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ON THE UNIFORM CONVEXITY OF L^p SPACES, $1 < p \leq 2$

BY

A. MEIR

1. A normed linear vector space E is called uniformly convex (Clarkson [1]) or uniformly rotund (Day [2]) if for every ε, $0 < \varepsilon < 2$,

$$\delta(\varepsilon) = \delta_E(\varepsilon) \doteq \inf \{1 - \|\tfrac{1}{2}(x + y)\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}$$

is positive. The function $\delta(\varepsilon)$ is called modulus of rotundity. It was proved by Clarkson [1] (see also Köthe [5], pp. 358–362 and Day [2], pp. 144–149) that the classical real or complex Lebesgue spaces L^p are uniformly convex for $1 < p < \infty$. Both the proof of this result as well as the explicit determination of the modulus $\delta_p(\varepsilon)$ is easy when $p \geq 2$. For, in this case, elementary arguments yield that for $\|x\|_p \leq 1$, $\|y\|_p \leq 1$ we have

$$\left\| x + y \right\|_p^p + \left\| x - y \right\|_p^p \leq 2^p \tag{1.1}$$

Hence, if $\|x - y\|_p \geq \varepsilon$,

$$\left\| \frac{1}{2}(x + y) \right\|_p < 1 - \frac{1}{p}\left(\frac{\varepsilon}{2}\right)^p.$$

The proofs given in [1], [3], [5] for the uniform convexity of L_p and for the calculation of $\delta_p(\varepsilon)$, when $1 < p \leq 2$, are much more complicated. In a recent paper Jakimovski and Russell [4] established an inequality which (when $\lambda = 1/2$) is of the same form as (2.1) but with $p(p - 1)/8$ replaced by an unknown constant. When $1 < p \leq 2$, the inequality of [4] yields the uniform convexity of L^p but not the evaluation of $\delta_p(\varepsilon)$. The purpose of this paper is to prove for $1 < p \leq 2$, a more precise inequality by a much simpler argument. As a corollary it yields not only the uniform convexity of L^p for $1 < p \leq 2$, but also a very simple proof of Hanner's result [3], namely, that

$$\delta_p(\varepsilon) = (p - 1)\varepsilon^2/8 + O(\varepsilon^3), \quad \text{as } \varepsilon \to 0.$$

2. We shall prove the following:

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THEOREM 1. *Let $1 < p \le 2$ and let L^p denote the real or complex Lebesgue space over a measure space Ω. Then for every $f, g \in L^p$, we have*

$$\left\|\frac{1}{2}(|f|+|g|)\right\|_p^{2-p}\left\{\frac{1}{2}\left\|f\right\|_p^p+\frac{1}{2}\left\|g\right\|_p^p-\left\|\frac{1}{2}(f+g)\right\|_p^p\right\} \tag{2.1}$$
$$\ge \frac{p(p-1)}{8}\left\|f-g\right\|_p^2.$$

COROLLARY 1. *If $1 < p \le 2$, $\|f\|_p \le 1$, $\|g\|_p \le 1$, then*

$$\left\|\frac{1}{2}(f+g)\right\|_p \le 1 - \frac{p-1}{8}\left\|f-g\right\|_p^2, \tag{2.2}$$

i.e.,

$$\delta_p(\varepsilon) \ge \frac{p-1}{8}\varepsilon^2.$$

Moreover, this estimate for $\delta_p(\varepsilon)$ is asymptotically best possible, as $\varepsilon \to 0$.

Remark. (i). Inequalities (2.1) and (2.2) become trivial for $p = 1$. Indeed, there exist $f, g \in L^1[0, 1]$ such that

$$\|f\|_1 = \|g\|_1 = \|\tfrac{1}{2}(f+g)\|_1 = 1 \quad \text{and} \quad \|f-g\|_1 = 1.$$

Remark (ii). From (2.1) one can deduce also the inequality

$$2\left\|f\right\|_p^2 + 2\left\|g\right\|_p^2 > \left\|f+g\right\|_p^2 + \frac{p(p-1)}{2}\left\|f-g\right\|_p^2 \tag{2.3}$$

for every $f, g \in L^p$, $1 < p \le 2$. Observe that for $p = 2$, the equality sign holds in (2.3), as well as in (2.1).

3. *Proof.* For the proof we shall need only the following simple facts: If $-1 \le u \le 1$ and $1 < p \le 2$, then

$$\frac{1}{2}(1+u)^p + \frac{1}{2}(1-u)^p \ge 1 + \frac{p(p-1)}{2}u^2. \tag{3.1}$$

If α, β are complex numbers, then

$$|\alpha-\beta|^2 + |\alpha+\beta|^2 = (|\alpha|-|\beta|)^2 + (|\alpha|+|\beta|)^2. \tag{3.2}$$

If $0 \le v \le 1$ and $1 < p \le 2$, then

$$\frac{1}{p}(1-v^p) \ge \frac{1}{2}(1-v^2). \tag{3.3}$$

If F, G, H are non-negative functions in L^p, $p > 1$, and $F^{1-r}G^r \geq H$ everywhere, with some r, $0 < r < 1$, then

$$\left\|F\right\|_p^{1-r} \cdot \left\|G\right\|_p^r \geq \left\|H\right\|_p. \tag{3.4}$$

The inequalities (3.1) and (3.3) can be proven by calculus; (3.2) is the parallelogram rule and (3.4) follows from Hölder's inequality applied to $F^{p(1-r)}G^{pr}$.

