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THE DISTRIBUTION OF POWERFUL INTEGERS

BY

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1. Introduction and statement of results

Let k be a fixed integer greater than unity. A positive integer is said to be powerful if it contains only powers of primes as factors; more precisely, let $G(k)$ denote the set of all positive integers with the property that if a prime p divides an element of $G(k)$, then p^k divides it also. In other words the set of powerful (or k-full) numbers $G(k)$ contains numbers whose canonical representation is

$$n = a_1^k a_2^{k+1} \cdots a_k^{2k-1}, \tag{1.1}$$

and this representation is unique if we stipulate that $a_2 \cdots a_k$ is square-free. If we set

$$f_k(n) = \begin{cases} 1 & n \in G(k) \\ 0 & n \notin G(k) \end{cases}, \quad F_k(s) = \sum_{n=1}^{\infty} f_k(n) n^{-s}, \tag{1.2}$$

it follows that for $\operatorname{Re} s > 1/k$,

$$F_k(s) = \prod_p (1 + p^{-ks} + p^{-(k+1)s} + \cdots) = \prod_p \left(1 + \frac{p^{-ks}}{1 - p^{-s}}\right). \tag{1.3}$$

For $x \geq 1$ we denote by $A_k(x)$ the number of k-full integers not exceeding x, so that from (1.1) and (1.2) we have

$$A_k(x) = \sum_{n \leq x, n \in G(k)} 1 = \sum_{n \leq x} f_k(n) = \sum_{a_1^k a_2^{k+1} \cdots a_k^{2k-1} \leq x} \mu^2(a_2 \cdots a_k), \tag{1.4}$$

where $\mu(n)$ is the Möbius function. For further factoring of (1.3) we note that for $k \geq 2$ and $K = \frac{1}{2}(3k^2 + k - 2)$ there are constants $a_{r,k}$ $(2k + 2 < r \leq K)$ such that

$$\left(1 + \frac{v^k}{1 - v}\right)(1 - v^k)(1 - v^{k+1}) \cdots (1 - v^{2k-1}) = 1 - v^{2k+2} + \sum_{r=2k+3}^{K} a_{r,k} v^r. \tag{1.5}$$

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This follows when we note that the product of the first two factors on the left-hand side equals

$$(1 - v + v^k)(1 + v + \cdots + v^{k-1}) = 1 + v^{k+1} + v^{k+2} + \cdots + v^{2k-1},$$

and multiplying out the remaining factors we obtain (1.5). If we substitute $v = p^{-s}$ in (1.5) and take the product over all primes, then using (1.3) and the product representation $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ (Re $s > 1$) for the Riemann zeta function it follows that

$$\begin{aligned} F_k(s) &= \zeta(ks)\zeta((k+1)s) \cdots \zeta((2k-1)s) \prod_p \left(1 - p^{-(2k+2)s} + \sum_{r=2k+3}^{K} a_{r,k} p^{-rs}\right) \\ &= \zeta(ks)\zeta((k+1)s) \cdots \zeta((2k-1)s)\zeta^{-1}((2k+2)s)\phi_k(s), \end{aligned} \tag{1.6}$$

where $\phi_2(s) = 1$ and $\phi_k(s)$ has a Dirichlet series with the abscissa of absolute convergence equal to $1/(2k+3)$ if $k > 2$. Therefore we may write

$$F_k(s) = G_k(s)H_k(s), \tag{1.7}$$

where

$$H_k(s) = \sum_{n=1}^{\infty} h_k(n)n^{-s} = \zeta(ks)\zeta((k+1)s) \cdots \zeta((2k-1)s) \tag{1.8}$$

and

$$G_k(s) = \sum_{n=1}^{\infty} g_k(n)n^{-s} = \phi_k(s)/\zeta((2k+2)s) \tag{1.9}$$

is a Dirichlet series converging absolutely for Re $s > 1/(2k+2)$. From (1.7) we infer that

$$A_k(x) = \sum_{a_1^k a_2^{k+1} \cdots a_k^{2k-1} \le x} \mu^2(a_2 \cdots a_k) = \sum_{mn \le x} g_k(m)h_k(n), \tag{1.10}$$

so that $A_k(x)$ is closely related to the unweighted sum

$$S_k(x) = \sum_{n \le x} h_k(n) = \sum_{a_1^k a_2^{k+1} \cdots a_k^{2k-1} \le x} 1, \tag{1.11}$$

where the summation is taken over positive integers $a_1, \ldots, a_k$. Following standard procedures (e.g., the inversion formula for Dirichlet series used in Section 2) we may write

$$S_k(x) = \sum_{r=k}^{2k-1} C_{r,k} x^{1/r} + \Delta_k^*(x), \tag{1.12}$$

where

$$C_{r,k} = \prod_{j=k, j \ne r}^{2k-1} \zeta(j/r),$$

and $\Delta_k^*(x)$ may be considered as an error term. If we define ρ_k^* as the infimum of all ρ satisfying

