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THE ACTION OF THE STABLE OPERATIONS OF COMPLEX K-THEORY ON COEFFICIENT GROUPS

BY

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Introduction

A stable operation of degree 0 on complex K-theory is a natural transformation

$$\phi^*: K^*(\ ) \to K^*(\ )$$

and may be identified with a map of spectra $\phi: K \to K$. Adams and Clarke [1] showed that the set of such operations is large, in fact uncountable. Since the coefficient groups $K^*(S^0) = \pi_*(K)$ are shown to be $\mathbf{Z}$ for $*$ even, 0 for $*$ odd, it is natural to ask what the action of ϕ on these groups might be. The present paper answers this question, both for K-theory and K-theory localized at a prime.

In [4] and [3], Lance and Clarke respectively considered the corresponding unstable question, i.e., the action induced in homotopy by a self H-map of BU or $BU_{(p)}$. Our results are of the same form as theorem 4 of [3], but in the stable case we must consider $\pi_i(K)$ for $i < 0$ as well.

We will define integers $\gamma_p(n)$, $\Gamma(n)$, $t_p(n,i)$, $v(n,i)$ for $n \in \mathbf{Z}^+$, $0 \le i \le n$ and show:

THEOREM 1. *If the action of $\phi: K_{(p)} \to K_{(p)}$ on $\pi_{2i}(K_{(p)}) = \mathbf{Z}_{(p)}$ is multiplication by λ_i, then*

$$\sum_{i=0}^{n} t_p(n,i) \cdot \lambda_{i-m} \equiv 0 \mod p^{\gamma_p(n)}$$

for all $n \in \mathbf{Z}^+$, $m \in \mathbf{Z}$. Furthermore every sequence $\{\lambda_i\}$ satisfying these congruences for the special cases $m = [n/2]$ arises from a unique map of spectra.

THEOREM 2. *If the action of $\phi: K \to K$ on $\pi_{2i}(K) = \mathbf{Z}$ is multiplication by λ_i, then*

$$\sum_{i=0}^{n} v(n,i) \cdot \lambda_{i-m} \equiv 0 \mod \Gamma(n)$$

for all $n \in \mathbf{Z}^+$, $m \in \mathbf{Z}$. Furthermore every sequence $\{\lambda_i\}$ satisfying these congruences for the special cases $m = [n/2]$ arises from a unique map of spectra.

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The functions γ_p and Γ are easily described:

$$\gamma_p(n) = \nu_p((n + [n/p - 1])!)$$

where $[x]$ denotes the largest integer less than or equal to x, and $\nu_p(x)$ denotes the p-adic valuation, i.e., the largest integer m for which p^m divides x; and $\Gamma(n)$ is the unique integer with $\nu_p(\Gamma(n)) = \gamma_p(n)$ for all primes p.

The integers $t_p(n,i)$ can be described as follows: For a given prime p, let $a_1, a_2, \ldots$ denote the sequence $1, 2, 3, \ldots, p-1, p+1, \ldots$ of integers prime to p. Then $t_p(n,i)$ is defined by the equation

$$(w - a_1)\ldots(w - a_n) = \sum_{i=0}^{n} t_p(n,i) \cdot w^i$$

For the integers $v(n,i)$, we first choose a sequence $a_{1,n}, \ldots, a_{n,n}$ of integers subject to the conditions that for each prime p for which $\gamma_p(n) > 0$, we have

$$a_{i,n} \equiv a_i \mod p^m$$

where m is the least integer for which $p^m > a_n$. Note that a_i depends on the prime p being considered. This is a finite set of conditions, and can always be satisfied, by the Chinese Remainder Theorem. The $v(n,i)$ are defined by the equation

$$(w - a_{1,n})\ldots(w - a_{n,n}) = \sum_{i=0}^{n} v(n,i) \cdot w^i.$$

The proof of these theorems is based on the fact that $(K_0K)_{(p)}$ and K_0K are free over $\mathbf{Z}_{(p)}$ and $\mathbf{Z}$ respectively. This was established in [1], and implies that the Kronecker pairing induces an isomorphism. In §1 we construct explicit bases for $(K_0K)_{(p)}$ and K_0K and use this in §2 to prove the theorems.

Section 1

We begin by recalling from [2] the description of the hopf algebra K_*K. There it was pointed out that the natural map $K_*K \to K_*K \otimes Q$ is an injection, and that $K_*K \otimes Q$ equals $Q[u, v, u^{-1}, v^{-1}]$, i.e., finite Laurent series in two variables [2, Propositions 2.1 and 2.2]. Thus it suffices to describe those series lying in the image of this map, and this was done in [2] by giving a certain integrality condition (Theorem 2.3).