In order to prove (2.1), let f, $g \in L^p$, $1 < p \leq 2$. We set

$$u = \frac{|f| - |g|}{|f| + |g|}, \quad v = \frac{|f + g|}{|f| + |g|}, \quad w = \frac{|f - g|}{|f| + |g|},$$

if $|f| + |g| > 0$, otherwise $u = v = w = 1$. Then we clearly have $-1 \leq u \leq 1$, $0 \leq v \leq 1$ and, by (3.2), $u^2 + 1 = v^2 + w^2$. Hence, by (3.1),

$$\begin{aligned}\frac{1}{2}(1+u)^p + \frac{1}{2}(1-u)^p &\geq 1 + \frac{p(p-1)}{2}u^2 \\ &\geq 1 + \frac{p(p-1)}{2}w^2 - \frac{p(p-1)}{2}(1-v^2) \\ &\geq 1 + \frac{p(p-1)}{2}w^2 - (p-1)(1-v^p),\end{aligned}$$

where we used (3.3) in the last inequality. Thus we obtained

$$\frac{1}{2}(1+u)^p + \frac{1}{2}(1-u)^p - [2 - p + (p-1)v^p] \geq \frac{p(p-1)}{2}w^2 \tag{3.5}$$

and, a fortiori,

$$\frac{1}{2}(1+u)^p + \frac{1}{2}(1-u)^p - v^p \geq \frac{p(p-1)}{2}w^2, \tag{3.6}$$

since $0 \leq v \leq 1$.

Substitution of u, v, w into (3.6) and multiplication of both sides by $\frac{1}{4}(|f| + |g|)^2$ yields

$$\left(\frac{1}{2}|f| + \frac{1}{2}|g|\right)^{2-p} \cdot \left[\frac{1}{2}|f|^p + \frac{1}{2}|g|^p - \left|\frac{1}{2}(f+g)\right|^p\right] \geq \frac{p(p-1)}{8}|f-g|^2.$$

Taking square root on both sides, applying (3.4) to the left hand product with $r = p/2$, and then squaring both sides, we obtain (2.1).

Proof of Corollary 1. If $\|f\|_p \leq 1$, $\|g\|_p \leq 1$, then, by (2.1),

$$1 - \left\|\frac{1}{2}(f+g)\right\|_p^p \geq \frac{p(p-1)}{8}\|f-g\|^2.$$

Since $p(1 - c) \ge 1 - c^p$ for $0 \le c \le 1$, the last inequality implies (2.2). In order to prove that the estimate for $\delta_p(\varepsilon)$ is best possible, we let $\Omega = [0, 1]$ with the usual Lebesgue measure, $f(2) = 1$ for $0 \le t \le 1$ and for given $\varepsilon > 0$,

$$g(t) = \begin{cases} 1 + \varepsilon, & 0 \le t \le 1/2 \\ 1 - \eta, & 1/2 \le t \le 1, \end{cases}$$

where $\eta > 0$ is so chosen that $(1 + \varepsilon)^p + (1 - \eta)^p = 2$. Then it is easy to see that

$$\eta = \varepsilon + O(\varepsilon^2) \quad \text{and} \quad \left\| \frac{1}{2}(f + g) \right\|_p^p = 1 - \frac{p(p-1)}{8}\varepsilon^2 + O(\varepsilon^3),$$

as $\varepsilon \to 0$. This proves our claim.

Remark (iii). Inequality (2.1) could be somewhat strengthened by using (3.5) instead of the weaker inequality (3.6). We then obtain for every $f, g \in L^p$, $1 \le p \le 2$,

$$\left\| \frac{1}{2}(|f| + |g|) \right\|_p^{2-p} \cdot \left[\frac{1}{2}\left\| f \right\|_p^p + \frac{1}{2}\left\| g \right\|_p^p - (p-1)\left\| \frac{1}{2}(f+g) \right\|_p^p \right]$$
$$\ge (2 - p)\left\| \frac{1}{2}(|f| + |g|) \right\|_p^2 + \frac{p(p-1)}{8}\left\| f - g \right\|_p^2.$$

Remark (iv). Inequality (2.1) remains valid also if f and g are Hilbert space valued functions over a measure space Ω and the L^p-norm is defined by $(\int_\Omega \|f\|_H^p)^{1/p}$. This follows from the fact that the only property of the complex numbers used in the proof is (3.2), which is true in a Hilbert space if absolute values are replaced by the Hilbert space norms.

4. By a mild modification of the proof we can obtain the following, more precise version of Theorem 1 in [4] for the case $1 < p \le 2$:

THEOREM 2. *Let $1 < p \le 2$, $0 < \lambda < 1$. Then for every $f, g \in L^p$ we have*

$$\left\| \frac{1}{2}(|f| + |g|) \right\|_p^{2-p} \cdot \left\{ \lambda\left\| f \right\|_p^p + (1 - \lambda)\left\| g \right\|_p^p - \left\| \lambda f + (1 - \lambda) g \right\|_p^p \right\}$$
$$\ge \frac{1}{4}p(p-1)\left\| f - g \right\|_p^2 \cdot \min(\lambda, 1 - \lambda)$$

We omit the proof.

COROLLARY. *If* $\|f\|_p \le 1$, $\|g\|_p \le 1$, $\|f-g\|_p \ge \varepsilon$, *then*

$$\left\| \lambda f + (1-\lambda) g \right\|_p < 1 - \frac{p-1}{4} \mu \varepsilon^2$$

where $\mu = \min(\lambda, 1-\lambda)$.

We conjecture that this estimate is asymptotically best possible as $\varepsilon \to 0$, $\lambda \to 0$. Note that for $\lambda = 1/2$ it reduces to our earlier inequality (2.2)

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UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA