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ON FUNCTIONS THAT ARE UNIVERSALLY PETTIS INTEGRABLE

BY

LAWRENCE H. RIDDLE AND ELIAS SAAB[1]

I. Introduction

It is very important to recognize when a scalarly measurable function f from a probability space $(\Omega, \Sigma, \lambda)$ to a Banach space is Pettis integrable. A number of authors studied the above problem and related questions about the Pettis integral [2], [4]–[13], [16]–[20].

In this paper we are going to study a class of functions defined on a compact Hausdorff space with values in a dual Banach space and we shall prove in Part III that they are universally Pettis integrable whenever they are universally scalarly measurable.

In [11] the notion of Banach spaces having the universal Pettis integrability property was introduced (UPIP). A Banach space E has the UPIP if for every compact Hausdorff space K every bounded function $f\colon K \to E$ that is universally scalarly measurable is universally Pettis integrable. It was shown in [11], [13] that the dual of a separable Banach space has the UPIP. It was also shown that if E is a WCG Banach space and if $f\colon K \to E^*$ is a bounded universally scalarly measurable function whose range is weak*-separable, then f is universally Pettis integrable. In this paper we are going to show that if a bounded function

$$f\colon K \to \big(E^*, \sigma(E^*, E)\big)$$

is universally Lusin measurable, then it is universally Pettis integrable whenever it is universally scalarly measurable. Several applications are given, namely, if C is a weak*-compact subset of the dual E^* of a Banach space E such that the identity map

$$I\colon \big(C, \sigma(E^*, E)\big) \to E^*$$

is universally scalarly measurable, then the same is true for the identity map,

$$I\colon \big(w^*(\mathrm{conv}(C)), \sigma(E^*, E)\big) \to E^*.$$

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We also show that if the linear span of such a set C is norm dense in E^*, then E^* has the weak Radon-Nikodym property. If

$$f\colon K \to \left(E^*, \sigma(E^*, E)\right)$$

is universally Lusin measurable, we show that f is universally Pettis integrable if and only if for every Radon probability measure λ on K, the set $\{\langle f, x\rangle :$ $x \in E, \|x\| \leq 1\}$ is almost weakly precompact in $L_\infty(K, \lambda)$. This shows that a function $f\colon K \to l_\infty$ that is weak*-scalarly universally measurable is universally Pettis integrable if and only if for every Radon measure λ on K the set of coordinate functions f_n of f is almost weakly precompact in $L_\infty(K, \lambda)$.

The relations between Pettis integrability of a function and the Bourgain property (see definition) are discussed in detail in Part IV of this paper. The Bourgain property of a function always implies that the function is Pettis integrable. The converse is not always true. However, we will show that if K is a compact metric space and E is a separable Banach space, then saying that a function $f\colon K \to E^*$ is universally Pettis integrable is equivalent to saying that f has the Bourgain property for every Radon probability measure on K.

We would like to thank Professor J. Bourgain for allowing us to include the proofs of some of his unpublished results which form the basis of Part IV of this paper.

II. Definitions and notations

Let $(\Omega, \Sigma, \lambda)$ be a finite measure space and let f be a function from Ω into E that is Pettis integrable [4], [11]. If $A \in \Sigma$, we will denote by $\int_A f d\lambda$ the Pettis integral of f over A.

If $E = F^*$ is a dual Banach space and the function f is only w^*-integrable [11] we denote its w^*integral over a set A by w^*-$\int_A f d\lambda$.

The closed unit ball of a Banach space E will be denoted by B_E.

If L is a subset of F^*, the norm closed convex hull of L will be denoted by $n(\mathrm{conv}(L))$ and the w^*-closed convex hull of L will be denoted by $w^*(\mathrm{conv}(L))$.

Let C be a w^*-compact subset F^*. The set C is said to be a Pettis set if the identity map $(C, \sigma(E^*, E)) \to E^*$ is universally Pettis integrable [11]. If, in addition the set C is convex, then C will have the Weak Radon Nikodym property [17].

If K is a compact Hausdorff space, we denote by $M_+^1(K)$ the set of all probability Radon measures on K.

Let $\lambda \in M_+^1(K)$ and let (X, τ) be a completely regular space, if $f\colon K \to (X, \tau)$ is λ-Lusin measurable [11], the image of λ by f is a Radon probability measure defined by $h(\lambda)(B) = \lambda(h^{-1}(B))$ for every Borel subset of (X, τ).

All notions and notations used in this paper and not defined can be found in [4] and [11].

III. Pettis integrability

THEOREM 1. *Let E be a Banach space and E^* be its dual. Let K be a compact Hausdorff space and f a bounded and universally Lusin measurable from K to $(E^*, \sigma(E^*, E))$. Then the following statements are equivalent.*

(i) *The function f is universally scalarly measurable.*
(ii) *The function f is universally Pettis integrable.*
(iii) *For every compact subset K_1 of K such that*

$$f: K_1 \to \big(E^*, \sigma(E^*, E)\big)$$

is continuous, the set $\{\langle f, x\rangle | K_1 : \|x\| \leq 1\}$ is weakly precompact in $C(K_1)$.

Proof. *Case* A. *f continuous.* Consider (iii) $\Rightarrow$ (ii). Let $T: E \to C(K)$ be defined by $Tx = \langle f, x\rangle$. The operator T is clearly bounded and linear. By [11] and (iii), it factors through a Banach space not containing l_1. Consider the adjoint T^* from $M(K) = C(K)^*$ to E^*. It is easy to check that $T^*(\varepsilon_k) = f(k)$ for every k in K and therefore $T^*(M_+^1(K)) = w^*(\mathrm{conv}(f(K))$. Hence this set is a weak Radon-Nikodym set (see [11], [13], [17]). This implies that every x^{**} in X^{**} is universally measurable on

$$\big(w^*(\mathrm{conv}(f(K)), \sigma(E^*, E)\big)$$

and satisfies the barycentric formula [11]. Let λ be in $M_+^1(K)$ and $x^{**} \in X^{**}$. The function $x^{**}: (f(K), \sigma(E^*, E)) \to R$ is $f(\lambda)$-measurable. Hence $x^{**}f$ is λ-measurable [1] and therefore f is λ-scalarly measurable. On the other hand

$$\begin{aligned} x^{**}\left(w^* - \int_K f\,d\lambda\right) &= x^{**}\left(w^* - \int_{f(K)} x^*\,df(\lambda)\right) \\ &= \int_{f(K)} x^{**}(x^*)\,df(\lambda) \\ &= \int_K x^{**}f(x^*)\,d\lambda. \end{aligned}$$

The second equality holds because x^{**} satisfies the barycentric formula [11]. If A is any Borel subset of K of positive measure, define μ in $M_+^1(K)$ by

$\mu(B) = \lambda(B \cap A)/\lambda(A)$. Then

$$x^{**}\left(w^* - \int_A f\,d\lambda\right) = \lambda(A)x^{**}\left(w^* - \int_K f\,d\mu\right)$$
$$= \lambda(A)\int_K x^{**}f\,d\mu$$
$$= \int_A x^{**}f\,d\lambda.$$

This shows that f is Pettis integrable and $\int_A f\,d\lambda = w^* - \int_A f\,d\lambda$ for every Borel subset A of K.

It is clear that (ii) $\Rightarrow$ (i). To show that (i) $\Rightarrow$ (iii) let

$$A = \{\langle x, f\rangle : \|x\| \leq 1\}.$$

Let $\langle x_\alpha, f\rangle$ be a net in A. Choose a subnet (x_β) of (x_α) that converges weak* to an element x^{**} in X^{**}. This implies that

$$\langle x_\beta, f\rangle \to \langle x^{**}, f\rangle \quad \text{pointwise.}$$

The function $\langle x^{**}, f\rangle$ is universally measurable. Therefore the set A is relatively compact in the space of all universally measurable functions on K with the topology of pointwise convergence. By [3, Theorem 2F], every sequence in A has a pointwise convergent subsequence, hence A is weakly precompact in $C(K)$.

Case B. f universally Lusin measurable. To see that in this case (iii) $\Rightarrow$ (ii) let $\lambda \in M_+^1(K)$ and $\varepsilon > 0$. Choose $K_1 \subseteq K$ such that $f: K_1 \to (E^*, \sigma(E^*, E)$ is continuous and $\lambda(K \setminus K_1) \leq \varepsilon$. By case A, f is λ-Pettis integrable when restricted to K_1, an exhaustive argument shows that f is Pettis integrable. The implication (i) $\Rightarrow$ (iii) can be done verbatim as in Case A.

The proof of Case A shows that if f is continuous then (i), (ii) and (iii) are equivalent to:

(iv) The set $w^*(\text{conv}(f(K))$ has the weak Radon Nikodym property.

It is worth mentioning that any bounded linear operator T from a Banach space E to any $C(K)$ space can be represented by a continuous function $f: K \to (E^*, \sigma(E^*, E))$ such that $Tx = \langle f, x\rangle$ for every x in E (see Dunford and Schwartz, p. 490). It was proved there that T is weakly compact if and only if $f: K \to (E^*, \sigma(E^*, E^{**}))$ is continuous. It was also shown that T is compact if and only if $f: K \to (E^*, \|\cdot\|)$ is continuous. It follows from Theorem 1 that T is weakly precompact if and only if f is universally Pettis integrable. The same method shows that T is an Asplund operator if and only if $f: K \to (E^*, \|\cdot\|)$ is universally Bochner integrable.

The following theorem deals with weak*-compact sets that are not necessarily convex and is an easy consequence of Theorem 1 and the results of [11] and [17].

THEOREM 2. *Let H be a weak*-compact subset of E^*. Then the following statements are equivalent*:

(i) *The set H is a Pettis set.*

(ii) *The identity map $I: (H, \sigma(E^*, E)) \to E^*$ is universally scalarly measurable.*

(iii) *For every $x^{**} \in X^*$ and every weak*-compact subset M of H, the function x^{**} restricted to $(M, \sigma(E^*, E))$ has a point of continuity.*

(iv) *The set $\{x|_H: \|x\| \leq 1\}$ is weakly precompact in $C(H)$.*

(v) *The set $L = w^*$-conv(H) has the weak Radon-Nikodym property.*

Condition (v) in Theorem 2 implies the following stability result.

COROLLARY 3. *Let $H_1, H_2, \dots, H_n$ be n weak*-compact subsets of E^*. If each $H_i, 1 \leq i \leq n$, satisfies any one of the equivalent conditions* (i) *through* (v) *of Theorem* 2, *so does the weak*-closed convex hull L of the union of the $H_i, 1 \leq i \leq n$. In particular L has the weak Radon-Nikodym property.*

Proof. Let $H = \cup_{i=1}^n H_i$. Choose x^{**} in E^{**} and $\varepsilon > 0$. If $\lambda \in M_+^1(H)$, then there exists for each $1 \leq i \leq n$ a weak*-compact subset $M_i \subset H_i$ such that

$$\lambda(H_i \setminus M_i) \leq \varepsilon/n$$

and the restriction of x^{**} to $(M_i, \sigma(E^*, E))$ is continuous. Let $M = \cup_{i=1}^n M_i$. Then x^{**} restricted to $(M, \sigma(E^*, E))$ is continuous and $\lambda(H \setminus M) \leq \varepsilon$. This shows that x^{**} restricted to $(H, \sigma(E^*, E))$ is universally measurable. Apply Theorem 2 (v) to conclude that the set $L = W^*$-conv(H) has the weak Radon-Nikodym property.

COROLLARY 4. *Let H be a weak*-compact subset of the dual E^* of a Banach space E. If the linear span of H is norm dense in E^*, then E^* has the weak Radon-Nikodym property if and only if H is a Pettis set.*

Proof. Let $(x_n)_{n \geq 1}$ be a sequence in B_E. By (iv) of Theorem 2, there is a subsequence (x_{n_k}) such that $\lim_k \langle h, x_{n_k} \rangle$ exists for every $h \in H$. Let $x^* \in E^*$ and $\varepsilon > 0$. Choose a linear combination $\sum_{i=1}^p \lambda_i h_i$ of elements in H such that $\|x^* - \sum_{i=1}^p \lambda_i h_i\| \leq \varepsilon$. This implies that

$$|\langle x^*, x_{n_k} \rangle - \sum_{i=1}^p \lambda_i \langle h_i, x_{n_k} \rangle| < \varepsilon.$$

For every $1 \leq i \leq p$, the sequence $(\lambda_i\langle h_i, x_{n_k}\rangle)$ is a Cauchy sequence. Now it is easy to see that the sequence $\langle x^*, x_{n_k}\rangle$ is a Cauchy sequence. This shows that the unit ball of E is weakly precompact and that therefore E^* has the weak Radon-Nikodym property.

As E. Granirer pointed out to us, in the above corollary one cannot replace norm dense by w^*-dense because the dual of any separable Banach space can be written as the w^*-closure of a subspace that is generated by a norm compact subset.

The following theorem offers a universal converse of Lemma 5 in [11] when f is defined on a compact space.

THEOREM 5. *Let K be a compact Hausdorff space, let E be any Banach space and let $f: K \to (E^*, \sigma(E^*, E))$ be universally Lusin measurable. Then the following statements are equivalent*:

(i) *The function f is universally Pettis integrable.*

(ii) *For every $\lambda \in M_+^1(K)$, the set $\{\langle f, x\rangle : \|x\| \leq 1\}$ is almost weakly precompact in $L_\infty(K, \lambda)$.*

Before proving this theorem, we need the following.

LEMMA 6. *Let $f:(K,\lambda) \to E^*$ be λ-w^*-integrable and λ-w^*-Lusin measurable. Then for λ-almost every t in K*

$$f(t) \in C = w^*\left(\operatorname{conv}\left\{\frac{w^* - \int_A f\,d\lambda}{\lambda(A)} : A \text{ Borel and } \lambda(A) > 0\right\}\right).$$

Proof. Let K_1 be a compact subset of K such that

$$f: K_1 \to \big(E^*, \sigma(E^*, E)\big)$$

is continuous. Let $H_1 \subset K_1$ be the support of the measure λ restricted to K_1.