$$\Delta_k^*(x) \ll x^\rho \tag{1.13}$$

as $x \to \infty$, then an application of E. Landau's classical results concerning lattice point problems (see [16], [17]) gives, for $k \geq 2$,

$$\frac{k-1}{k(3k-1)} \leq \rho_k^* \leq \frac{1}{k+2}. \tag{1.14}$$

From (1.10) and (1.12) it is seen that the asymptotic formula for $A_k(x)$ may be written as

$$A_k(x) = \gamma_{0,k} x^{1/k} + \gamma_{1,k} x^{1/(k+1)} + \cdots + \gamma_{k-1,k} x^{1/(2k-1)} + \Delta_k(x), \tag{1.15}$$

where, for $i = 0, 1, \ldots, k-1$,

$$\gamma_{i,k} = \operatorname*{Res}_{s=1/(k+i)} F_k(s)s^{-1} = C_{k+i,k}\,\phi_k(1/(k+i))/\zeta((2k+2)/(k+i)),$$

and $\Delta_k(x)$ may be considered as an error term. The estimation of $\Delta_k(x)$ will be the main goal of this paper, and in analogy with (1.13) we define ρ_k to be the infimum of all ρ satisfying

$$\Delta_k(x) \ll x^\rho \tag{1.16}$$

as $x \to \infty$.

The investigation of powerful numbers began in 1935, when P. Erdös and G. Szekeres [4] proved in an elementary way that $\rho_k \leq 1/(k+1)$ for $k \geq 2$. Their result was sharpened in 1958 by P. Bateman and E. Grosswald [1], who proved that $\rho_2 \leq \frac{1}{6}$, $\rho_3 \leq \frac{7}{46}$, $\rho_k \leq 1/(k+2)$ for $k \geq 2$ and

$$\rho_k \leq \max\,(r/k(r+2),\ 1/(k+r+1)), \quad r = [\sqrt{2k}], \quad k \geq 4. \tag{1.17}$$

Further improvements may be found in the work of E. Krätzel [15], whose results include the estimate

$$\rho_k \leq 1/(k+H(k)), \quad \sqrt{\frac{8k}{3}} < H(k) < \left(1+\sqrt{\frac{7}{3}}\right)\sqrt{\frac{8k}{3}}, \tag{1.18}$$

which is valid if k is sufficiently large. The sharpest results for $3 \leq k \leq 5$ were obtained by A. Ivić in [9] (further improvements in [11]), where he proved

$$\begin{aligned} \rho_3 &\leq \tfrac{655}{4643} = 0.1410\ldots, \\ \rho_4 &\leq \tfrac{257}{2072} = 0.1240\ldots \\ \rho_5 &\leq \tfrac{6656613}{62279997} = 0.1068\ldots. \end{aligned} \tag{1.19}$$

It was also proved in [9] that $\rho_k \leq 1/2k$ for $k > 2$ if the (so far unproved) Lindelöf hypothesis that $\zeta(\frac{1}{2} + it) \ll t^\varepsilon$ holds.

For small values of k we shall improve on existing bounds for ρ_k by proving the following result.

THEOREM 1.

$$\rho_3 \leq \tfrac{263}{2052} = 0.128167\ldots, \quad \rho_4 \leq \tfrac{3091}{25981} = 0.118971\ldots,$$
$$\rho_5 \leq \tfrac{1}{10}, \quad \rho_6 \leq \tfrac{1}{12}, \quad \rho_7 \leq \tfrac{1}{14}.$$

We conjecture that $\rho_k \leq 1/2k$, and apart from the absence of suitable power moments for the zeta function, our methods would give this for $k \leq 13$. The proof of values for ρ_k when $k = 5, 6, 7$ will be given by complex integration, while the values of ρ_3 and ρ_4 will follow from an estimate for the general three-dimensional problem. If $1 \leq a \leq b \leq c$ are integers, then we have

$$\begin{aligned} D(a, b, c; x) &= \sum_{n_1^a n_2^b n_3^c \leq x} 1 \\ &= \zeta(b/a)\zeta(c/a)x^{1/a} + \zeta(a/b)\zeta(c/b)x^{1/b} \\ &\quad + \zeta(a/c)\zeta(b/c)x^{1/c} + \Delta(a, b, c; x), \end{aligned} \tag{1.20}$$

where $\Delta(a, b, c; x)$ may be regarded as an error term, and the main terms are evaluated most conveniently by residues. In case some of the numbers a, b, c are equal, the main terms are obtained by taking the appropriate limit. It will be seen from Lemma 1 of Section 2 that $\Delta_3(x)$ is essentially of the same order of magnitude as $\Delta(3, 4, 5; x)$, so that $\rho_3 \leq \tfrac{262}{2052}$ is a special case of the following.

