ILLINOIS JOURNAL OF MATHEMATICS
Volume 28, Number 4, Winter 1984

A CHARACTERIZATION THEOREM FOR L.C.S. VALUED FUNCTIONS

BY

JOSE L. DE MARÍA

Introduction. One of the key theorems in the study of the Bochner integration is the Pettis theorem, because it gives a complete description of the behavior of Bochner measurable functions.

Chi [1] (1973), Gilliam [2] (1976) and Rodriguez-Salinas [4] (1979), [5] (1982) have successively extended integration theory by means of the introduction of a more general class of the vector functions. These lead to new Radon–Nikodym theorems and help to explain the geometric structure of some locally convex spaces that are more complicated than Banach spaces.

We are going to study the $\bar{\mu}$-measurable functions which have been introduced by Rodriguez-Salinas in [4] and which are the most general among those appearing in the papers above.

In this work we obtain a characterization of these functions which is similar to the Pettis Theorem; it enables us to obtain an Egorov theorem for functions with values in a locally convex space (l.c.f.) which is LF.

Also, as a consequence, it is easy to prove that the almost everywhere limit of a sequence of $\bar{\mu}$-measurable functions is a $\bar{\mu}$-measurable function. The theorem has been used in a later integration theory on strictly localizable measure spaces where it has been a good tool because it simplifies several proofs.

Throughout this paper (Ω, Σ, μ) will be a finite complete measure space, E will be a Hausdorff locally convex space and f, f_α functions from Ω to E.

DEFINITIONS. A function f is *simple* ($f \in S_0(\Sigma, E)$) if

$$f = \sum_{i=1}^{n} y_i \chi_{A_i}$$

where $y_i \in E$, $A_i \in \Sigma$ and χ_{A_i} is the characteristic function of A_i.

A function f is *u-simple* ($f \in S(\Sigma, E)$) if it is the uniform limit of a net $(f_\alpha)_{\alpha \in \Lambda}$ in $S_0(\Sigma, E)$.

We say that f is *μ-measurable* ($f \in M(\Sigma, \mu, E)$) if for every $\epsilon > 0$, there exists $K_\varepsilon \in \Sigma$ such that $\mu(\Omega \backslash K_\varepsilon) < \varepsilon$ and $f \cdot \chi_{K_\varepsilon}$ is μ-simple.

Next we define the functions which are the subject of our study.

Received May 27, 1982.

A function f is *$\bar{\mu}$-measurable* ($f \in \bar{M}(\Sigma, \mu, E)$) if it is the uniform limit of a net $(f_\alpha)_{\alpha \in \Lambda}$ in $M(\Sigma, \mu, E)$.

These functions have been introduced by Rodriguez-Salinas [4].

It is easy to deduce that if E is metrizable then $M(\Sigma, \mu, E) = \bar{M}(\Sigma, \mu, E)$, and there also exists an example where E is not metrizable and $M(\Sigma, \mu, E) \neq \bar{M}(\Sigma, \mu, E)$.

It is clear that a function f is u-simple if and only if $f(\Omega)$ is precompact and for every continuous seminorm p on E, for every element x in E, the function $p(f - x)$ is measurable. This last condition is weaker than Borel measurability and shows that f is measurable for the σ-algebra generated for the convex neighborhoods of E.

DEFINITION 1. Let f be a function from Ω to E; f is called *ω-precompact* if for every V neighborhood of O in E, there exist a μ-null subset $Z_V \subset \Omega$ and a countable set $M \subset E$ such that $f(\Omega \backslash Z_V) \subset M + V$.

The following theorem generalizes the Pettis theorem for the Bochner measurability.

THEOREM 2. *A function f is $\bar{\mu}$-measurable if and only if*

(2.1) *f is ω-precompact, and*

(2.2) *for every continuous seminorm p on E and for every element x of E, the function $p(f - x)$ is measurable.*

Proof ($\Rightarrow$ (2.1)). Let f be a $\bar{\mu}$-measurable function; there exists a net $(f_\alpha)_{\alpha \in \Lambda}$ in $M(\Sigma, \mu, E)$ that converges uniformly to f.

Given V, a convex neighborhood of O in E, there exists α_0 such that if $\alpha > \alpha_0$ and $t \in \Omega$, then $f(t) - f_\alpha(t) \in \frac{1}{2}V$.

Fix $\alpha > \alpha_0$; since f_α is μ-measurable there exists a sequence of disjoint measurable sets $(K_n)_{n=1}^{\infty}$, such that

$$\mu\left(\Omega \backslash \bigcup_n K_n\right) = 0 \quad \text{and} \quad f_\alpha = \sum_{n=1}^{\infty} f_\alpha \cdot \chi_{K_n} + f_\alpha \cdot \chi_Z$$

where $Z = \Omega \backslash \bigcup_n K_n$, and $f_\alpha \cdot X_{K_n} \in S(\Sigma, E)$ (This is a direct consequence of the definitions).