Claim. If $t \in H_1$ then $f(t) \in C$. If not, there exists an $x \in E$ such that

$$\langle f(t), x\rangle \geq \alpha > \frac{\int_A \langle f, x\rangle\, d\lambda}{\lambda(A)}$$

for every Borel subset A with $\lambda(A) > 0$. The map $t \to \langle f(t), x\rangle$ is continuous on H_1. Let V be a w^*-open neighborhood of t in H_1 such that $u \in V \Rightarrow \langle f(u), x\rangle \geq \alpha$.

Notice that $\lambda(V) > 0$ since H_1 is the support of λ restricted to K_1. Now

$$\alpha > \frac{\int_V \langle f, x\rangle \, d\lambda}{\lambda(V)} \geq \frac{\alpha\lambda(V)}{\lambda(V)} = \alpha.$$

This contradiction finishes the proof of the claim.

Since $f\colon K \to (E^*, \sigma(E^*, E))$ is λ-Lusin measurable, the claim and a standard measure theory argument show that K can be written as $K = H \cup M$ where $\lambda(M) = 0$ and $f(H) \subseteq C$. Hence $f(t) \in C$ λ-almost everywhere.

Proof of Theorem 5. Let $\lambda \in M_1^+(K)$ and $\varepsilon > 0$. Choose $K_1 \subset K$ such that $\lambda(K \setminus K_1) \leq \varepsilon$ and $f\colon K_1 \to E^*$ is continuous. By Theorem 1, the set

$$H = \left\{\langle f, x\rangle|_{K_1} : \|x\| \leq 1\right\}$$

is weakly precompact in $C(K_1)$ and therefore the set $\{\langle f, x\rangle \chi_{K_1} : \|x\| \leq 1\}$ is weakly precompact in $L_\infty(K, \lambda)$ since the inclusion map $C(K_1) \to L_\infty(K, \lambda)$ is a contraction. To see that (ii) $\Rightarrow$ (i) let $\lambda \in M_+^1(K)$ and $\varepsilon > 0$. Choose a compact subset K_1 in K such that $\lambda(K \setminus K_1) < \varepsilon$ and the set $\{\langle f, x\rangle|_{K_1} : \|x\| \leq 1\}$ is weakly precompact in $L_\infty(K_1, \lambda)$. Now consider the operator

$$T\colon L_1(K_1, \lambda) \to X^*$$

defined by $T(g) = w^* - \int_K g f \, d\lambda$ and consider the adjoint operator

$$T^*\colon X \to L_\infty(K_1, \lambda).$$

It is easy to see that $T^*(B_X) = \{\langle f, x\rangle|_{K_1} : \|x\| \leq 1\}$. Hence $T^*(B_X)$ is weakly precompact and consequently T^* factors through a Banach space not containing l_1. Therefore

$$T^{**}\left(B_{L_\infty^*(K_1, \lambda)}\right)$$

is a weak Radon-Nikodym set in X^* [11]. The function $f\colon K_1 \to X^*$ takes its values almost surely in the set

$$C = w^* - \left(\overline{\operatorname{conv}}\left\{\frac{w^* - \int_A f \, d\lambda}{\lambda(A)} : \text{A Borel in } K_1 \text{ and } \lambda(A) > 0\right\}\right),$$

by Lemma 6. Since

$$T\left(\frac{\chi_A}{\lambda(A)}\right) = T^{**}\left(\frac{\chi_A}{\lambda(A)}\right) = \frac{w^* - \int_A f\,d\lambda}{\lambda(A)},$$

we see that

$$C \subset T^{**}\left(B_{L_\infty^*(K_1,\lambda)}\right).$$

Let x^{**} be in X^{**}. The function x^{**} restricted to the set C is $f(\lambda|_{K_1})$ measurable [11], [13], [17]. Therefore the function $x^{**}f$ restricted to K_1 is $\lambda|_{K_1}$ measurable [1]. Since $\lambda(K \setminus K_1) < \varepsilon$, this shows that $x^{**}f$ is λ-measurable. Hence f is universally scalarly measurable and therefore f is universally Pettis integrable by Theorem 1.

As an application of Theorem 5 we offer the following.

COROLLARY 7. *Let K be a compact Hausdorff space and let*

$$g: K \to (l_\infty, \text{weak}^*)$$

be a universally Lusin measurable map. Then the following are equivalent.

(i) *For every $\mu \in M_+^1(K)$ the set $\{g_n = \langle g, e_n\rangle : n \geq 1\}$ is almost weakly precompact in $L_\infty(\mu)$ ($(e_n)_{n\geq 1}$ denotes the usual basis of l_1).*

(ii) *The function g is universally Pettis integrable.*

Proof. For each $\alpha > 0$, there is a Borel subset B in K such that $\mu(\Omega \setminus B) < \alpha$ and such that the set $\{g_n\chi_B : n \in N\}$ is weakly precompact in $L_\infty(\mu)$. Observe that the set $A = \{\langle g, \pm e_n\rangle\chi_B : n \in N\}$ is weakly precompact in $L_\infty(\mu)$. Therefore the closed convex hull of A is also weakly precompact in $L_\infty(\mu)$ and consequently the set

$$\{\langle g, x\rangle\chi_B : x \in \overline{\text{co}}(\pm e_n)\}$$

is weakly precompact in $L_\infty(\mu)$. The Krein-Mil'man Theorem now ensures that the set $\{\langle g, x\rangle\chi_B : \|x\| \leq 1\}$ is weakly precompact in $L_\infty(\mu)$. Finally, appeal to Theorem 5 to conclude that the function g is μ-Pettis integrable. Therefore g is universally Pettis integrable. This shows that (i) $\Rightarrow$ (ii). To see that (ii) $\Rightarrow$ (i), notice that

$$\{g_n = \langle g, e_n\rangle : n \geq 1\}$$

is included in the set

$$\{\langle g, x\rangle : x \in l_1, \|x\| \leq 1\}$$

and apply Theorem 5.

Corollary 7 has to be compared to the following theorem.

THEOREM 8. *Let $T: L_1(\mu) \to l_\infty$ have the representation $T\phi = (\int_\Omega \phi g_n \, d\mu)$ where (g_n) is a uniformly bounded sequence in $L_\infty(\mu)$.*

(a) *The operator T is Bochner representable if and only if the sequence (g_n) is almost relatively weakly compact in $L_\infty(\mu)$.*

(b) *The operator T is a Dunford-Pettis operator if and only if the sequence (g_n) is relatively (norm) compact in $L_1(\mu)$.*

Proof. A quick calculation reveals that $T^*x = \Sigma x_n g_n$ for each x in the ball of l_1. Therefore $T^*(B_{l_1})$ is contained in the closed convex hull of $\{\pm g_n : n \in N\}$. If the sequence (g_n) is almost relatively weakly compact in $L_\infty(\mu)$, then the set $T^*(B_{l_1})$ is also almost relatively weakly compact in $L_\infty(\mu)$ and consequently T is Bochner representable; but (g_n) is clearly contained in $T^*(B_{l_1})$, so the converse implication also holds. This establishes part (a).