For our purposes it is sufficient to only consider K_0K, and to give a slightly different description. Letting $w = v/u$, we see that

$$K_0K \otimes Q = Q[w, w^{-1}].$$

Let A denote the ring of polynomials $f \in Q[w]$ which take integral values at the integers. Proposition 5.3 and Theorem 2.3 of [2] can be restated as the following description of K_0K in terms of A:

PROPOSITION 3. *The image of K_0K in $Q[w, w^{-1}]$ equals the union of the subrings $(1/w^n) \cdot A$ for $n = 0, 1, 2, \ldots$.*

In [12], Adams and Clarke show that K_0K is actually a free abelian group. This isn't obvious from the description above, even though A is easily seen to be free. The difficulty is that $K_0K \cap Q[w]$ contains more than just A, for example it contains $(w^2 - 1)/24$.

We will see that this problem does not arise in the p-local case, and so we consider it first. Let us fix a prime p, and let B denote the subring of $Q[w]$ consisting of those polynomials f for which $f(k) \in \mathbf{Z}$, if k is an integer prime to p. Also let us denote by $G_{(p)}$ the p-localization of an abelian group G.

LEMMA 4. *$B \supseteq A$, and for any $f \in B$ there exists an integer n such that $w^n \cdot f \in A_{(p)}$.*

PROOF. The first statement is immediate. For the second, take n to be the maximum of the p-exponents of the denominators of the coefficients of the polynomial f.

The inclusion $K_0K \to Q[w, w^{-1}]$ extends uniquely to an inclusion

$$(K_0K)_{(p)} \to Q[w, w^{-1}].$$

The previous lemma implies the following p-local analog of Proposition 3:

PROPOSITION 5. *The image of $(K_0K)_{(p)}$ in $Q[w, w^{-1}]$ equals the union of the subrings $(1/w^n) \cdot B_{(p)}$.*

In contrast with A, the ring B has the following useful property:

LEMMA 6. If $w^n \cdot f \in B_{(p)}$, *and* $f \in Q[w]$, *then* $f \in B_{(p)}$.

Proof. It suffices to show that if $w^n \cdot f \in B$, then there exists a non-zero integer b prime to p for which $b \cdot f \in B$. There certainly exists some non-zero integer for which $b \cdot f \in B$, for example the product of the denominators of the coefficients of f. Order the non-zero integers with this property by divisibility and choose a minimal one, b.

Suppose b were divisible by p, and let $b = p \cdot b'$. If $(k,p) = 1$, then we have $b \cdot f(k) = p \cdot b' \cdot f(k) \in \mathbf{Z}$ and also $k^n \cdot f(k) \in \mathbf{Z}$. Thus $b' \cdot f(k) \in \mathbf{Z}$, and so $b' \cdot f \in B$, contradicting the minimality of b.

This lemma will allow us to construct a basis for $(K_0K)_{(p)}$ from one for $B_{(p)}$. A basis for $B_{(p)}$ can be constructed as follows:

DEFINITION 7. Define polynomials $q_n(w) \in Q[w]$ by

$$q_0(w) = 1, \quad q_n(w) = (w - a_1)\dots(w - a_n)/(a_{n+1} - a_1)\dots(a_{n+1} - a_n)$$

where $a_1, a_2, \dots$ are as defined in the introduction. Note that the p-adic norm of the denominator of $q_n(w)$ is $\gamma_p(n)$.

PROPOSITION 8. *$\{q_n \mid n = 0, 1, 2, \dots\}$ is a basis over $\mathbf{Z}_{(p)}$ for $B_{(p)}$.*

Proof. Since

(a) degree $(q_n) = n$,

(b) $q_n(a_i) = 0$ if $i \leq n$ and $q_n(a_i) = 1, i = n+1$,

it is clear that any polynomial f of degree n in $B_{(p)}$ can be expressed as a $\mathbf{Z}_{(p)}$ linear combination of $q_o, \ldots, q_n$.

It remains, therefore, to show that $q_n \in B_{(p)}$. For this we note that $B_{(p)}$ can be described as those $f \in Q[w]$ for which $f(k) \in \mathbf{Z}_{(p)}$ for all integers k prime to p. Thus we must show that for such k,

$$\nu_p((k-a_1)\ldots(k-a_n)) \geq \nu_p((a_{n+1}-a_1)\ldots(a_{n+1}-a_n)).$$

Since $(k,p) = 1$, we note that

$$\begin{aligned} \nu_p((k-a_1)\ldots(k-a_n)) &= \nu_p((k-1)\cdot(k-2)\ldots(k-a_n)) \\ &= \nu_p((k-1)!/(k-a_n-1)!) \end{aligned}$$

and

$$\begin{aligned} \nu_p((a_{n+1}-a_1)\ldots(a_{n+1}-a_n)) &= \nu_p((a_{n+1}-1)!/(a_{n+1}-a_n-1)!) \\ &= \nu_p((a_{n+1}-1)!). \end{aligned}$$

