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FINITE GROUPS THAT ACT ON SPHERES IN WHICH A CENTRAL ELEMENT ACTS FREELY

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Introduction

It is known that if a finite group, G, acts freely on a homotopy $(2N-1)$-sphere then $H^*(G; Z)$ has period $2N$. In this paper we show that if G is a finite group with a central element T of order p (p a prime) and if G acts on a homotopy $(2N-1)$-sphere in such a way that T acts freely then this puts certain restrictions on the Hochschild–Serre spectral sequence for computing $H^*(G; Z_p)$, and in particular we obtain an element

$$\xi \in H^{2p^{\mu(N)}}(G; Z_p) \quad \text{where } \mu(N) = \max\{i: p^i \mid N\},$$

such that ξ is a non-zero divisor in the ring $H^*(G; Z_p)$. We can use this to prove that $H^*(G; Z_p)$ has period $2p^{\mu(N)}$ in the case that G acts freely.

In Section 1 we establish some relevant homological algebra. In Section 2 we describe a splitting lemma: namely if K acts on a homotopy lense space L then $H^*(EK \times_K L; Z_p)$ is a $H^*(K; Z_p)$ direct summand of $H^*(G; Z_p)$ where G and K are related by an extension $1 \to Z_p \to G \to K \to 1$. In Section 3 we prove the main theorems of the paper and discuss the example of extra-special 2-groups acting on a homotopy sphere in which the central element of order 2 acts freely.

1. Homological algebra

Let K be a field and Λ a finitely generated augmented K-algebra, assumed to be commutative. Let C_* always denote a Λ-chain complex such that each C_n is a finitely generated free Λ-module. Let $H_* C_*$ be the usual homology groups of C_*.

We will let $C_*^{[N]}$ denote the free Λ-chain complex constructed by killing off the cycles of C_* in dimensions $N+1$ and larger [1]. Then $C_*^{[N]}$ comes equipped with a chain map

$$j: C_* \longrightarrow C_*^{[N]}$$

and satisfies the following properties.

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(1.1) (a) $H_i C_*^{[N]} = 0$ for $i > N$.

(b) $j_*\colon H_i C_* \to H_i C_*^{[N]}$ is an isomorphism for $i \leq N$.

(c) $j\colon C_* \to C_*^{[N]}$ satisfies the following universal mapping property: Suppose D_* is a Λ-chain complex and $H_i D_* = 0$ for $i > N$ and $\phi\colon C_* \to D_*$ is a Λ-chain map, then there is a factorization, $\bar{\phi}$,

$$\begin{array}{ccc} C_* & \xrightarrow{\;j\;} & C_*^N \\ \phi \big\downarrow & \swarrow{\scriptstyle \phi} & \\ D_* & & \end{array}$$

such that $\bar{\phi} \circ j = \phi$. Furthermore if $\phi'\colon C_*^{[N]} \to D_*$ is any other chain map with $\phi' j \simeq \phi$ then $\phi' \simeq \phi$ ∎

Spectral Sequences. For C_*, K and Λ as above let $H^* C_*$ (resp. $H^*_\Lambda C_*$) denote the cohomology of the cochain complex $\mathrm{Hom}_K(C_*, K)$ (resp. $\mathrm{Hom}_\Lambda(C_*, K) \cong \mathrm{Hom}_K(K \otimes_\Lambda C_*, K)$). Then, by [2], there is a spectral sequence of $\mathrm{Ext}^*_\Lambda(K, K)$-modules

$$E_2^{**} = \mathrm{Ext}^*_\Lambda(K, H^* C_*) \Rightarrow H^*_\Lambda C_*. \tag{1.2}$$

Furthermore this spectral sequence is natural with respect to maps of Λ-chain complexes.

If X is a CW-complex endowed with a free cellular action by a finite group G and $C_*(X; K)$ is the CW-chain complex of X, $\Lambda = K[G]$ the group ring, then we may identify

$$K \otimes_{K[G]} C_*(X; K) \cong C_*(X/G; K) \quad \text{and} \quad X/G \simeq EG \times_G X.$$

Then the spectral sequence (1.2) coincides with the Serre spectral sequence associated to the fibration

$$X \longrightarrow EG \times_G X \longrightarrow BG.$$

To establish notation we recall the mod p cohomology rings of the cyclic groups.