For part (b), recall that if T is a Dunford-Pettis operator, then T^* maps bounded sequences in l_1 into sequences with almost everywhere convergent subsequences. Therefore each subsequence of (g_n) has an $L_1(\mu)$-convergent subsequence; that is, the sequence (g_n) is relatively compact in $L_1(\mu)$. Conversely, if (g_n) is relatively compact in $L_1(\mu)$, then as above, the set $T^*(B_{l_1})$ is also relatively compact in $L_1(\mu)$. Therefore T^* maps bounded sequences into sequences with $L_1(\mu)$-convergent subsequences, which in turn have almost everywhere convergent subsequences. Hence T is a Dunford-Pettis operator and this completes the proof.

Let $f: (\Omega, \Sigma, \mu) \to Y$ be a Pettis integrable function into a Banach space Y and let Γ be a sub-σ-algebra of Σ. A Pettis integrable function $g: (\Omega, \Gamma, \mu) \to Y$ is said to be a Pettis conditional expectation of f with respect to the σ-algebra Γ if g is scalarly Γ-measurable and if $\int_A g \, d\lambda = \int_A f \, d\lambda$ for each set A in Γ. The following theorem provides a sufficient condition for a bounded dual-valued Pettis integrable function to have Pettis conditional expectation. It naturally makes use of weakly precompact sets.

THEOREM 9. *Let $f: (\Omega, \Sigma, \mu) \to E^*$ be a bounded Pettis integrable function. If the set $\{\langle f, x\rangle : \|x\| \le 1\}$ is weakly precompact in $L_\infty(\mu)$, then f has Pettis conditional expectation with respect to all sub-σ-algebras of Σ.*

Proof. Let Γ be a sub-σ-algebra of Σ and define an operator

$$T: X \to L_\infty(\Gamma, \mu)$$

by

$$Tx = \xi(\langle f, x\rangle | \Gamma)$$

for each x in E. Since the set $\{\langle f, x\rangle : \|x\| \leq 1\}$ contains no copy of the l_1-basis in $L_\infty(\Sigma, \mu)$ and the conditional expectation operator ξ is a contraction from $L_\infty(\Sigma, \mu)$ into $L_\infty(\Gamma, \mu)$, we may conclude that $T(B_E)$ contains no copy of the l_1-basis in $L_\infty(\Gamma, \mu)$. Consequently $T(B_E)$ is weakly precompact in $L_\infty(\Gamma, \mu)$ and there is a Pettis integrable kernel $g:(\Omega, \Gamma, \mu) \to E^*$ for the operator

$$T^*: L_1(\Gamma, \mu) \to E^*.$$

Then $\langle g, x\rangle = Tx = \xi(\langle f, x\rangle | \Gamma)$ a.e. for every x in E. Therefore

$$\int_B \langle g, x\rangle \, d\mu = \int_B \xi(\langle f, x\rangle | \Gamma) \, d\mu = \int_B \langle f, x\rangle \, d\mu$$

for every set B in Γ and hence $\int_B g \, d\mu = \int_B f \, d\mu$ for every set B in Γ. This shows that g is a Pettis conditional expectation of f for the σ-algebra Γ.

In view of Theorems 5 and 9, one can ask the following.

Question. If, in Theorem 9, we suppose that the set

$$\{\langle f, x\rangle : \|x\| \leq 1\}$$

is almost weakly precompact in $L_\infty(\mu)$, does f have a Pettis conditional expectation with respect to all sub-σ-algebras of Σ?

If the above were true, then any function satisfying the conditions of Theorem 5 would have a Pettis conditional expectation with respect to all Radon measurers on all sub-σ-algebras of the Borel σ-algebra of K.

IV. The Bourgain property

So far we have seen that the family $\{\langle f, x\rangle : \|x\| \leq 1\}$ plays a strong role in determining Pettis integrability for a bounded scalarly measurable function f from Ω into a dual space E^*. We continue this approach in this part, but, rather than viewing such families as subsets of $L_\infty(\mu)$, we now consider them simply as families of real-valued functions on Ω. A property of real-valued functions formulated by J. Bourgain [2] is the cornerstore of our discussion.

DEFINITION 10. Let (Ω, Σ, μ) be a measure space. A family Ψ of real-valued functions on Ω is said to have the *Bourgain property* if the following condition is satisfied: For each set A of positive measure and for each $\alpha > 0$, there is a finite collection F of subsets of positive measure of A such that for each function f in Ψ, the inequality $\sup f(B) - \inf f(B) < \alpha$ holds for some member B of F.

Let $f:\Omega \to E^*$ be a bounded scalarly measurable function. Fix x^{**} in E^{**} and use Goldstine's Theorem to find a bounded net (x_β) in E that converges to x^{**} in the weak*-topology. Let x_A^* be the weak*-integral of f over a set A in Σ and note that

$$x^{**}(x_A^*) = \lim_\beta x_A^*(x_\beta) = \lim_\beta \int_A \langle f, x_\beta \rangle \, d\mu.$$

Now if we could take the last limit underneath the integral sign, then we would have

$$x^{**}(x_A^*) = \int_A \lim_\beta \langle f, x_\beta \rangle \, d\mu = \int_A x^{**} f \, d\mu,$$

proving that f is Pettis integrable. Unfortunately, it is not always possible to take the limit underneath the integral sign but it is always possible to do so if the net (x_β) can be replaced by a sequence. The next theorem, which is due to Bourgain [2], essentially allows us to do this for some functions f.

THEOREM 11. *If (Ω, Σ, μ) is a finite measure space and Ψ is a family of real-valued functions on Ω satisfying the Bourgain property, then:*

(i) *the pointwise closure of Ψ satisfies the Bourgain property;*

(ii) *each element in the pointwise closure of Ψ is measurable;*

(iii) *each element in the pointwise closure of Ψ is the almost everywhere pointwise limit of a sequence from Ψ.*

Proof. The proof of (i) is completely straightforward. Towards verifying (ii) and (iii), take a function g belonging to the pointwise closure of Ψ and an ultrafilter U on Ψ that has g as a cluster point. For A in Σ and $\alpha > 0$, let

$$\Psi(A;\alpha) = \{ f \in \Psi : \sup f(A) - \inf(A) < \alpha \}.$$

It follows from the definition of the Bourgain property that if A has positive measure and $\alpha > 0$, then there exists a subset B of A of positive measure with $\Psi(B;\alpha)$ belonging to U. Now for each $\alpha > 0$, use Zorn's Lemma to find a maximal set P_α of mutually disjoint sets of positive measure such that $\Psi(A;\alpha) \in U$ for each $A \in P_\alpha$. Note that each P_α is necessarily countable. Moreover,

(a) the set $\Omega \setminus \cup P_\alpha$ has measure 0 for each $\alpha > 0$, and

(b) if F is a finite subset of positive reals and Q_α is a finite subset of P_α for each α in F, then g belongs to the pointwise closure of $\cap_{\alpha \in F} \cap_{A \in Q_\alpha} \Psi(A;\alpha)$.

