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PRODUCT OF SUBGROUPS IN LIE GROUPS

Dedicated to Professor Gail S. Young on His Seventieth Birthday

BY

TA-SUN WU[1]

Introduction

In the study of analytic subgroups of an analytic group, sometimes it is important to know the structure of the product of two analytic subgroups and the closure of this product. There are two fundamental theorems in this aspect: one is a theorem of Mostow's and another is known as the Auslander-Wang-Zassenhaus theorem. Mostow's theorem (cf. [5]) says that if D is a closed uniform subgroup of a solvable analytic group G such that D contains no non-trivial normal analytic subgroup of G, then DN is closed in G and $D \cap N$ is a closed uniform subgroup of N, where N is the nilradical of G. (Let Y be a subset of a topological space X. Then we denote by $\overline{Y}$ the closure of Y in X. Let Z be a subgroup of a topological group K. Then, by definition, Z is a uniform subgroup of K or Z is uniform in K if $K/\overline{Z}$ is compact. If K is an analytic group, then the maximal nilpotent normal analytic subgroup of K is called the nilradical of K.) Clearly, Mostow's theorem holds for discrete uniform subgroups. A natural question is what happens if the discrete subgroup D is not uniform. In fact, in this case, it is easy to find examples of D so that DN is not closed in G and $D \cap N$ is not uniform in N. However, in these notes, we prove the following result.

THEOREM 1. *Let G be a simply connected solvable analytic group with nilradical N. If D is a discrete subgroup of G such that DN is dense in G, then D is nilpotent.*

By definition, if D is a discrete subgroup of a locally compact group G and if α: $G \to G/D$ denotes the canonical map, then D is called an L-subgroup of G provided that for every neighborhood U of the identity element e of G the

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closure of $\alpha(\{g \in G\colon gDg^{-1} \cap U = \{e\}\})$ is compact in G/D. As an application of Theorem 1, we give a variation of a proof of a theorem of S.P. Wang concerning L-subgroups (cf. [8]).

THEOREM 2. *Let G be a solvable analytic group with nilradical N. If D is an L-subgroup of G, then $D \cap N$ is uniform in N.*

The Auslander-Wang-Zassenhaus theorem (cf. [6]) states that if G is a Lie group, R is a closed connected solvable normal subgroup of G, and H is a closed subgroup of G such that H_0 is solvable, then $(\overline{RH})_0$ is also solvable. (If Z is a subgroup of a topological group K, then we denote Z_0 the connected component of Z containing the identity.) In these notes, we use the theorem above to prove the following result.

THEOREM 3. *Let G be an analytic group with radical R, let H be a closed subgroup of G, and let $F = \overline{RH}$. If $R(F_0)$ denotes the radical of F_0, then $F = R(F_0)H$ and $F_0 = R(F_0)H_0$. (By the radical of an analytic group K, we mean the maximal solvable normal analytic subgroup of K.)*

Using this theorem, we are able to prove that if H is a uniform and unimodular subgroup of a locally compact group G, then G is unimodular. This result was known for those groups that satisfy the second axiom of countability (cf. [4]). Since it appears in [9], we will not give the detail here.

The letter R, Q, Z denote the collection of all real numbers, rational numbers, integers, respectively.

1. Discrete subgroups of simply connected solvable analytic groups

Our main result in this section is the following theorem.

THEOREM 1. *Let G be a simply connected solvable analytic group with nilradical N. If D is a discrete subgroup of G such that DN is dense in G, then D is nilpotent.*

Since the notion of semisimple slitting is used intensively in the proof, we shall recall some of its results as follows (cf. [1]):

Let G, N, D be as in Theorem 1.

(1) There exists a group G_s and an abelian analytic group T of automorphisms of G such that the differential of each element of T is semisimple (i.e., completely reducible), $G_s = G \rtimes T$ (the semidirect product), and $G_s = M \rtimes T$, where M is the nilradical of G_s. G_s is called a semisimple splitting of G.