THEOREM 2. *If a, b, c are integers such that $1 \leq a < b \leq c$, $c \leq a + b$, $92b \leq 171a$ or if $(a, b, c) = (1, 2, 2)$, then as $x \to \infty$ we have*

$$\Delta(a, b, c; x) \ll x^{263/171(a+b+c)} \log^2 x. \tag{1.21}$$

We shall prove Theorem 2 in Section 3, where some applications and remarks concerning (1.21) are given, and we devote Section 4 to certain additive problems concerning powerful numbers, focusing our attention on

$$B_{k,m}(x) = \sum_{n \leq x} R_{k,m}(n),$$

where $R_{k,m}(n)$ is the number of ways n can be written as a sum of m k-full numbers.

In concluding this section, let us make the following two remarks. Firstly, from (1.7), (1.8) and (1.9) it is seen that $\rho_k^* < 1/(2k+2)$ (at present known by [21] to hold only for $k = 2$) would give

$$\Delta_k(x) \ll x^{1/(2k+2)} \exp(-c_k \delta(x)), \tag{1.22}$$

where $c_k > 0$, $\delta(x) = \log^{3/5} x \cdot (\log\log x)^{-1/5}$. The case $k = 2$ was settled by Bateman and Grosswald in [1]. The general estimate in the case $\rho_k^* <$

$1/(2k+2)$ could be obtained following their proof [1], or it could be obtained directly by applying the convolution theorem of [10]. Thus, apart from unproved conjectures like Riemann's or Lindelöf's, the estimate (1.22) appears to be the limit of present methods, since there is no way to remove $1/\zeta((2k+2)s)$ from the product representation (1.6) of $F_k(s)$.

Another remark is that, for $k \geq 3$, the line Re $s = 0$ is a natural boundary for the function $G_k(s)$ given by (1.9). To see this we need a lemma of T. Estermann [5] which states that, for small x,

$$1 - x + x^k = \prod_{n=1}^{\infty} (1 - x^n)^{l_k(n)}. \tag{1.23}$$

Here $l_k(n)$ is an integer given by

$$l_k(n) = \frac{1}{n} \sum_{ab=n} \mu(a) \sum_{r=1}^{k} \lambda_r^b,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the roots of $\lambda^k - \lambda^{k-1} + 1 = 0$. From the product representation of $F_k(s)$ and (1.5) we have

$$G_k(s) = \frac{\zeta(s)}{H_k(s)} \prod_p (1 - p^{-s} + p^{-ks}),$$

so that from (1.23) we see that $G_k(s)$ can be written as an infinite product of the Riemann zeta-functions. For example, we have

$$(1, 0, -1, -1, -1, 0, 0, 1, 1, 1, 0, 0, -1, -1,$$
$$0, 0, 1, 1, 1, 0, -1, -2, -2, -1, 1, 3, \ldots)$$

for the sequence $(l_3(n))$, so that

$$G_3(s) = \frac{\zeta(13s)\zeta(14s)\zeta(21s)\zeta^2(22s)\zeta^2(23s)\zeta(24s)\cdots}{\zeta(8s)\zeta(9s)\zeta(10s)\zeta(17s)\zeta(18s)\zeta(19s)\zeta(25s)\zeta^3(26s)\cdots}.$$

If we assume the truth of the Riemann hypothesis we can deduce easily that the zeros of $G_k(s)$ $(k \geq 3)$ are dense in the line Re $s = 0$. If we follow the proof of the main theorem in Estermann's paper we can give an unconditional proof using only simple zero-density estimates for the Riemann zeta-function. We shall not require this result in our proofs of the theorems, and shall therefore omit the proof.

2. Proof of Theorem 1

In this section we shall prove Theorem 1, except for bounds for ρ_3 and ρ_4 which will follow from Theorem 2. We require first the following result,

LEMMA 1. *If, as* $x \to \infty$,

$$\Delta_k^*(x) \ll x^{\eta_k} \log^{\lambda_k} x \tag{2.1}$$

for $1/(2k+2) \le \eta_k < 1/(2k-1)$ *and* $\lambda_k \ge 0$, *where* $\Delta_k^*(x)$ *is given by* (1.12), *then*

$$\Delta_k(x) \ll x^{\eta_k} \log^{\lambda_k'} x \tag{2.2}$$

for $\Delta_k(x)$ *defined by* (1.15), *where* $\lambda_k' = \lambda_k$ *for* $1/(2k+2) < \eta_k < 1/(2k-1)$, *and* $\lambda_k' = \lambda_k + 1$ *for* $\eta_k = 1/(2k+2)$.

