mathscr{R}(X_1)$ is dense in $L^p(X)$ for every $p < p_0$.

(B) Construction of X_2. Let $p_j \searrow p_0$. For each j, choose n_j such that

$$\sum_{n_j}^{\infty} F_2 2^{2n} 2^{-2n(2-q_j)/(2-q_0)}$$

is sufficiently small. It is possible by the above construction to remove open disks $B_n^{(i)}$ of radius $\delta_n = 2^{-2n/(2-q_0)}$ for all $n \ge n_j$, from

$$A_j(0) = \{2^{-(j+1)} \le |z| \le 2^{-j}\}, j = 1, 2, \dots,$$

to obtain a nowhere dense set Y_j such that $\mathscr{R}(Y_j)$ is not dense in $L^{p_j}(Y_j)$ but is dense in $L^{p_0}(Y_j)$ since

$$\sum_{nj}^{\infty} 4^n C_{q_j}(B_n^{(i)}) \leq F_2 \sum_{nj}^{\infty} 2^{2n} 2^{-2n(2-q_j)/(2-q_0)}$$

but

$$2^{2n} C_{q_0}(B_x(2^{-n}) \setminus Y_j) \geq c > 0 \quad \text{for all } x \text{ in } Y_j.$$

Let $X_2 = \bigcup_1^\infty Y_j \cup \{0\}$. It is easy to verify that this X_2 is the desired set.

With this lemma, we are now in a position to prove the remaining part of each theorem.

Proof of (ii) *Theorem 1.* Let $2/(1-\alpha) < p < \infty$. Take X_1 in the lemma, with $p_0 = p$. Then $\mathscr{R}(X_1)$ is not dense in $L^p(X_1)$ but is dense in $L^{p'}(X_1)$ for all $2/(1-\alpha) \leq p' < p$. Part (ii) of Theorem 2 thus implies that $\mathscr{R}(X_1)\bar{p}_1$ is dense in lip (α, X_1).

Proof of (i) *of Theorem 2.* Let $2 \leq p < 2/(1-\alpha)$. Take X_2 in the lemma, with $p_0 = p$. Then $\mathscr{R}(X_2)$ is dense in $L^p(X_2)$ but is not dense in any $p < p' < 2/(1-\alpha)$. Part (i) of Theorem 1 thus implies that $\mathscr{R}(X_2)\bar{p}_1$ is not dense in lip (α, X_2).

4. Rational modules of other type

Let g be a smooth function. We denote by $\mathscr{R}(X) + \mathscr{R}(X)g$ the rational modules $\{r_0 + r_1 g\}$. In [9], Trent and the author have investigated the rational modules of this type. When X has no interior, it is proved that $\mathscr{R}(X) + \mathscr{R}(X)g$ is uniformly dense in $C(X)$ if and only if $\mathscr{R}(Z)$ is uniformly dense in $C(Z)$, where $Z = \{x \in X, \bar{\partial} g(X) = 0\}$. A necessary condition for $\mathscr{R}(X) + \mathscr{R}(X)g$ to be dense in lip (α, X) is that $\mathscr{R}(Z)$ is dense in lip (α, Z) (cf. [6]). Thus one may wonder whether $\mathscr{R}(X) + \mathscr{R}(X)g$ is dense in lip (α, X) if X has no interior and if $\mathscr{R}(Z)$ is dense in lip (α, Z). Again this is proved negatively. It turns out that Theorem 1 and 2 are both valid when $\mathscr{R}(X)\bar{p}_1$ is replaced by $\mathscr{R}(X) + \mathscr{R}(X)g$ with this restriction on g: that $\mathscr{R}(Z)$ is dense in lip (α, Z). The last condition can be verified in terms of Hausdorff content [5].

To see part (ii) of Theorem 2 for the module $\mathscr{R}(X) + \mathscr{R}(X)g$, we make the following observation. A distribution T with compact support annihilates $\mathscr{R}(X) + \mathscr{R}(X)g$ if and only if $(\bar{\partial} g)\hat{T}$ annihilates $\mathscr{R}(X)$. When $T \in \text{Lip}(\alpha, X)^*$, $\hat{T}$ is a measure, and T annihilates $\mathscr{R}(Z)$ if and only if support $\hat{T} \subset Z$ [4].

To see part (i) of Theorem 1 for the module $\mathscr{R}(X) + \mathscr{R}(X)g$, we may assume that the set X is essential as defined in [1] without loss of generality. Then a "localization" procedure will allow us a measure (or a L^q function) $\mu \perp \mathscr{R}(X)$ with support μ disjoint from Z, and hence the distribution $\bar{\partial}\,(\mu/(\bar{\partial} g))$ will be the desired nonzero continuous linear functional on lip (α, X) that annihilates $\mathscr{R}(X) + \mathscr{R}(X)g$.

The rest of Theorem 1 and 2 for the module $\mathscr{R}(X) + \mathscr{R}(X)g$ just follows easily from the lemma.

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