(2) G and M generate G_s, and $N \subseteq M$.

(3) Let $p\colon M \rtimes T \to M$ be the projection. Then $p|G$, the restriction of p to G, is a homeomorphism of G onto M.

If Z is a subgroup of an algebraic group K, then the smallest algebraic subgroup of K containing Z is called the algebraic hull of Z and is denoted by $A_h(Z)$.

(4) Let $H_D = A_h(D \cap N)$. Then, $D \cap N$ is uniform in H_D, H_D is a closed analytic subgroup of N, H_D is normalized by D. And hence, $D_R = DH_D$ is a closed subgroup of G.

(5) Let $p_T\colon M \rtimes T \to T$ be the projection. Then, T can be so chosen that $T_D = p_T(D_R)$ acts on D_R by conjugation. Hereafter, we always assume that T is so chosen. Put $D_S = D_R \rtimes T_D$ and let M_D be the maximal nilpotent normal subgroup of D_S. Then, $D_S = M_D \rtimes T_D$ and M_D is contained in M.

(6) $p|D_R$, the restriction of p to D_R, is a homeomorphism of D_R onto M_D.

If Z is a subgroup of a group K, then we denote $[Z, Z]$ the subgroup of K generated by elements of the form $aba^{-1}b^{-1}$ with a, b in Z.

(7) $[D_s, D_s]$ is contained in H_D.

(8) If K is a vector subgroup of D_s normalized by D_R, then K is a normal subgroup of D_s.

Now, we are ready to prove our theorem. We shall use the same notations as above.

Proof of Theorem 1. If H_D is trivial, then so is $D \cap N$. Together with the fact that $[G, G]$ is contained in N (since G is a solvable analytic group), we see that $[D, D]$ is trivial. In particular, D is nilpotent. So, from now on, we shall assume that H_D is not trivial.

Since H_D is a simply connected nilpotent analytic *gp*, the last non-trivial term V of the lower central series of H_D is a vector group. Because H_D is normalized by D and V is a characteristic subgroup of H_D, V is normalized by D. So, we have an action of D on V by conjugation. Let d be an element of D. If v is an element of V, we denote dvd^{-1} by $I(d)(v)$. Then, $I(d)$ is an element of $GL(V)$. Denote by $I(d)_u$ the unipotent part and $I(D)_s$ the semisimple part in the Jordan decomposition of $I(d)$. We also denote the collection of $I(d), I(d)_u, I(d)_s$ as d ranges over D by $I(D), I(D)_u, I(D)_s$, respectively.

By a result in [6] (Corollary 1 of Theorem 2.3), $D \cap V$ is a discrete uniform subgroup of V. Since $D \cap N$ is a discrete uniform subgroup of H_D. Thus, $D \cap V$ is a free abelian subgroup of V generated by an $\mathbf{R}$-basis

$$\mathscr{B} = \{v_1, \ldots, v_n\}$$

for V [2, III 1.2]. Let V_Q be the Q-span of $v_1, \ldots, v_n$, and let V_1 be the eigenvalue one Q-subspace of V_Q relative to $I(D)_u$. We claim that $D \cap V_1$ is not trivial.

Lying in the center of H_D, V is pointwise fixed by $I(d)$ for every d in $D \cap N$. Together with the fact that $D/(D \cap N)$ is abelian, $I(D)$ is an abelian group; and hence, $I(D)_u$ is an abelian group. On the other hand, since $D \cap V$

is clearly $I(D)$-invariant, $I(D)$ lies in $GL(n, Z)$ relative to $\mathscr{B}$; and hence, $I(D)_u$ lies in $GL(n, Q)$ relative to $\mathscr{B}$ [3, pp. 62, 63]. Let L be the Q-span of the elements of the form $I(d)_u - \mathrm{id}$, where id denotes the identity matrix. By the commutativity of $I(D)_u$, we see that L is a Lie subalgebra of $M(n, Q)$ consisting of nilpotent matrices. It follows that there is a non-zero element v_0 of V_Q anniliated by L [7, Theorem 3.5.2]. Clearly, v_0 is in V_1. Expressing v_0 as a finite Q-linear combination of $v_1, \ldots, v_n$, we see that there is a large enough integer a such that av_0 is a non-zero element in $D \cap V_1$. This proves that $D \cap V_1$ is nontrivial as we claimed.