Proof. The proof of this simple and useful result is essentially given in [1] and [9], but we shall give a sketch for the sake of completeness. From (1.10) and (1.11) we have

$$A_k(x) = \sum_{m \le x} g_k(m) \sum_{n \le x/m} h_k(n)$$
$$= \sum_{r=k}^{2k-1} C_{r,k} x^{1/r} \sum_{m \le x} g_k(m) m^{-1/r} + \sum_{m \le x} g_k(m) \Delta_k^*(x/m), \tag{2.3}$$

where we have used (1.12) for $S_k(x) = \sum_{n \le x} h_k(n)$. From (1.9) and the fact that $\phi_k(s)$ converges absolutely for Re $s > 1/(2k+3)$ we infer that

$$\sum_{n \le x} g_k(n) \ll x^{1/(2k+2)},$$

whence by partial summation

$$\sum_{m \le x} g_k(m) m^{-1/r} = G_k(1/r) + \sum_{m > x} g_k(m) m^{-1/r} = G_k(1/r) + O(x^{1/(2k+2) - 1/r}). \tag{2.4}$$

Substituting (2.4) in (2.3) we obtain

$$A_k(x) = \sum_{r=k}^{2k-1} C_{r,k} G_k(1/r) x^{1/r} + O(x^{1/(2k+2)}) + \sum_{m \le x} g_k(m) \Delta_k^*(x/m)$$
$$= \gamma_{0,k} x^{1/k} + \gamma_{1,k} x^{1/(k+1)} + \cdots + \gamma_{k-1,k} x^{1/(2k-1)} + O(x^{\eta_k} \log^{\lambda_k'} x) \tag{2.5}$$

with

$$\gamma_{i,k} = C_{k+i,k} G_k(1/(k+i)) = C_{k+i,k} \phi_k(1/(k+i))/\zeta((2k+2)/(k+i)),$$

since

$$\sum_{m \le x} g_k(m) \Delta_k^*(x/m) \ll x^{\eta_k} \log^{\lambda_k} x \sum_{m \le x} |g_k(m)| m^{-\eta_k} \ll x^{\eta_k} \log^{\lambda_k} x,$$

because the second sum above is $O(\log x)$ if $\eta_k = 1/(2k+2)$ and it is bounded if $\eta_k > 1/(2k+2)$.

Lemma 1 is therefore proved; we have not considered the case $\eta_k < 1/(2k+2)$, since this would lead to (1.22), as remarked in Section 1.

We now proceed with the proof of Theorem 1 supposing $4 < k < 8$. By Lemma 1 it will be sufficient to prove $\Delta_k^*(x) \ll x^{1/2k+\varepsilon}$, but we remark that taking more care we could obtain $\Delta_k^*(x) \ll x^{1/2k} \log^c x$, with explicit $c = c(k) \ge 0$. The classical method of contour integration is applied to the function $H_k(s)$, which is regular except for simple poles at

$$s = 1/k,\ 1/(k+1),\ \ldots,\ 1/(2k-1).$$

Proceeding similarly as in [24, Lemma 3.12], we have, for x half a large odd integer, and $b > 1/k$,

$$\sum_{n\le x} h_k(n) = (2\pi i)^{-1} \int_{b-iT}^{b+iT} H_k(s)x^s s^{-1}\, ds + O(x^b T^{-1}(b-1/k)^{-1}) + O(\phi(2x)x^{1/k}T^{-1}\log x). \tag{2.6}$$

Here $\phi(x)$ denotes a non-decreasing positive function for which $h_k(n) = O(\phi(n))$, so that from the definition of $h_k(n)$ it is seen that one may take $\phi(x) = x^\varepsilon$ for any $\varepsilon > 0$. Therefore for fixed $1 > b > 1/k$, $\varepsilon > 0$,

$$\sum_{n\le x} h_k(n) = (2\pi i)^{-1} \int_{b-iT}^{b+iT} H_k(s)x^s s^{-1}\, ds + O(x^{1/k+\varepsilon}T^{-1}) + O(x^b T^{-1}). \tag{2.7}$$

Moving the line of integration to $\operatorname{Re} s = 1/2k$, we obtain, by the residue theorem,

$$(2\pi i)^{-1} \int_{b-iT}^{b+iT} H_k(s)x^s s^{-1}\, ds = \sum_{r=k}^{2k-1} \operatorname*{Res}_{s=1/r} H_k(s)x^s s^{-1} + (2\pi i)^{-1}(I_1 + I_2 + I_3), \tag{2.8}$$

where

$$I_1 = \int_{1/2k-iT}^{1/2k+iT} H_k(s)x^s s^{-1}\, ds,\quad I_2 = \int_{b-iT}^{1/2k-iT} H_k(s)x^s s^{-1}\, ds,\quad I_3 = \int_{1/2k+iT}^{b+iT} H_k(s)x^s s^{-1}\, ds. \tag{2.9}$$