The maximality of P_α yields condition (a), and condition (b) follows because g is a cluster point of U.

Now let $(A_{m,n})_n$ be an enumeration of $P_{1/m}$, and set

$$B = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} A_{m,n}.$$

By condition (a), we have $\mu(\Omega \setminus B) = 0$. Pick some point $\omega_{m,n}$ in each set $A_{m,n}$ and define

$$f_m = \sum_{n=1}^{\infty} g(\omega_{m,n}) \chi_{A_{m,n}}.$$

Each f_m is measurable and a quick computation using (b) shows that the sequence (f_m) converges to g uniformly on B. Therefore g is measurable.

Unfortunately, the functions f_m may not belong to Ψ. To establish (iii), therefore, use condition (b) to pick for each integer m a function h_m belonging to $\cap_{i=1}^{m} \cap_{n=1}^{m} \Psi(A_{i,n}; 1/i)$ such that

$$|h_m(\omega_{i,n}) - g(\omega_{i,n})| < 1/i$$

for each $1 \le i, n \le m$. The triangle inequality now ensures that $(h_m(\omega))$ converges to $g(\omega)$ for each ω in B. This completes the proof.

It is worth remarking here that a uniformly bounded family Ψ of real-valued functions has the Bourgain property if and only if the following condition holds:

(*) For each non-null measurable set A in Σ and for each pair $a < b$ of real numbers, there is a finite collection F of non-null measurable subsets of A such that for each f in Ψ, either $\inf f(B) \ge a$ or $\sup f(B) \le b$ for some member B of F.

Indeed, the Bourgain property for Ψ with $\alpha = b - a$ clearly implies property (*); conversely, the Bourgain property for Ψ can be obtained by finitely many successive applications of property (*).

In the sequel we study the family $\{\langle f, x\rangle : \|x\| \le 1\}$ for a bounded function $f : \Omega \to X^*$ and use the Bourgain property to determine the Pettis integrability of the function. We shall say that f has the *Bourgain property* if the family $\{\langle f, x\rangle : \|x\| \le 1\}$ has the Bourgain property.

Example 12. All strongly measurable functions into E^* have the Bourgain property. In particular, all Bochner integrable functions into E^* have the Bourgain property.

To see this, suppose $f : \Omega \to E^*$ is strongly measurable and let (s_n) be a sequence of simple functions for which

$$\lim_n \|f - s_n\| = 0 \quad \text{a.e.}$$

Let A be a measurable subset of Ω with $\mu(A) > 0$, and let $\alpha > 0$. Egorov's theorem ensures the existence of a set B with $\mu(\Omega \setminus B) < \mu(A)$ such that the sequence (s_n) converges uniformly to f on B. Choose an integer n so that $\|f(\omega) - s_n(\omega)\| < \alpha/4$ for all ω in B. Since $\mu(A \cap B) > 0$ we can find a set C on which s_n is constant and for which $\mu(A \cap B \cap C) > 0$. Let x be in the ball of E. Then for all ω_1, ω_2 in $A \cap B \cap C$, the triangle inequality shows that

$$|\langle f(\omega_1), x\rangle - \langle f(\omega_2), x\rangle| \leq \alpha/4 + 0 + \alpha/4 = \alpha/2.$$

Therefore

$$\sup_{A\cap B\cap C}\langle f, x\rangle - \inf_{A\cap B\cap C}\langle f, x\rangle < \alpha$$

for all x in the ball of E.

The following theorem gives a sufficient condition for Pettis integrability. Its converse is not true in general.

THEOREM 13. *A bounded function $f\colon \Omega \to E^*$ that has the Bourgain property is Pettis integrable.*

Proof. While no *a priori* hypothesis about the measurability of f is assumed, the Bourgain condition does show immediately that $\langle f, x\rangle$ is measurable for each x in E. Fix x^{**} in the unit ball of X^{**} and fix a set A in Σ. Let x_A^* be the weak*-integral of f over A, so that

$$(1)\qquad x_A^*(x) = \int_A \langle f, x\rangle\, d\mu \quad \text{for all } x \in E.$$

We must show that $x^{**}f$ is measurable and that $x^{**}(x_A^*) = \int_A x^{**}f\,d\mu$. Accordingly, let $\alpha > 0$ and set

$$\Psi = \{\langle f, x\rangle : \|x\| \leq 1, |\langle x^{**} - x, x_A^*\rangle| < \alpha\}.$$

Goldstine's theorem ensures that $x^{**}f$ lies in the pointwise closure of Ψ. Since Ψ has the Bourgain property, the function $x^{**}f$ is measurable by Theorem 11(ii), and statement (iii) of the same theorem shows that $x^{**}f$ is the almost everywhere limit of a sequence $\langle f, x_n\rangle$ from Ψ; that is,

$$(2)\qquad \lim_n \langle f, x_n\rangle = x^{**}f \quad \text{a.e.},$$

where

$$(3)\qquad |x^{**}(x_A^*) - x_A^*(x_n)| < \alpha \quad \text{for each } n.$$

It now follows from equations (1), (2), (3) and the Dominated Convergence

Theorem that

$$|x^{**}(x_A^*) - \int_A x^{**} f \, d\mu| \leq \alpha.$$

Since α was arbitrary, we conclude that x_A^* is the Pettis integral of f over the set A.

The following example shows that an E^*-valued universally Pettis integrable function does not have the Bourgain property in general.

Example 14. *A universally Pettis integrable function without the Bourgain property.*

For each $t \in [0,1]$, define a subset D_t by

$$D_t = \{t \in [0,1] : |t - s| = \text{dyadic rational}\}$$

and define $f : [0,1] \to l_\infty[0,1]$ by $f(t) = \chi_{D_t}$. This function was constructed by R. S. Phillips in [10].

Claim 1. $x^* f = 0$ except on a countable set for every x^* in l_∞^* $[0,1]$.

Proof of Claim 1. Any x^* in $l_\infty^*[0,1]$ can be written as $x^* = \beta_1 + \beta_2$ where β_1 has countable support S and β_2 vanishes on null sets.

Let $N = \cup_{t \in S} D_t$. If $t_0 \notin N$, then $D_{t_0} \cap S = \emptyset$, since $t \in D_{t_0} \cap S$ implies that $t_0 \in D_t \subset N$, a contradiction. Therefore

$$\begin{aligned} x^* f(t_0) &= \langle \beta_1, \chi_{D_{t_0}} \rangle + \langle \beta_2, \chi_{D_{t_0}} \rangle \\ &= \beta_1(D_{t_0}) + \beta_2(D_{t_0}) \\ &= 0 + 0 \end{aligned}$$

since D_{t_0} is disjoint from the support of β_1 and β_2 vanishes on countable subsets. Thus $x^* f = 0$ off N and N is countable. Consequently f is universally Pettis integrable.