Now $(k-1)!/(k-a_n-1)!a_n!$ is a binomial coefficient, and so is an integer. Thus

$$\nu_p((k-1)!/(k-a_n-1)!) \geq \nu_p(a_n!).$$

If a_{n+1} is not congruent to 1 mod p, then $a_{n+1}-1 = a_n$, and we are finished. If $a_{n+1} \equiv 1 \bmod p$, then

$$\begin{aligned} \nu_p((k-a_1)\ldots(k-a_n)) &= \nu_p((k-1)!/(k-a_n)!) \\ &\geq \nu_p((a_n+1)!) \\ &= \nu_p((a_{n+1}-1)!). \end{aligned}$$

PROPOSITION 9. *$\{(1/w^{[n/2]})\cdot q_n \mid n = 0,1,2,\ldots\}$ is a basis for $(K_0K)_{(p)}$ over $\mathbf{Z}(p)$.*

Proof. We make use of the subgroups $F(n,m)$ introduced in [1]. Let

$$F(n,m) = (K_0K)_{(p)} \cap \operatorname{span}(w^n, w^{n+1}, \ldots, w^m)$$

and let $l,t : F(n,m) \to \mathbf{Q}$ be the homomorphisms $l(f) = a_m, t(f) = a_n$ if $f = \Sigma_{i=n}^{m} a_i w^i$. By Lemma 6, any element of $F(n,m)$ is of the form $w^n \cdot f$ with $f \in B_{(p)}$ of degree $m-n$. Since f is a linear combination of $q_o, \ldots, q_{m-n}$ we see that image(l) and image(t) are equal to $l(q_{m-n})\cdot\mathbf{Z}(p)$ and $t(q_{m-n})\cdot\mathbf{Z}(p)$ respectively. Also, l and t induce isomorphisms

$$F(n,m)/F(n,m-1) \simeq \text{Image}(l), \quad F(n,m)/F(n+1,m) \simeq \text{Image}(t).$$

Thus we see, by induction on n, that

$$q_o, \ldots, (1/w^{[n/2]})\cdot q_n$$

is a basis for $F(-[n/2], n-[n/2])$.

We return now to the question of finding a basis for K_0K. Our construction is based on the following observation:

PROPOSITION 10. *If $\{x_i \mid i = 0,1,2,\dots\}$ is a subset of a torsion free abelian group G with the property that for each prime p it forms a $\mathbf{Z}_{(p)}$ basis for $G_{(p)}$, then it forms a basis for G over $\mathbf{Z}$.*

Our candidates for basis elements for K_0K are the Laurent polynomials

$$(1/w^{[n/2]}) \cdot p_n(w),$$

where we define

$$p_o(w) = 1 \text{ and } p_n(w) = (w-a_{1,n})\dots(w-a_{n,n})/\Gamma(n)$$

with $\Gamma(n), a_{1,n}, \dots, a_{n,n}$ as in the introduction.

LEMMA 11. *For every prime p and nonnegative integer n, $p_n(w) \in B_{(p)}$.*

Proof. If $p > n + [n/p - 1]$, then $\gamma_p(n) = \nu_p(\Gamma(n)) = 0$ and the result is obvious. Otherwise, note that if k is an integer prime to p, then

$$\nu_p(k - a_{i,n}) \geq \nu_p(k - a_i)$$

since $a_{i,n} \equiv a_i \bmod p^m$ and $0 \leq a_i \leq p^m$. Thus

$$\nu_p((k-a_{1,n})\dots(k-a_{n,n})) \geq \nu_p((k-a_1)\dots(k-a_n)) \geq \gamma_p(n) = \nu_p(\Gamma(n)).$$

LEMMA 12. *$\{1/w^{[n/2]} \cdot p_n(x) \mid n = 0,1,\dots\}$ is a basis for $(K_0K)_{(p)}$ over $\mathbf{Z}_{(p)}$.*

Proof. By Lemma 11, the prospective basis elements are actually in $(K_0K)_{(p)}$. Consider the matrix expressing $1/w^{[n/2]} \cdot p_n(w)$ in terms of the basis elements of Proposition 9. Since

$$1/w^{[n/2]} \cdot p_n(w) \in F(-[n/2], n - [n/2]),$$

it is a $\mathbf{Z}_{(p)}$ linear combination of $q_o, \dots, 1/w^{[n/2]} \cdot q_n(w)$ and so the matrix is upper triangular. Furthermore the leading and trailing coefficients of both $p_n(w)$ and $q_n(w)$ have p-adic norm - $\gamma_p(n)$ so that the diagonal entries of the matrix are all units in $\mathbf{Z}_{(p)}$. Thus, by Cramer's rule, the matrix is invertible. The result follows.