We have

$$I_1 \ll x^{1/2k} \int_1^T \prod_{r=k}^{2k-1} |\zeta(r/2k + rit)|\, t^{-1}\, dt + x^{1/2k}, \tag{2.10}$$

and we now proceed to estimate

$$I_4 = \int_1^T \prod_{r=k}^{2k-1} |\zeta(r/2k + rit)|\, dt, \tag{2.11}$$

by repeated use of the Cauchy-Schwarz inequality for integrals, giving the detailed proof for $k = 7$, and omitting the easier cases $k = 5$ and $k = 6$. For this we shall need the following power moments for the zeta-function:

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^4\,dt \ll T\log^4 T,\quad \int_1^T |\zeta(\sigma+it)|^4\,dt \ll T,\quad \sigma>\tfrac{1}{2}, \tag{2.12}$$

$$\int_1^T |\zeta(\sigma+it)|^8\,dt \ll T^{1+\varepsilon}\quad \text{for } \sigma\geq\tfrac{5}{8}, \tag{2.13}$$

$$\int_1^T |\zeta(\sigma+it)|^{16}\,dt \ll T^{1+\varepsilon}\quad \text{for } \sigma\geq\tfrac{13}{16}. \tag{2.14}$$

The estimates (2.12) are to be found in 7.5 and 7.6 of [24], (2.13) was recently proved by D. R. Heath-Brown [7], and (2.14) follows with $\alpha=\frac{1}{2}$ from Theorem 7.9 of [24] when one uses

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^{16}\,dt \ll T^{8/3},$$

which follows trivially from $\int_1^T |\zeta(\frac{1}{2}+it)|^{12}\,dt \ll T^2\log^{17} T$, proved by D. R. Heath-Brown [6]. We obtain then, for $k=7$,

$$I_4 \leq \prod_{r=7}^{8}\left(\int_1^T |\zeta(r/14+rit)|^4\,dt\right)^{1/4}\prod_{r=9}^{11}\left(\int_1^T |\zeta(r/14+rit)|^8\,dt\right)^{1/8},$$

$$\prod_{r=12}^{13}\left(\int_1^T |\zeta(r/14+rit)|^{16}\,dt\right)^{1/16} \ll T^{1+\varepsilon}, \tag{2.15}$$

when we apply (2.12), (2.13), and (2.14), since $\frac{9}{14}>\frac{5}{8}$ and $\frac{12}{14}>\frac{13}{16}$, so that integrating by parts

$$I_1 \ll x^{1/2k}T^{\varepsilon}. \tag{2.16}$$

The integrals I_2 and I_3 are estimated in the same fashion; we give the details only for I_3. With $b=1/k+\varepsilon$ we obtain

$$I_3 \ll \int_{1/2k}^{1/k+\varepsilon} x^{\sigma}T^{-1}\prod_{r=k}^{2k-1}|\zeta(r\sigma+irT)|\,d\sigma. \tag{2.17}$$

If we use $\zeta(\sigma+it)\ll t^{(1-\sigma)/3}\log t$, $t\geq t_0$, $\frac{1}{2}\leq\sigma\leq 1$ (see [24]), then it follows that

$$I_2+I_3 \ll x^{1/k+\varepsilon}T^{q} = x^{1/k+\varepsilon}T^{(k-11)/12+k\varepsilon} \ll 1 \tag{2.18}$$

where

$$q=\frac{1}{3}\sum_{r=k}^{2k-1}\left(1-\frac{r}{2k}\right)+k\varepsilon-1$$

if $4 < k \le 10$ and $T = T(x)$ is sufficiently large. Using more refined estimates for the order of the zeta function in the critical strip, we could also obtain (2.18) for $k = 11$, 12 and 13. Our result that $\rho_k \le 1/2k$ for $4 < k < 8$ will follow from Lemma 1, (2.7), (2.8), (2.16) and (2.18) if one chooses $T = T(x)$ sufficiently large. For the estimates of ρ_3 and ρ_4 the reader is referred to Section 3.