(1.3) Proposition [2]. (a) *If $p = 2$ then*

$$H^*(Z_2; Z_2) \cong \mathrm{Ext}^*_{Z_2[Z_2]}(Z_2, Z_2) \cong Z_2[x_1]$$

where $x_1 \in H^1(Z_2; Z_2)$.

(b) *If p is an odd prime then*

$$H^*(Z_p; Z_p) \cong \mathrm{Ext}^*_{Z_p[Z_p]}(Z_p, Z_p) \cong Z_p[x_2] \otimes \Lambda(x_1)$$

where $x_i \in H^i(Z_p; Z_p)$ and $\beta x_1 = x_2$ where β is the Bockstein operator.

2. A splitting lemma

Fix a natural number N and a prime p. Let L be a finite CW-complex with $\pi_1 X \cong Z_p$ and

$$H^*(L; Z_p) \cong \begin{cases} Z_2[x_1]/(x_1^{2N}) & \text{if } p = 2, \\ Z_p[x_2]/(x_2^N) \otimes \Lambda(x_1) & \text{if } p \text{ is odd} \end{cases}$$

and $\beta x_1 = x_2$ in the latter case. Assume that K is a finite group and that K acts cellularly on L in such a way that the induced K action on $\pi_1 L$ is trivial. K acts diagonally on $EK \times L$ and it is a free action. Let $\omega \in H^2(K; Z_p)$ be the k-invariant [1] associated to the above free action. Let $K(Z_p, 1, \omega)$ be the K-Eilenberg–MacLane space with k-invariant ω. Then there is an equivariant map

$$f\colon EK \times L \longrightarrow K(Z_p, 1, \omega) \tag{2.1}$$

that induces an isomorphism on fundamental groups. It follows that the induced map f^* on (non-equivariant) mod p cohomology is an epimorphism with kernel (x_1^{2N}) for $p = 2$ and (x_2^N) for odd primes.

We will let $\mathscr{L}(\omega) = C_*(K(Z_p, 1, \omega); Z_p)$, regarded as a free $Z_p[K]$-chain complex. $\mathscr{L}^{[N]}(\omega)$ is the construction described in Section 1.

We define a map θ by the commutative diagram

$$\begin{array}{ccc} C_*(EK \times L; Z_p) & \xrightarrow{\ \theta\ } & \mathscr{L}^{[2N-1]}(\omega) \\ & {\scriptstyle f_\#}\searrow \quad \swarrow{\scriptstyle j} & \\ & C_*(K(Z_p, 1, \omega); Z_p) & \end{array}$$

where j is defined in Section 1.

(2.2) Proposition. *θ is a $Z_p[K]$-chain homotopy equivalence.*

Proof. Since both chain complexes are free $Z_p[K]$-complexes it is enough to show that θ induces an isomorphism on non-equivariant cohomology. We have already seen that f^* is an isomorphism in the range where $H^*(L; Z_p)$ is non-zero. Furthermore in this range j^* is the dual to an isomorphism (1.1). This proves the proposition. ∎

For a free $Z_p[K]$-chain complex C_* let $E_r(C_*)$ be the spectral sequence (1.2) for computing $H^*(Z_p \otimes_{Z_p[K]} C_*)$. For convenience let $C_* = C_*(EK \times L; Z_p)$. By naturality we have maps of spectral sequences

$$\begin{array}{ccccc} E_r(\mathscr{L}^{[2N-1]}(\omega)) & \xrightarrow{\ j^*\ } & E_r(\mathscr{L}(\omega)) & \xrightarrow{\ f^*\ } & E_r(C_*) \\ & & \underbrace{\qquad\qquad\qquad\qquad}_{\theta^*} & & \end{array}$$

that commute with the action of $H^*(K; Z_p)$ and where θ^* is an isomorphism for all $r \geq 2$.

(2.3) COROLLARY (SPLITTING LEMMA). *For each* $r \geq 2$, f^*: $E_r(\mathscr{L}(\omega)) \to E_r(C_*)$ *is a split epimorphic map of bigraded* $H^*(K; Z_p)$*-chain complexes.* ▮

The orbit space $K(Z_p, 1, \omega)/K$ is an Eilenberg–MacLane space of type $(G, 1)$ where G is the group given by the extension

$$1 \longrightarrow Z_p \longrightarrow G \longrightarrow K \longrightarrow 1$$

classified by $\omega \in H^2(K; Z_p)$.