Since V_1 is pointwise fixed by each element of $I(D)_u$, one checks directly that V_1 is $I(D)$-invariant; and hence, $D \cap V_1$ is $I(D)$-invariant. Together with the fact that $D \cap V_1$ is pointwise fixed by each element of $I(D)_U$, we conclude that

(9) $D \cap V_1$ is $I(D)_s$-invariant, and $I(g)_s|(D \cap V_1) = I(g)|(D \cap V_1)$ for every g in D.

From this, we claim that $D \cap V_1$ is T_D-invariant. That is, if $x = mt$ is an element of D_R with m in M_D and t in T_D, we must prove that

$$t(D \cap V_1)t^{-1} = D \cap V_1.$$

To this end, let x, m, t be as above, and write $x = dh$ with d in D and h in H_D. Let $\{w_1, \ldots, w_m\}$ be a Q-basis for V_1 and let W be the $\mathbf{R}$-span of $w_1, \ldots, w_m$. Expressing each w_i as a finite Q-linear combination of $v_1, \ldots, v_n$, one sees that there is a large enough integer b such that $bw_1, \ldots, bw_m$ are in $D \cap V_1$. By the linearities of $I(d)$ and $I(d)_s$, (9) shows that W is $I(d)_s$-invariant and $I(d)|W = I(d)_s|W$. Moreover, since clearly W is contained in V and V lies in the center of H_D, h acts on W trivially. Consequently, W is $I(x)$-invariant and

$$I(x)|W = I(d)_s|W, \tag{10}$$

where $I(x)$ denotes the action of x on G_s by conjugation.

Clearly, the above discussion of x holds for every element in D_R, applying (8) to the vector group W, W is therefore invariant under the action of every element in $(D_R)_s$ by conjugation. In particular, W is $I(m)$-invariant and $I(t)$-invariant, where $I(m), I(t)$ denote the actions of m, t on G_s by conjugation, respectively. Since W is a vector subgroup of the simply connected nilpotent analytic group M and m is contained in M (since M_D is contained in M), one checks directly that m acts on W unipotently. On the other hand, since T_D is contained in T, by the semisimplicity of T, t acts on W semisimply, Moreover, by (7), $mtm^{-1}t^{-1}$ lies in H_D. Since W is contained in the center of H_D, $mtm^{-1}t^{-1}$ acts on W trivially by conjugation; that is, $I(m)|W$ and $I(t)|W$ commute. Consequently, $I(m)|W$ is the unipotent part and $I(t)|W$ is the semisimple part in the Jordan decomposition of $I(x)|W$. Thus, (10) forces that m acts on W trivially; and hence, $I(x)|W = I(t)|W$. Together with (9) and (10), we obtain

$$t(D \cap V_1)t^{-1} = D \cap V_1,$$

as we wished. And thus, $D \cap V_1$ is T_D-invariant.

Next, we claim that T_D is dense in T from the hypothesis DN is dense in G. To see this, let t_1 be a fixed element of T, and let U be any open neighborhood of the identity in T. Since N is contained in M, the fact that DN is dense in G implies that G is contained in $\overline{MD}$. Together with the fact that G_s is generated by G and M, we have $G_s = \overline{MD}$. It follows that there is an element m_1 in M and an element d_1 in D such that $m_1 d_1$ lies in MUt_1 since MUt_1 is an open neighborhood of t_1 in G_s. On the other hand, d_1 can be expressed as a product $m_2 t_2$ with m_2 in M_D and t_2 in T_D. Consequently, $m_1 m_2 t_2$ is an element in MUt_1 with $m_1 m_2$ in M, t_2 in T, and Ut_1 contained in T. It follows that t_2 must be in Ut_1. This proves our claim that T_D is dense in T.