Since $f_\alpha(K_n)$ is precompact, there exists a finite set F_n such that

$$f_\alpha(K_n) \subset F_n + \tfrac{1}{2}V.$$

Hence $f(\Omega \backslash Z) \subset f_\alpha(\Omega \backslash Z) + \frac{1}{2}V \subset M + V$, where $M = \bigcup_n F_n$ is a countable subset of E.

($\Rightarrow$ (2.2)) Given a continuous seminorm p on E, and $x \in E$, the net $p(f_\alpha - x)$ converges uniformly to $p(f - x)$. It follows that $p(f - x)$ is measurable.

($\Leftarrow$) Let V be a closed absolutely convex neighborhood of O and let p be its Minkowski functional; there exist a μ-null subset Z, and a countable subset $M = (x_n)_{n=1}^{\infty}$ in E such that $f(\Omega \backslash Z) \subset M + V$. It is easily proved that there exists a partition $(A_n)_{n=1}^{\infty}$ of $\Omega \backslash Z$ in Σ such that $f(t) - x_n \in V$ for every $t \in A_n$. Hence we can construct a μ-measurable function

$$f_p = \sum_{n=1}^{\infty} x_n \cdot \chi_{A_n} + f \cdot \chi_Z$$

so that $p(f_p(t) - f(t)) < 1$ for every t in Ω.

It follows that the net $(f_p)_{p \in \Gamma}$ converges uniformly to f, where Γ is the set of the continuous seminorms on E and the order is the natural order for the seminorms. Hence f is $\bar{\mu}$-measurable. ∎

With this characterization it is very easy to prove that $\bar{M}(\Sigma, \mu, E)$ is closed for the almost everywhere limits of sequences.

PROPOSITION 3. *If $(f_n)_{n=1}^{\infty}$ is a sequence of functions in $\bar{M}(\Sigma, \mu, E)$ that converges almost everywhere to a function f, then f is $\bar{\mu}$-measurable.*

Proof. We can suppose that $f(t) = \lim_n f_n(t)$ for every $t \in \Omega$.

Let U be a closed absolutely convex neighborhood of O in E and $t \in \Omega$. There exists $n_0 \in \mathbf{N}$ such that $f(t) \in f_n(t) + \frac{1}{2}U$ for every $n > n_0$.

Since f_n is μ-measurable, there exist a subset Z_n in Ω, $\mu(Z_n) = 0$ and a countable subset M_n in E such that

$$f_n(\Omega \backslash Z_n) \subset M_n + \tfrac{1}{2}U$$

If $Z = \bigcup_n Z_n$ and $M = \bigcup_n M_n$ it follows that $\mu(Z) = 0$, M is countable, and $f(\Omega \backslash Z) \subset M + U$. Thus f is ω-precompact.

Let p be a continuous seminorm on E, and let x be in E; then

$$p(f - x) = \lim p(f_n - x) \quad \text{everywhere},$$

and as $p(f_n - x)$ is measurable, $p(f - x)$ is also measurable.

Therefore, by Theorem 2, f is $\bar{\mu}$-measurable. ∎

Next we obtain some Egorov theorems for μ-measurability as an application of the preceding theorems.

THEOREM 4. *Let E be a metrizable l.c.s. and let $(f_n)_{n=1}^{\infty}$ be a sequence of μ-measurable functions that converges almost everywhere to f. Then $(f_n)_{n=1}^{\infty}$ converges almost uniformly to f.*

Proof. We may assume that $(f_n)_{n=1}^{\infty}$ converges everywhere to f. Since E is metrizable, there exists a countable family of continuous seminorms $(p_n)_{n=1}^{\infty}$ which defines the E-topology. As $\bar{M}(\Sigma, \mu, E)$ is a vector space, we can suppose that $(f_n)_{n=1}^{\infty}$ converges everywhere to O.

Given $m \in \mathbf{N}$, since f_m is Borel measurable, by the Egorov theorem in the real case, there exists a subset Z_m such that $\mu(Z_m) < \varepsilon/2^m$ and $\lim p_m f_n = 0$ uniformly on $\Omega \backslash Z_m$, for every $\varepsilon > 0$.

If we let $Z = \bigcup_m Z_m$, then $(f_n)_{n=1}^{\infty}$ converges to O uniformly in $\Omega \backslash Z$. Indeed, let $\mathscr{P} = \{p_{m_1}, \ldots, p_{m_k}\}$ be a family of continuous seminorms on E, and let δ be positive; for every p_{m_i}, there exists $n_i \in \mathbf{N}$ such that if $n > n_i$ we have

$$p_{m_i}(f_n(t)) < \delta$$

for every $t \in \Omega \backslash Z$. If $n > \sup\{n_1, \ldots, n_k\}$, then $p(f_n(t)) < \delta$, for every $t \in \Omega \backslash Z$ and for every $p \in \mathscr{P}$. ∎

We state two lemmas about the behavior of the functions in $M(\Sigma, \mu, E)$ and $\bar{M}(\Sigma, \mu, E)$. The proofs are straightforward.