Claim 2. f fails the Bourgain property (with respect to the Lebesgue measure λ).

To prove claim 2, we need the following lemma:

LEMMA A. *Let $A \subset [0,1]$ and let $F = \cup_{t \in A} D_t$. Then either $\lambda^*(F) = 0$ or $\lambda^*(F) = 1$ (where λ^* is the outer measure associated with λ).*

Proof of Lemma A. Note that F satisfies the following property:

(*) If $x \in F, r$ a dyadic rational and $x + r \in [0,1]$, then $x + r \in F$.

This will help us prove that

$$\frac{\lambda^*(F \cap I)}{\lambda^*(I)} = \lambda^*(F)$$

for any interval I. To do this it is enough to show the above equality for I a dyadic interval,

$$I = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right].$$

Divide $[0,1]$ into 2^n equal intervals $I_1, \ldots, I_2n$. Then I is one of these intervals and

$$\lambda^*(F \cap I) = \lambda^*(F \cap I_i), \quad i = 1, \ldots, 2^n,$$

because F satisfies (*). Hence

$$\lambda^*(F) = \sum_{i=1}^{2^n} \lambda^*(F \cap I_i) = 2^n \lambda^*(F \cap I) = \frac{\lambda^*(F \cap I)}{\lambda^*(I)}$$

If $\lambda^*(F) \neq 0$, then F has a point of density x_0, (see [14, p. 194]). Therefore

$$1 = \lim_{\substack{\lambda(I) \to 0 \\ x_0 \text{ midpoint of } I}} \frac{\lambda^*(F \cap I)}{\lambda^*(I)} = \lambda^*(F).$$

Proof of Claim 2. Let $B_1, B_2, \ldots, B_n$ be arbitrary non-null subsets of $[0,1]$. Without loss of generality we may assume that $\cup_{i=1}^n B_i = [0,1]$ and that the B_i are disjoint.

First suppose that for every t in $[0,1]$, there exists some B_i such that $D_t \cap B_i = \varnothing$. Let

$$E_i = \{t \in [0,1] : D_t \cap B_i = \varnothing\}.$$

Then $\cup_{i=1}^n E_i = [0,1]$. Observe that $E_i \subset [0,1] \setminus B_i$. Also, if $t \in E_i$, then $D_t \subset [0,1] \setminus B_i$. Hence

$$F_i = \bigcup_{t \in E_i} D_t \subset [0,1] \setminus B_i$$

for each i. Now there exists some E_{i_0} such that $\lambda^*(E_{i_0}) \neq 0$; therefore

$\lambda^*(F_{i_0}) \neq 0$ and by Lemma A, $\lambda^*(F_{i_0}) = 1$ which yields $\lambda^*(B_{i_0}) = 0$, a contradiction. Therefore, there must exist $\hat{t} \in [0,1]$ such that for each $i = 1, 2, \ldots, n$, there is $s_i \in B_i$ with $s_i \in D_{\hat{t}}$. Choose $t_i \in B_i \setminus D_{\hat{t}}$ for each $i = 1, 2, \ldots, n$, and let $x \in l_1[0,1]$ be given by $x = \delta_{\hat{t}}$. Then

$$\langle f(t_i), x \rangle = \chi_{D_{t_i}}(\hat{t}) = 0$$

since $\hat{t} \notin D_{t_i}$ and therefore $\inf_{t \in B_i} \langle f(t), x \rangle = 0, i = 1, 2, \ldots, n$. On the other hand

$$\langle f(s_i), x \rangle = \chi_{D_{s_i}}(\hat{t}) = 1$$

since $t \in D_{s_i}$. This shows that $\sup_{t \in B_i} \langle f(t), x \rangle = 1, i = 1, 2, \ldots, n$. Hence f fails the Bourgain property for the Lebesgue measure.

In this sequel we will show that if, in addition, E is separable, any E^*-valued universally Pettis integrable function defined on a compact metric space has the Bourgain property for any Radon measure λ on K. To do this we will characterize the Bourgain property of an E^*-valued function f in terms of the associated family of martingales in the case where E and $(\Omega, \Sigma, \lambda)$ are separable. We will use this characterization to show that if the family $\{\langle f, x \rangle : \|x\| \leq 1\}$ is almost weakly precompact in $L_\infty(\Omega, \Sigma, \lambda)$ then f has the Bourgain property.

LEMMA 15. *Suppose $f: \Omega \to E^*$ and $g: \Omega \to E^*$ are equal almost everywhere. Then f has the Bourgain property if and only if g has the Bourgain property.*

Proof. Let N be a null set such that $f(\omega) = g(\omega)$ for all ω not in N. Clearly both $\langle f, x \rangle$ and $\langle g, x \rangle$ have the same supremum and infimum on the set $A \setminus N$ for any set A of positive measure. The conclusion now follows immediately.

For the rest of this paper we shall assume that (Ω, Σ, μ) is a finite separable measure space. This means that there is a sequence (π_n) of finite partitions of Ω consisting of sets in Σ such that

(1) each member of π_{n+1} is contained in a member of π_n (i.e., π_{n+1} refines π_n), and

(2) the union of the σ-algebras generated by the partitions π_n is dense in Σ.

For example, if $\Omega = [0,1]$ and μ is Lebesgue measure on the Borel sets Σ, then the dyadic partitions of $[0,1]$ would satisfy these assumptions. Let Σ_n denote the σ-algebra generated by π_n and let $\sigma = \cup_{n=1}^{\infty} \Sigma_n$.

LEMMA 16 (Bourgain). *Suppose A is a subset of Ω with positive measure and $0 < \alpha < 1$. Then there is an integer m and a measurable subset $B \subset A$ with*

$\mu(B) > (1 - \alpha)\mu(A)$ such that for every uniformly bounded by 1 real-valued martingale (g_n, Σ_n) and for every $n \geq m$.

(i) $\operatorname{ess\,inf} g(A) \leq \inf g_n(B) + \alpha$ *and*

(ii) $\operatorname{ess\,sup} g(A) \geq \sup g_n(B) - \alpha$ *where g is any almost everywhere limit of the sequence (g_n).*

Proof. Choose $a, b > 0$ so that $1 - \alpha/4 < a < 1, b < 1$, and $1 + a < 2b$. Choose an integer m and a set A_1 in Σ_m such that

$$\mu(A \Delta A_1) < (1 - b)^2 \mu(A).$$

Now let

$$\Pi = \{E \in \sigma : \mu(E \cap A_1 \setminus A) > (1 - a)\mu(E)\}$$

and set $W = \cup \Pi$. We can easily see that $\mu(W \cap A_1 \setminus A) > (1 - a)\mu(W)$. If we let $C = \Omega \setminus W$, then

$$\mu(C) = 1 - \mu(W) > 1 - \frac{\mu(A_1 \setminus A)}{1 - a} > 1 - (1 - b)\mu(A)$$

and

$$\mu(E \cap A_1 \setminus A) \leq (1 - a)\mu(E)$$

whenever E is in σ and $E \cap C \neq \varnothing$.