Since $D \cap V_1$ is discrete, and hence closed in G_s. The results that $D \cap V_1$ is T_D-invariant and T_D is dense in T therefore imply that $D \cap V_1$ is T-invariant. It follows from the connectedness of T and the discreteness of $D \cap V_1$, $D \cap V_1$ is pointwise fixed by every element of T. In particular, $D \cap V_1$ is pointwise fixed by every element of T_D; or equivalently, every element of T_D acts on $D \cap V_1$ trivially by conjugation.

If t_0 is any element of T_D, then there is an element x_0 in D_R such that $x_0 = m_0 t_0$ with m_0 in M_D. By the above discussion of x, we see that W is $I(t_0)$-invariant. Since t_0 acts on $D \cap V_1$ trivially, $bw_1, \dots, bw_m$ lie in $D \cap V_1$, and W is the $\mathbf{R}$-span of $w_1, \dots, w_m$, the linearity of $I(t_0)$ (over reals) therefore implies that t_0 acts on W trivially. Moreover, by the above discussion of x, we also see that m_0 acts on W trivially. Consequently, x_0 acts on W trivially. Since the above discussion holds for every element in T_D and every element in D_R, W is a nontrivial (since $D \cap V_1$ is non-trivial) vector group contained in H_D and contained in the center of $(D_R)_s$, where $(D_R)_s = D_s$.

Let

$$f\colon (D_R)_s \to (D_R)_s/W$$

be the canonical map. (Note that $(D_R)_s/W = (D_R/W) \rtimes T_D$.) If $f(H_D)$ is trivial, then H_D coincides with W. Thus, D_R/H_D is isomorphic with

$$D/(D \cap H_D) = D/(D \cap N).$$

Since the later is an abelian group and $H_D = W$ is contained in the center of D_R, D_R is nilpotent; and hence, D is nilpotent.

Suppose that $f(H_D)$ is not trivial. Then, again since $bw_1, \dots, bw_m$ are in $D \cap V_1$, and W is the $\mathbf{R}$-span of $w_1, \dots, w_m$, we see that $D \cap V_1$ is a discrete uniform subgroup of W. It follows that DW is closed in $(D_R)_s$. Thus, $f(D)$ is topologically isomorphic with $D/(D \cap W)$; in particular, $f(D)$ is a discrete subgroup of $f((D_R)_s)$. Similarly, one obtains that $f(D \cap N)$ is a discrete subgroup of $f(H_D)$. Moreover, one checks directly that $f(H_D)$ is a simply connected nilpotent analytic group and $f(D \cap N)$ is a discrete uniform subgroup in $f(H_D)$. Therefore, we may consider the last non-trivial term of the lower central series of $f(H_D)$ and go over exactly the same agreement as above to obtain a non-trivial vector group contained in $f(H_D)$ and contained in the center of $f((D_R)_s)$. Continuing this process, due to the finite-dimensionality of H_D, the process must terminate at a stage in which a similar argument in the last paragraph applies. So, we may conclude that D is nilpotent. This proves the theorem.

Remark. In fact, we have proven that D_R is nilpotent; and hence, so is D_s.

2. L-subgroups of solvable analytic groups

As an application of Theorem 1, we give another proof of the following result of S.P. Wang (cf. [8]).

THEOREM 2. *Let G be a solvable analytic group with nilradical N. If D is an L-subgroup of G, then $D \cap N$ is uniform in N.*

Proof. Let G' be a universal covering group of G and let $f\colon G' \to G$ be the group covering. Then $f^{-1}(D)$ is an L-subgroup of G' (Lemma 1.5 in [8]). Let N' be the nilradical of G'. If $f^{-1}(D) \cap N'$ is uniform in N', since $f(N') = N$, we see that $D \cap N$ is uniform in N. So, we may assume in addition that G is simply connected.