3. The general three-dimensional divisor problem

Let now $\Delta(a, b, c; x)$ denote the error term in the general three-dimensional divisor problem, as defined by (1.20), where $1 \le a \le b \le c$ are fixed integers, and for brevity we set $d = a + b + c$. In the same paper [4] where the investigation of powerful numbers was initiated, Erdös and Szekeres investigated the asymptotic formula for the number of non-isomorphic abelian groups whose order does not exceed x. Subsequent authors successfully carried on this research (see [21], [22], [23]), and the problem can be reduced to the estimation of $\Delta(1, 2, 3; x)$. A useful formula for $\Delta(1, 2, 3; x)$ was discovered by P. G. Schmidt in [22], involving sums with the function

$$\psi(x) = x - [x] - \tfrac{1}{2}. \tag{3.1}$$

Following Schmidt's method of proof, the following generalization of his result may be obtained (see [11] for a proof):

LEMMA 2. *If $1 \le a < b \le c$, $b \le 2a$ are integers, then*

$$\Delta(a, b, c; x) = -\sum_{(u,v,w)} S_{u,v,w}(x) + O(x^{1/d}), \tag{3.2}$$

where $d = a + b + c$,

$$S_{u,v,w}(x) = \sum_{n \le x^{1/d}} \sum_{n < m \le (xn^{-w})^{1/(u+v)}} \psi((xm^{-v}n^{-w})^{1/u}), \tag{3.3}$$

and (u, v, w) is any permutation of (a, b, c).

We may write

$$S_{u,v,w}(x) \ll \max_{M,N} |S_{u,v,w}(x; M, N)| \log^2 x$$

where the maximum is taken over M, N satisfying $N \le x^{1/d}$, $N \le 2M$, and $M^{u+v}N^w \le x$, and

$$S_{u,v,w}(x; M, N) = \sum_{M < m \le 2M, N < n \le 2N, m^{u+v}n^w \le x, m > n} \psi((xm^{-v}n^{-w})^{1/u}). \tag{3.5}$$

To estimate the above sum we shall apply the following result of B. R. Srinivasan [23, Theorem 5], which enabled him to prove

$$\Delta(1, 2, 3; x) \ll x^{105/407} \log^2 x,$$

and which we state as follows:

LEMMA 3.

$$S_{u,v,w}(x; M, N) \ll (F^{1/2-\theta}M^{3/4-\theta/2}N^{5/4-3\theta/2})^{1/(3/2-\theta)} + F^{1/4}M^{1/4}N + F^{-1/2}MN, \tag{3.6}$$

where $F = (xM^{-v}N^{-w})^{1/u}$, $\theta \le \frac{33}{250}$.

We are now ready to prove Theorem 2. Since $N \ll M$ in $S_{u,v,w}(x; M, N)$, the condition $c \le a + b$ of Theorem 2 gives

$$(MN)^{d/2} \le M^{u+v}N^w \le x. \tag{3.7}$$

The conditions $c \le a + b$, $92b \le 171a$ ensure $3u - d/2 \ge 3a - d/2 \ge 0$, and so we obtain

$$F^{1/4}M^{1/4}N = (x(M^4N^8)^{u/2}(M^{u+v}N^w)^{-1})^{1/4u} \ll (x(MN)^{3u-d/2})^{1/4u} \ll x^{3/2d}, \tag{3.8}$$

and similarly

$$F^{-1/2}MN \ll x^{3/2d} = x^{3/2(a+b+c)}. \tag{3.9}$$

With $\theta = \frac{33}{250}$, the first term on the right-hand side of (3.6) becomes

$$\begin{aligned}(F^{92}M^{171}N^{263})^{1/342} &= (x^{92}(MN)^{263u}(M^{u+v}N^w)^{-92})^{1/342u} \\ &\ll (x^{92}(MN)^{263u-46d})^{1/342u} \\ &\ll (x^{526u/d})^{1/342u} = x^{263/171(a+b+c)},\end{aligned} \tag{3.10}$$

since $263u - 46d \ge 263a - 46(a + b + c) \ge 0$ if $(a, b, c) = (1, 2, 2)$ or if $c \le a + b$, $92b \le 171a$. Formula (1.21) follows from the above estimates since $\frac{3}{2} < \frac{263}{171}$.

With $(a, b, c) = (3, 4, 5)$, we obtain

$$\Delta(3, 4, 5; x) \ll x^{263/2052} \log^2 x,$$

which in view of Lemma 1 gives $\rho_3 \le \frac{263}{2052}$.

Finally to prove $\rho_4 \le \frac{3091}{25981}$ we use a result of [9] based on the work of E. Krätzel [15]. If $\Delta(\bar{a}_{k,m}; x)$ denotes the error term in the asymptotic formula for

$$\sum_{n_0^k n_1^{k+1} \cdots n_m^{k+m} \le x} 1$$

and if $\Delta(\bar{a}_{k,m}; x) \ll x^{\alpha_{k,m}}$, then (2.11) of [9] gives, with the exponent pair $(\frac{2}{7}, \frac{4}{7})$,

$$\alpha_{k,m} = \frac{2 + k\alpha_{k,m-1}}{5k + 2m - 2k(k + m)\alpha_{k,m-1}}$$

if $27k\beta_{k,m} \le 14 + 13k\alpha_{k,m-1}$, where $\beta_{k,m}$ is precisely defined in [9]. From Lemma 1 it is seen that ρ_4 is essentially $\alpha_{4,3}$ and using Theorem 2 we obtain

$\alpha_{4,2} \le \frac{263}{2565}$ which gives (after verifying that $108\beta_{4,3} \le 14 + 56\alpha_{4,2}$)

$$\rho_4 \le \frac{1 + 2\alpha_{4,2}}{13 - 28\alpha_{4,2}} \le \frac{3091}{25981} = 0.118971 \ldots,$$

as claimed.