We claim that the integer m and the set $B = A \cap A_1 \cap C$ satisfy the stated conditions. First of all, notice that

$$\begin{aligned} \mu(B) &\geq \mu(A \cap A_1) - \mu(\Omega \setminus C) \geq \mu(A \cap A_1) - (1 - b)\mu(A) \\ &\geq \mu(A) - (1 - b)^2\mu(A) - (1 - b)\mu(A) > a\mu(A) \geq (1 - \alpha)\mu(A). \end{aligned}$$

We next verify condition (i) (the argument for (ii) follows by replacing g_n with $-g_n$). Suppose $n \geq m$ and β is any number satisfying $\inf g_n(B) < \beta < 1 + \alpha$. Since g_n is constant on the members of π_n, there is some element I in π_n such that $I \cap B$ is non-empty and $g_n < \beta$ on I. Moreover, we have $I \subset A_1$ since $I \cap A_1$ is non-empty and A_1 is a union of sets in π_n. But because $I \cap C \neq \varnothing$, we see that

$$\mu(I \cap A) = \mu(I) - \mu(I \cap A_1 \setminus A) \geq a\mu(I).$$

Now suppose $\operatorname{ess\,inf} g(A) > \beta + \alpha$. Because g_n is the conditional expectation of g with respect to the σ-algebra Σ_n, we have

$$\begin{aligned}\beta\mu(I) &> \int_I g_n\,d\mu = \int_I g\,d\mu = \int_{I\cap A} g\,d\mu + \int_{I\setminus A} g\,d\mu \\ &> (\beta+\alpha)\mu(I\cap A) - \mu(I\setminus A) = (\beta+\alpha+1)\mu(I\cap A) - \mu(I) \\ &\geq (\beta+\alpha+1)a\mu(I) - \mu(I).\end{aligned}$$

Hence $(\beta+\alpha+1)a - 1 < \beta$, and this implies $\beta > 3 - \alpha$, a contradiction. Therefore $\operatorname{ess\,inf} g(A) \leq \beta + \alpha$, and the proof is complete.

Let $f: \Omega \to E^*$ be a bounded weak*-scalarly measurable function and define an E^*-valued martingale (f_n, Σ_n) by

$$f_n(\cdot) = \sum_{A\in\pi_n} \frac{w^*\text{-}\int_A f\,d\mu}{\mu(A)}\chi_A(\cdot)$$

where $w^*\text{-}\int_A f\,d\mu$ is the weak*-integral of f over the set A. Without loss of generality we may assume that $\|f\| \leq 1$ pointwise. Then for each x in E, the sequence $(\langle f_n, x\rangle, \Sigma_n)$ is a real-valued martingale, uniformly bounded by 1, with $\lim_n\langle f_n, x\rangle = \langle f, x\rangle$ a.e., where the exceptional null set may, of course, vary with x.

Lemma 17. *Let E be a separable Banach space. Then f has the Bourgain property if and only if the family $\{\langle f_n, x\rangle : n \in N, \|x\| \leq 1\}$ has the Bourgain property.*

Proof. Let (x_m) be a dense sequence in E. For each integer m there exists a null set N_m satisfying

$$\lim_n \langle f_n(\omega), x_m\rangle = \langle f(\omega), x_m\rangle$$

for each ω that is not in the null set N_m. Because the sequence (x_n) is dense, it follows easily that for each $x \in E$,

$$\lim_n \langle f_n(\omega), x\rangle = \langle f(\omega), x\rangle$$

for each ω that is not in the null set $N = \bigcup_{m=1}^{\infty} N_m$.

First suppose that the family $\{\langle f_n, x\rangle : n \in N, \|x\| \leq 1\}$ has the Bourgain property. When the ball of E^* is equipped with the weak*-topology the space of functions from Ω into B_{E^*} is compact for the topology of pointwise convergence. Therefore, there exists a pointwise weak*-cluster point $g: \Omega \to E^*$

of the sequence (f_n). This means that the family $\{\langle g, x\rangle : \|x\| \leq 1\}$ belongs to the pointwise closure of the family $\{\langle f_n, x\rangle : n \in N, \|x\| \leq 1\}$. Consequently, the function g has the Bourgain property by Theorem 11 (i). A moment's reflection, however, shows that $\langle f(\omega), x\rangle = \langle g(\omega), x\rangle$ for each ω not in N and for each x in E. Hence $f = g$ almost everywhere. Now invoke Lemma 15 to see that f has the Bourgain property.

Conversely, suppose that the family $\{\langle f, x\rangle : \|x\| \leq 1\}$ has the Bourgain property. Let A be a set of positive measure and let $a < b$. Choose $\alpha < 0$ such that $a + \alpha < b - \alpha$. There exist non-null subsets $A_1, \ldots, A_k$ of A such that for each x in the ball of E either $\sup_{A_i}\langle f, x\rangle \leq b - \alpha$ or $\inf_{A_i}\langle f, x\rangle \geq a + \alpha$ for some i.

According to Lemma 16, there is for each set A_i an integer m_i and a non-null subset B_i of A_i such that

(a) $\operatorname{ess\,inf}_{A_i}\langle f, x\rangle \leq \inf_{B_i}\langle f_n, x\rangle + \alpha$ and

(b) $\operatorname{ess\,sup}_{A_i}\langle f, x\rangle \geq \sup_{B_i}\langle f_n, x\rangle - \alpha$ for every x in the ball of E and for every $n \geq m_i$. Let

$$m = \max\{m_i : 1 \leq i \leq k\}.$$

Let $n \geq m$, let x be in the ball of E, and note that there exists an A_i such that either

$$b - \alpha \geq \sup_{A_i}\langle f, x\rangle \geq \operatorname{ess\,sup}_{A_i}\langle f, x\rangle \geq \sup_{B_i}\langle f_n, x\rangle - \alpha$$

or

$$a + \alpha \leq \inf_{A_i}\langle f, x\rangle \leq \operatorname{ess\,inf}_{A_i}\langle f, x\rangle \leq \inf_{B_i}\langle f_n, x\rangle + \alpha.$$

That is, either $b \geq \sup_{B_i}\langle f_n, x\rangle$ or $a \leq \inf_{B_i}\langle f_n, x\rangle$. Therefore the sets $B_1, \ldots, B_k$ will work for the set A for the family

$$\{\langle f_n, x\rangle : n \geq m, \|x\| \leq 1\}.$$

However, the functions $f_1, \ldots, f_{m-1}$ are just simple functions, so that for each $i = 1, \ldots, m-1$ there exists a set C_i on which f_i is constant and $\mu(A \cap C_i) > 0$. Thus the sets $B_1, \ldots, B_k, C_1 \cap A, \ldots, C_{m-1} \cap A$ will work to show that the family

$$\{\langle f_n, x\rangle : n \in N, \|x\| \leq 1\}$$

has the Bourgain property,

THEOREM 18. *Let E be a separable Banach space and let $f : \Omega \to E^*$ be a bounded weak*-scalarly measurable function. If the family*

$$\{\langle f, x\rangle : \|x\| \leq 1\}$$

is almost weakly precompact in $L_\infty(\mu)$, then f has the Bourgain property, and hence f is Pettis integrable.