(2.4) COROLLARY. f^*: $H^*(G; Z_p) \to H^*(EK \times_K L; Z_p)$ *is a split epimorphic map of* $H^*(K; Z_p)$*-modules.* ▮

3. Periodicity

We assume that Σ^{2N-1} is a finite CW-complex, the homotopy type of a $(2N-1)$-sphere, and endowed with a cellular action by a finite group, G, containing an element, T, of order p (p a prime) and such that

(a) T is in the center of G and
(b) $T\colon \Sigma^{2N-1} \to \Sigma^{2N-1}$ is fixed point free.

T generates a normal subgroup $Z_p \lhd G$. Let L denote the orbit space Σ^{2N-1}/Z_p.

(3.1) PROPOSITION. (a) For $p = 2$, $H^*(L; Z_2) \cong Z_2[z_1]/(z_1^{2N})$, (b) *For* p *odd,*

$$H^*(L; Z_p) \cong Z_p[z_2]/(z_2^N) \otimes \Lambda(z_1)$$

where $\beta z_1 = z_2$ *and* $z_i \in H^i(L; Z_p)$.

Proof. This is routine from the Serre spectral sequence associated to the fibration

$$\Sigma^{2N-1} \longrightarrow EZ_p \times_{Z_p} \Sigma^{2N-1} \longrightarrow BZ_p$$

and (1.3). ▮

Let $\{E_r\}$ be the spectral sequence for the group extension

$$1 \longrightarrow Z_p \longrightarrow G \longrightarrow K \longrightarrow 1$$

where Z_p is generated by T. We recall that the transgression operator, τ, commutes with the action of the Steenrod algebra.

Since

$$x_1^{2^i} = Sq^{2^{i-1}}(x_1^{2^{i-1}}) \quad \text{for } p = 2$$

and

$$x_2^{p^i} = \mathscr{P}^{p^{i-1}}(x_2^{p^{i-1}}) \quad \text{for } p \text{ odd}$$

in $H^*(Z_p; Z_p)$, an inductive argument establishes that $x_1^{2^i}$ (resp. $x_2^{p^i}$) is transgressive in the above group extension spectral sequence.

(3.2) PROPOSITION. (a) *If $p = 2$ and $t \geq 0$ then the map induced by cup product*

$$x_1^{2^i} \cup \text{—}: E_{2^i-t}^{*,*} \longrightarrow E_{2^i-t}^{*,*+2^i}$$

is a first quadrant chain complex isomorphism.

(b) *If p is odd and $t \geq 0$ then*

$$x_2^{p^i} \cup \text{—}: E_{2p^i-t}^{*,*} \longrightarrow E_{2p^i-t}^{*,*+2p^i}$$

is a first quadrant chain complex isomorphism.

Proof. Since part (a) is similar to (b) we prove only (b). Since $x_2^{p^i}$ is transgressive it follows that $x_2^{p^i} \cup$ — is a chain map in the stated range by the derivation property of the differentials. It is clearly an isomorphism because $x_2 \cup$ — is an isomorphism. This completes the proof. ∎

(3.3) COROLLARY. (a) *For $p = 2$,*

$$x_1^{2^i} \cup \text{—}: E_{2^i+1} \longrightarrow E_{2^i+1}$$

is a first quadrant vector space isomorphism.

(b) *For p odd,*

$$x_2^{p^i} \cup \text{—}: E_{2p^i+1} \longrightarrow E_{2p^i+1}$$

is a first quadrant vector space isomorphism. ∎

Let $\mu_p(k) = \max\{i: p^i \mid k\}$. Recall that we are assuming that G acts on Σ^{2N-1} as described above and τ is the transgression operator in the group extension spectral sequence.

(3.4) PROPOSITION. (a) *If $p = 2$ then $\tau(x_1^{2^{\mu_2(N)+1}}) = 0$.*

(b) *If p is odd then $\tau(x_2^{p^{\mu_p(N)}}) = 0$.*

Proof. We prove part (b); part (a) is similar. Write $\mu_p(N) = \mu(N)$, and $N = p^{\mu(N)}m$. Thus m is relatively prime to p. Then $d_{2p^{\mu(N)}+1}$ is defined on x_2^N and

$$d_{2p^{\mu(N)}+1}(x_2^{p^{\mu(N)}m}) = m \cdot x_2^{p^{\mu(N)}(m-1)} \cup \tau(x_2^{p^{\mu(N)}}).$$

Let $\{E_r(L)\}$ be the spectral sequence for the fibration

$$L \longrightarrow EK \times_K L \longrightarrow BK.$$

Then according to the splitting lemma (2.3) the map $f^*: E_r \to E_r(L)$ is a split epimorphism, with splitting map $\alpha: E_r(L) \to E_r$.