Under suitable conditions on a, b, c one can replace $\frac{263}{171}$ in the exponent of (1.21) with

$$(5 + 6\lambda_0 - 6\lambda_1)/(3 + 2\lambda_0 - 2\lambda_1),$$

where (λ_0, λ_1) is any two-dimensional exponent pair satisfying $3\lambda_0 + \lambda_1 \le \frac{1}{2}$ (for the definition and properties of two-dimensional exponent pairs see [22]). If $(\frac{1}{12}, \frac{1}{4})$ were a two-dimensional exponent pair, then one would obtain a sharpening of Theorem 2 in the form

$$\Delta(a, b, c; x) \ll x^{3/2(a+b+c)} \log^2 x. \tag{3.11}$$

However, (3.11) certainly cannot hold for arbitrary values of a, b, c, since E. Krätzel has shown in [14] that

$$\Delta(a, b, c; x) = \Omega(x^{1/2(a+b)}), \quad c > 2(a + b), \tag{3.12}$$

where as usual $\Omega(f(x))$ is the negation of $o(f(x))$ as $x \to \infty$. E. Landau's classical theorems on lattice point problems (see [16], [17]) yield

$$\Delta(a, b, c; x) \ll x^{1/2a}, \quad \Delta(a, b, c; x) = \Omega(x^{1/(a+b+c)}). \tag{3.13}$$

The estimate $\Delta(a, b, c; x) \ll x^{1/2a}$ can also be obtained easily (with a factor $\log^{3/2} x$) following our proof of Theorem 1, and it supersedes the conjectured estimate (3.11) if $b + c < 2a$.

We may further remark that Theorem 2 gives

$$\Delta(1, 2, 2; x) \ll x^{263/855} \log^2 x \ll x^{0.3076024}, \tag{3.14}$$

which is an improvement of

$$\Delta(1, 2, 2; x) \ll x^{577/1740} \ll x^{0.3316092}, \tag{3.15}$$

proved in [11]. As shown in [11], (3.15) gives for some suitable constant $d_k \ge 0$

$$A_k(x, h) = \sum_{x < n \le x+h,\, a(n)=k} 1 = (d_k + o(1))h, \quad h \ge x^{581/1744} \log x, \tag{3.16}$$

where k is fixed, $x \to \infty$ and $a(n)$ is the number of non-isomorphic abelian groups of order n. By the method of proof of [11], the estimate (3.14) would improve the range for h in (3.16) to

$$h \ge x^{877/2653} \log^c x, \quad c \ge 5, \quad \frac{877}{2653} = 0.3305 \ldots .$$

4. Additive problems involving powerful numbers

Let $q_{k,n}$ denote the *nth* element of $G(k)$. The weak asymptotic formula

$$A_k(x) = \sum_{n \le x, n \in G(k)} 1 = \gamma_{0,k} x^{1/k} + O_k(x^{1/(k+1)}) \tag{4.1}$$

implies, for $x = q_{k,n}$,

$$n = \gamma_{0,k} q_{k,n}^{1/k} + O_k(q_{k,n}^{1/(k+1)}), \tag{4.2}$$

and therefore

$$q_{k,n} = (n/\gamma_{0,k})^k + O_k(n^{(k^2+k-1)/(k+1)}), \tag{4.3}$$

where O_k means that the implied constant depends on k, and, by (1.15) and (1.3),

$$\gamma_{0,k} = \prod_p \left(1 + \frac{1 - p^{(1-k)/k}}{p^{(k+1)/k} - p}\right). \tag{4.4}$$

Defining

$$R_{k,m}(n) = \sum_{x_1 + \cdots + x_m = n, x_1, \ldots, x_m \in G(k)} 1 \tag{4.5}$$

as the number of ways n can be written as a sum of m k-powerful numbers (0 and 1 are considered as k-powerful) it is seen from (4.3) that the summatory function

$$B_{k,m}(x) = \sum_{n \le x} R_{k,m}(n) \tag{4.6}$$

may be written as

$$B_{k,m}(x) = \sum_{x_1 + \cdots + x_m \le x, x_1, \ldots, x_m \in G(k)} 1 = \sum_{n_1^k + \cdots + n_m^k \le Y}' 1, \tag{4.7}$$

where

$$Y = \gamma_{0,k}^k x + O_{k,m}(x^{(k^2+k-1)/(k^2+k)}), \tag{4.8}$$

and $\sum'$ denotes summation over non-negative integers $n_1, \ldots, n_m$.