LEMMA 5. *If E_0 is a vector subspace of E, $f \in \bar{M}(\Sigma, \mu, E)$ (resp. $f \in M(\Sigma, \mu, E)$) and $f(\Omega) \subset E_0$, then f is $\bar{\mu}$-measurable from Ω to E_0 (resp. f is μ-measurable from Ω to E_0).*

LEMMA 6. *If E is a l.c.s., $f \in \bar{M}(E, \mu, E)$ (resp. $f \in M(\Sigma, \mu, E)$, $f \in S(\Sigma, E)$, $f \in S_0(\Sigma, E)$ and $\Omega' \subset \Omega$, $\Omega' \in \Sigma$, then the restriction of f to Ω', $f|_{\Omega'}$ is a $\bar{\mu}$-measurable function (resp. μ-measurable, u-simple, simple).*

With aid of these lemmas we prove the following analogue of Egorov's Theorem when E is an LF-space. (E is a strict inductive limit of a sequence $(E_n)_{n=1}^{\infty}$ of locally convex Frechet spaces).

THEOREM 7. *Let E be an LF-space. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $M(\Sigma, \mu, E)$. If $(f_n)_{n=1}^{\infty}$ converges almost everywhere to f, then $(f_n)_{n=1}^{\infty}$ converges almost uniformly to f.*

Proof. By Lemma 13.1 in [6], since E_k is metrizable, there exists a decreasing sequence $(U_m)_{m=1}^{\infty}$ of closed, absolutely convex neighborhoods of O in E such that $(U_m \cap E_k)_{m=1}^{\infty}$ is a base of neighborhoods of O in E_k, for every k.

We will prove first that every E_k belongs to the smallest σ-algebra generated by the family of scalar-valued uniformly continuous functions defined in E.

Let p_n be the Minkowski functional of U_n, then the function

$$\rho_n(x) = \inf\{p_n(x - y) : y \in E_1\}$$

has the following property: If $x \in E_1$ then $\rho_n(x) = 0$ for every $n \in \mathbf{N}$, and if $x \in E_k \backslash E_1$ then there exists n_0 such that $(x + U_n) \cap E_1 = \emptyset$ for every $n \geq n_0$; hence $\rho_n(x) \geq 1$ if $n \geq n_0$.

We define $h_n(x) = \min\{1, \rho_n(x)\}$ and $h(x) = \lim_n h_n(x)$. They are uniformly continuous. As $h(x)$ is 0 if $x \in E_1$ and 1 if $x \notin E_1$, then E_1 is the zeroset for a function which is a limit of a sequence of uniformly continuous functions. Thus E_1 is in the desired σ-algebra. Clearly this proof can be repeated with any E_k in place of E_1.

We shall now prove the theorem. Assume pointwise convergence. Since the composition of a $\bar{\mu}$-measurable function with a scalar uniformly continuous function on E is a measurable function [4, page 375], we see that $f_n^{-1}(E_k)$ is measurable for every k, and thus the sets

$$\Omega_p = \bigcap_{k=1}^{\infty} f_k^{-1}(E_p)$$

are measurable.

It is clear that $\Omega = \bigcup_{p=1}^{\infty} \Omega_p$ and $\Omega_1 \subset \Omega_2 \subset \cdots$ hence given $\varepsilon > 0$ there exists p such that $\mu(\Omega \backslash \Omega_p) < \varepsilon/2$ and $f_m(\Omega_p) \subset E_p$ for every m.

Using Lemma 6 it is inmediate that $f_m|_{\Omega p}: \Omega_p \to E$ is $\bar{\mu}$-measurable, and as $f_m|_{\Omega p}(\Omega_p) \subset E_p$ and E_p is metrizable, we can conclude, using Lemma 5, that $f_m|_{\Omega_p}$ is μ-measurable from Ω_p to E_p. Hence, by Theorem 4, there exists $H_0 \subset \Omega_p$ measurable such that $\mu(\Omega_p \backslash H_0) < \varepsilon/2$ and $f_m|_{\Omega_p}$ converges uniformly on H_0. Since $\mu(\Omega \backslash H_0) < \varepsilon$ and $(f_n)_{n=1}^{\infty}$ converges uniformly on H_0, the proof is finished. ∎

REFERENCES

1. G. Y. CHI, *On the Radon Nikodym theorem in locally convex spaces*, Lectures Notes in Mathematics, no. 541, Springer, N.Y., 1976, pp. 199–209.
2. D. GILLIAM, *On integration and the Radon Nikodym theorem in quasicomplete l.c.t.s.*, J. Reine Angew Math., vol. 292 (1977), pp. 125–137.
3. J. DE MARÍA, *Extensiones al teorema de Egorov*, Univ. Complutense, Tesis Doctoral, 1981.
4. B. RODRIGUEZ-SALINAS, *Integración de funciones con valores en un espacio localmente convexo*, Real Acad. Cienceias de Madrid, Tomo LXXIII, 1979.
5. ———, *Integración de funciones con valores en un espacio locamente convexo respecto de medidas infinitas*, Real Acad. de Ciencias de Madrid, to appear.
6. F. TREVES, *Topological vector spaces, distribution and kernels*, Academic Press, New York, 1967.

UNIVERISIDAD COMPLUTENSE
MADRID, SPAIN