Now, $f^*(x_2^N) = 0$ for dimension reasons. Thus letting $\gamma = \tau(x_2^{p^{\mu(N)}})$ we have (in $E_{2p^{\mu(N)}+1}(L)$)

$$\begin{aligned} 0 &= d_{2p^{\mu(N)}+1} f^*(x_2^N) \\ &= f^*(m \cdot x_2^{p^{\mu(N)}(m-1)} \cup \gamma) \\ &= m \cdot z_2^{p^{\mu(N)}(m-1)} \cup \gamma, \end{aligned}$$

since f^* is a map of $H^*(K; Z_p)$-modules and γ is represented by an element of $H^*(K; Z_p)$. Since $(m, p) = 1$, we have

$$0 = z_2^{p^{\mu(N)}(m-1)} \cup \gamma.$$

Now, α is a map of $H^*(K; Z_p)$-modules and $\alpha(z_2^i) = x_2^i$ for $i < N$. Thus we have (in $E_{2p^{\mu(N)}+1}$)

$$0 = \alpha(z_2^{p^{\mu(N)}(m-1)} \cup \gamma) = x_2^{p^{\mu(N)}(m-1)} \cup \gamma.$$

Now proposition (3.3) implies $\gamma = 0$, proving the proposition. ▮

(3.5) COROLLARY. (a) *If $p = 2$ then*

$$x_1^{2[\mu_2(N)+1]} \cup —: E_r \longrightarrow E_r$$

is a first quadrant chain complex isomorphism for all r.

(b) *For p odd,*

$$x_2^{p[\mu_p(N)]} \cup —: E_r \longrightarrow E_r$$

is the first quadrant chain complex isomorphism for all r. ▮

Now choose $\xi \in H^{2p[\mu_p(N)]}(G; Z_p)$ that is represented by the infinite cycle $x_1^{2[\mu_2(2N)]}$ if $p = 2$ or $x_2^{p[\mu_p(2N)]}$ if p is odd.

(3.6) COROLLARY (Periodicity). (a) *Multiplication by ξ in $H^*(G; Z_p)$ is an injection.*

(b) *If the G action on Σ^{2N-1} is free then multiplication by ξ is an isomorphism.*

Proof. (a) According to (3.5) there is a filtration on $H^*(G; Z_p)$ such that multiplication by ξ is filtration preserving and induces an injection on the associated graded groups

$$\xi \cup —: \operatorname{gr} H^*(G; Z_p) \rightarrowtail \operatorname{gr} H^*(G; Z_p).$$

It follows that $\xi \cup —$ must be an injection on $H^*(G; Z_p)$.

(b) It is known that if G acts freely then $H^*(G; Z_p)$ has period $2N$. Thus $\xi^m \cup$ — injects the finite dimensional vector space $H^i(G; Z_p)$ into a vector space of the same dimension. It follows that $\xi^m \cup$ — is an isomorphism. Consequently $\xi \cup$ — is an isomorphism. ∎

Extra special 2-groups. Suppose G is a group containing a central element, T, of order 2 and such that $G/\langle T\rangle \cong Z_2^l$. These are the extra-special 2-groups studied in [4]. Pairs (G, T) are classified by the elements of $H^2(Z_2^l; Z_2)$, the vector space of Z_2-quadratic forms, Q, in l variables.

A subspace of Z_2^l is called Q-isotropic if Q restricted to that subspace is identically zero. Let h be the codimension in Z_2^l of a Q-isotropic subspace of maximum dimension. The possible values for h are computed in [4].

(3.7) COROLLARY. *If (G, T) is an extra special 2-group that acts on Σ^{2N-1} such that T is fixed point free then $h \leq \mu_2(2N)$.*

Proof. Under the hypothesis $\tau(x_1^{2[\mu_2(N)]}) = 0$ in the group extension spectral sequence. But in [4] it is shown that $\tau(x_1)$, $\tau(x_2^2), \ldots, \tau(x_1^{2^{h-1}})$ are non-zero and $\tau(x_1^{2^h}) = 0$. Consequently $\mu_2(2N) \geq h$. ∎

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