Our main goal is an asymptotic formula for $B_{k,m}(x)$, and from (4.7) it is seen that this problem is transformed into the well-known problem of determining the number of lattice points in certain well-defined multi-dimensional regions. It is well known (see [25], [12]) that $B_{k,m}(x)$ is approximated by the volume of the corresponding region, so that as $x \to \infty$, we have

$$B_{k,m}(x) \sim C_{k,m} x^{m/k}, \tag{4.9}$$

where $C_{k,m}$ may be explicitly evaluated. In case $k = 2$ we have

$$\gamma_{0,k} = \zeta(\tfrac{3}{2})/\zeta(3) = 2.1732 \ldots,$$

and from the asymptotic formula (see [25], [26], [2])

$$\sum_{n_1^2+\cdots+n_m^2\le x} 1 = \frac{\pi^{m/2}x^{m/2}}{\Gamma(m/2+1)} + O_m(x^{c_m}), \tag{4.10}$$

where $c_2 < \frac{1}{3}$, $c_3 < \frac{3}{4}$, $c_m = m/2 - 1$ for $m \ge 4$, we obtain

$$B_{2,m}(x) = \frac{\zeta^m(\frac{3}{2})\pi^{m/2}x^{m/2}}{2^m\zeta^m(3)\Gamma(m/2+1)} + O_m(x^{(m^2+m-1)/(2m+2)}). \tag{4.11}$$

Therefore, on the average there are $(\zeta(\frac{3}{2})/\zeta(3))^m$ more representations of an integer as a sum of m square-full numbers, than as a sum of m non-negative integer squares. Similarly from (4.7), (4.8), and the asymptotic formula for

$$\sum_{n_1^k+n_2^k\le x} 1$$

(see [13]), we obtain

$$\begin{aligned} B_{k,2}(x) = \prod_p \left(1 + \frac{1-p^{(1-k)/k}}{p^{(k+1)/k}-p}\right)^2 \cdot \frac{\Gamma^2(k+1/k)}{\Gamma(k+2/k)}\, x^{2/k} \\ + O_k(x^{2(k^2+k-1)/(k^3+k^2)}) \end{aligned} \tag{4.12}$$

The above result for $k = 2$ was obtained in [8] by an application of an additive theorem of V. Tašbaev [19, pp. 102–104]; the same theorem would improve the error term in (4.12) to $O_k(x^{(2k+1)/(k^2+k)})$, and it may be mentioned that starting from $B_{2,2}(x)$ and proceeding inductively an improvement for the error term in (4.11) may be also obtained.

From (4.12) it is seen that for $k > 2$ there exist arbitrarily large integers which are not representable as a sum of two k-powerful numbers. In case $k = 2$ the density of integers which are a sum of two square-full numbers is zero according to P. Erdös, and in the other direction it was proved recently by R. Odoni [18] that the number of integers not exceeding x which are a sum of two square-full numbers is much greater than

$$x(\log x)^{-1/2} \exp\,(C \log\log x/\log\log\log x)$$

for some constant $C > 0$ and $x > x_0$. A problem similar to Waring's may be proposed: find an integer $M(k)$ such that all but finitely many numbers n are a sum of $M(k)$ numbers from $G(k)$. Since a perfect kth power certainly belongs to $G(k)$, it follows from the work on Waring's problem that $M(k)$ is finite for all $k \ge 2$. P. Erdös conjectured that $M(k) = k + 1$. In particular, this conjecture asserts $M(2) = 3$, and it seems that all numbers except 7, 15, 23, 87, 111 and 119 are a sum of three square-full numbers (this was verified for $n \le 32761$). Integers not representable as a sum of three squares are numbers of the form $4^a(8K + 7)$, therefore only integers of the form $8K + 7$ are possibly not representable as a sum of three square-full numbers, since if $n = 4^a(8K + 7)$,

$a \geq 1$, then $8K + 7 = x^2 + y^2 + 2z^2$, where x, y, z are integers (see [3]) and therefore

$$n = (2^a x)^2 + (2^a y)^2 + 8(2^{a-1}z)^2,$$

which is a sum of three square-full numbers. By considering various quadratic forms it may be shown that certain types of integers are a sum of three square-full numbers. For instance, if $N = 25n$, the only case when N might not be a sum of three square-full numbers is when $n = 8K + 7$. But in this case $n = x^2 + y^2 + 5z^2$, since (see [3]) this quadratic form represents integers not of the form $4^a(8L + 3)$. Thus we have $N = (5x)^2 + (5y)^2 + 5^3z^2$, which is a sum of three square-full numbers.

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