COROLLARY 13. *$\{1/w^{[n/2]} \cdot p_n(w) \mid n = 0,1,\dots\}$ is a basis for K_0K over $\mathbf{Z}$.*

Section 2

Now that we have constructed a basis for $(K_0K)_{(p)}$ and K_0K we may conclude, as in [1], Theorem 2.1:

PROPOSITION 14. *The Kronecker Pairing induces isomorphism*

$$(K^0K)_{(p)} \simeq \mathrm{Hom}_{\pi_*K_{(p)}}((K_*K)_{(p)}, \pi_*(K_{(p)}))^0 \simeq \mathrm{Hom}(K_0K_{(p)}, \mathbf{Z}_{(p)})$$

and

$$K_0K \simeq \mathrm{Hom}_{\pi_*K}(K_*K\pi_*K)^0 \simeq \mathrm{Hom}(K_0K, \mathbf{Z}).$$

PROPOSITION 15. *The action of*

$$\phi \in \mathrm{Hom}(K_0K, \mathbf{Z}) \text{ or } \phi \in \mathrm{Hom}((K_0K)_{(p)}, \mathbf{Z}_{(p)})$$

on $\pi_{2i}(K)$ *or* $\pi_{2i}(K)_{(p)}$ *is multiplication by* $\phi(w^i)$.

Proof. Since $\pi_*(K) = \mathbf{Z}[t, t^{-1}]$, where t is of degree 2, the action of a homomorphism $f: \pi_{2i}(K) \to \pi_{2i}(K)$ is multiplication by $f(t^i)$. Recall from [2] that the elements $w, u, v \in K_*K$ are defined by $w = v/u$, $v = \eta_R(t)$, $u = \eta_L(t)$ where η_R and η_L are the right and left actions of π_*K on K_*K. If $\phi \in K^0K$, then its action on π_*K is given by

$$\phi(x) = <\phi, \eta_R(x)>$$

where $< \ , \ >$ denotes the Kronecker product.

In Proposition 14 the isomorphism $\mathrm{Hom}_{\pi_*K}(K_*K, \pi_*K) \to \mathrm{Hom}(K_0K, \mathbf{Z})$ is a restriction to K_0K. Using the fact that K_*K is an extended π_*K module [2] we see that an inverse to this isomorphism is given by

$$\chi(f)(x) = f(u^{-1} \cdot x)$$

if $f \in \mathrm{Hom}(K_0K, \mathbf{Z})$, $x \in K_*K$ is of degree $2i$. If $\phi \in K^0K$ is the element whose image under the isomorphisms of Proposition 14 is f, then the action of ϕ on $\pi_{2i}(K)$ is multiplication by

$$\phi(t^i) = <\phi, \eta_R(t^i)> = <\phi, v^i> = f(u^{-i}v^i) = f(w^i).$$

The p-local case is similar.

Proof of Theorem 1. Identify $\phi \in (K^0K)_{(p)}$ with an element of Hom $((K_0K)_{(p)}, \mathbf{Z}_{(p)})$ via Proposition 14. Since $q_n \in (K_0K)_{(p)}$ for all n, we must have

$$\phi(w^{-m} \cdot q_n) \in \mathbf{Z}_{(p)} \quad \text{for any } m.$$

In other words,

$$\phi(w^{-m}(w - a_1)\dots(w - a_n)/(a_{n+1} - a_1)\dots(a_{n+1} - a_n)) \in \mathbf{Z}_{(p)}$$

or

$$\phi(w^{-m}(w - a_1)\dots(w - a_n)) \in p^{\gamma_p(n)} \cdot \mathbf{Z}_{(p)}.$$

Using the definition of $t(n, i)$, this is

$$\sum t_p(n,i) \cdot \phi(w^{i-m}) \in p^{\gamma_p(n)} \cdot \mathbf{Z}(p)$$

or

$$\sum t_p(n,i) \cdot \lambda_{i-m} \in p^{\gamma_p(n)} \cdot \mathbf{Z}(p).$$

as required.

Conversely suppose that $\{\lambda_i\}$ is a sequence of elements $\mathbf{Z}_{(p)}$ satisfying the congruences for the cases $m = [n/2]$. Then

$$x_i = \left(\sum t(n,i) \cdot \lambda_{i-[n/2]})/(p^{\gamma_p(n)}\right)$$

lies in $\mathbf{Z}_{(p)}$. Define an element ϕ of $\mathrm{Hom}((K_0K)_{(p)}, \mathbf{Z}_{(p)})$ by

$$\phi((1/w^{[n/2]}) \cdot q_n) = x_n.$$

Since these elements form a basis for $(K_0K)_{(p)}, \phi$ is uniquely defined, and has the required property.

The proof of Theorem 2 is similar.

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