Proof. Observe first that the family $\{\langle f, x\rangle : \|x\| \le 1\}$ has the Bourgain property if and only if for each $\alpha > 0$ there exists a set A in Σ with $\mu(\Omega \setminus A) < \alpha$ such that the family

$$\{\langle f, x\rangle \chi_A : \|x\| \le 1\}$$

has the Bourgain property. To see this, take a set B in Σ with $\mu(B) = \alpha > 0$ and apply the Bourgain condition to the non-null set $A \cap B$, where A satisfies the above hypothesis. Without loss of generality, therefore, we may delete the "almost" and assume that $\{\langle f, x\rangle : \|x\| \le 1\}$ is weakly precompact in $L_\infty(\mu)$. We will also assume that $\|f\| \le 1$.

By Lemma 17 it suffices to show that the family

$$\{\langle f_n, x\rangle : n \in N, \|x\| \le 1\}$$

has the Bourgain property. Suppose it does not. Then an argument due to Bourgain [2] produces a sequence (x_n) in the ball of E, a system $(A_{n,m})$, $n \in N, 1 \le m \le 2^n$, of sets of positive measure, and constants $\delta > \beta$ such that

(1) $A_{n+1,2m-1} \subset A_{n,m}$ and $A_{n+1,2m} \subset A_{n,m}$,
(2) $\langle f(\omega), x_{n+1}\rangle < \delta$ if $\omega \in A_{n+1,2m-1}$,
(3) $\langle f(\omega), x_{n+1}\rangle > \beta$ if $\omega \in A_{n+1,2m}$.

We sketch the inductive step in the construction. Let $A \in \Sigma$ and $a < b$ be reals for which property (*) (page 520) cannot be obtained. For each $m = 1, \dots, 2^n$, Lemma 16 provides an integer k_m and a subset $B_m \subset A_{n,m}$ of positive measure such that for $k \ge k_m$ and x in the ball of E,

$$\operatorname{ess\,inf}_{A_{n,m}} \langle f, x\rangle \le \inf_{B_m} \langle f_k, x\rangle + \alpha,$$
$$\operatorname{ess\,sup}_{A_{n,m}} \langle f, x\rangle \ge \sup_{B_m} \langle f_k, x\rangle - \alpha$$

where $\alpha > 0$ has been chosen so that $a + \alpha < b - \alpha$. Set

$$j = \max\{k_m : 1 \le m \le 2^n\}$$

and for each $m = 1, \dots, 2^n$ choose a subset C_m of B_m that has positive measure and is contained in a member of the partition π_j. The negation of the Bourgain property produces some integer k and x_{n+1} in the ball of E such that

$$\inf_{C_m} \langle f_k, x_{n+1}\rangle < a \quad \text{and} \quad \sup_{C_m} \langle f_k, x_{n+1}\rangle > b$$

for all $m = 1, \dots, 2^n$. Since f_k is constant on each member of π_k, it is clear that $k > j$ and therefore

$$\operatorname{ess\,inf}_{A_{n,m}}\langle f, x_{n+1}\rangle \leq \inf_{B_m}\langle f_k, x_{n+1}\rangle + \alpha < a + \alpha = \delta,$$
$$\operatorname{ess\,sup}_{A_{n,m}}\langle f, x_{n+1}\rangle \geq \sup_{B_m}\langle f_k, x_{n+1}\rangle - \alpha > b - \alpha = \beta$$

for each $m = 1, \ldots, 2^n$. Consequently, the sets

$$A_{n+1,2m-1} = \{\omega \in A_{n,m} : \langle f(\omega), x_{n+1}\rangle < \delta\}$$

and

$$A_{n+1,2m} = \{\omega \in A_{n,m} : \langle f(\omega), x_{n+1}\rangle > \beta\}$$

have positive measure.

Let $O_n = \cup_{m=1}^{2^{n-1}} A_{n,2m-1}$ and $E_n = \cup_{m=1}^{2^{n-1}} A_{n,2m}$ for each integer n. Then the sequence of pairs (O_n, E_n) is independent in the sense of Rosenthal [15]. More, however, is is true in this case, for we actually have

$$\left(\bigcap_{n\in G} O_n \cap \bigcap_{n\in B} E_n\right) \setminus N \neq \varnothing$$

for every pair of disjoint finite non-empty subsets G and B of integers and for every null set N. Rosenthal's argument (see [15]) therefore shows that the sequence $(\langle f, x_n\rangle)$ is a copy of the l_1-basis in the $L_\infty(\mu)$-norm, rather than in just the supremum norm. Since this contradicts the assumption that the family $\{\langle f, x\rangle : \|x\| \leq 1\}$ is weakly precompact in $L_\infty(\mu)$, we conclude that the family

$$\{\langle f_n, x\rangle : n \in N, \|x\| \leq 1\}$$

has the Bourgain property.

COROLLARY 19. *Let K be a compact metric space, E a separable Banach space and $f: K \to E^*$ a bounded function. Then the following statements are equivalent*:

(i) *The function f is universally Pettis integrable.*

(ii) *For every $\lambda \in M_+^1(K)$, f has the Bourgain property for (K, Σ, λ) where Σ is the Borel σ-algebra of K.*

Proof. The implication (ii) $\Rightarrow$ (i) is Theorem 13. To see that (i) $\Rightarrow$ (ii). Fix $\lambda \in M_+^1(K)$. Then $\lambda = \lambda_1 + \lambda_2$ where λ_1 is diffuse and λ_2 is purely atomic.

The function $f: (K, \Sigma, \lambda_2) \to E^*$ is Bochner integrable and therefore it has the Bourgain property for (K, Σ, λ_2) (see Example 12). The measure space (K, Σ, λ_1) is separable. By Theorem 5, $\{\langle f, x\rangle : \|x\| \leq 1\}$ is almost weakly precompact in $L_\infty(\lambda_1)$. Hence f has the Bourgain property for (K, Σ, λ_1).

It is easy now to see that f has the Bourgain property for (K, Σ, λ), because if $A \in \Sigma$ and $\lambda(A) > 0$, then either $\lambda_1(A) > 0$ or $\lambda_2(A) > 0$.

THEOREM 20. *Let K be a compact Hausdorff space and let $f: K \to E^*$ be bounded and universally Lusin measurable when E^* is equipped with its weak*-topology. Then the following statements are equivalent*:

(i) *The function f is universally scalarly measurable.*
(ii) *The function f is universally Pettis integrable.*
(iii) *For every Radon measure λ on K, the set*

$$\{\langle f, x\rangle : x \in E \|x\| \leq 1\}$$

is almost weakly precompact in $L_\infty(K, \lambda)$. If in addition K is metric and E is separable, the above statements are equivalent to:

(iv) *The function f has the Bourgain property for every Radon probability measure on K.*

As Example 14 shows, (ii) does not imply (iv) if the separability condition is dropped.

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ATLANTA, GEORGIA
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