If DN is closed in G, then $(DN)/N$ is topologically isomorphic with the discrete group $D/(D \cap N)$; and hence, N is open in DN. Since the closedness of DN in G implies that D is an L-subgroup of DN (Lemma 1.8 in [8]), we have that $D \cap N$ is an L-subgroup of N (Lemma 1.6 in [8]). Consequently, $D \cap N$ is uniform in N by Theorem 2.6 in [8]. And the theorem is proved.

Next, suppose that DN is not closed in G. We shall prove that this is impossible.

Let $F = \overline{DN}$, and $D' = D \cap F_0$. As we saw before, D' is an L-subgroup of F_0. Because F_0 is open in F, and $N \subseteq F_0$,

$$\overline{D'N} = \overline{(D \cap F_0)N} = \overline{(DN)} \cap F_0 = \overline{F_0} = F_0.$$

Suppose that $D'N$ is closed in F_0. Then, the last result implies that $D'N = F_0$; and hence, $F_0 = (D \cap F_0)N = (DN) \cap F_0$; i.e., F_0 is contained in DN. By the openness of F_0 in F again, DN is closed in F; and hence, DN is closed in G, a contradiction. This shows that $D'N$ is not closed in F_0. Next, we observe that

F_0 is not nilpotent. For if it is, $N \subseteq F_0$ and $[G,G] \subseteq N$ imply that $[G,G] \subseteq F_0$; i.e., F_0 is a normal nilpotent analytic subgroup of G containing N. Thus, $F_0 = N$. Again, the openness of F_0 in F shows that $DN = DF_0$ is open in F; and hence, DN is closed in F (and therefore in G). This contradiction proves that F_0 is not nilpotent.

Replacing F_0 and D' by G and D, respectively, the last paragraph tells us that we may assume in addition that $\overline{DN} = G$, DN is not closed in G. and G is not nilpotent. By Lemma 3.6 in [8], we may assume also that G lies in some general linear group so that N is unipotent.

Using the same notations in the beginning of Section 1, since $T_D(M_D) = M_D$, $T_D(A_h(M_D)) = A_h(M_D)$. Thus, by the semisimplicity of T_D, there is a subspace V of $L(M)$ complementary to $L(A_h(M_D))$ that is invariant under the differential of each element of T_D. (If K is any analytic group, then $L(K)$ denotes the Lie algebra of K). We claim that there is a non-zero element v_1 in V such that $\exp v_1$ lies in N. Clearly, it suffices to show that $A_h(M_D) \cap N \neq N$. To see this, suppose that $A_h(M_D) \cap N = N$. Then, N is contained in $A_h(M_D)$. Since $G = \overline{DN}$, by (3), $\overline{NM_D} = M$. Consequently,

$$M = A_h(M_D). \tag{11}$$

By Theorem 1, D_s is nilpotent; and hence, $D_s = M_D \times T_D$ (a direct product). Thus, $A_h(M_D) \times T_D$ and also $A_h(M_D) \times T_G$ (by the proof of Theorem 1 T_D is dense in T_G). Thus, (11) implies that $G_s = M \times T_G$; and hence, G_s is nilpotent. In particular, G is nilpotent, a contradiction. So, $A_h(M_D) \cap N \neq N$; and hence, there is a non-zero element v_1 in V such that $\exp v_1$ lies in N.

On the other hand, the set $\{x \in L((A_h(D))_0)\colon \exp x \in D\}$ generates a discrete subgroup $D_{\mathbf{Z}}$ of $L((A_h(D))_0)$ (Proposition 2.5 in [8]). By Margulis' Lemma [8], there is a neighborhood U of 0 in $L(A_h(G))$ such that

$$\{t \in \mathbb{R}\colon (\operatorname{Ad}\exp tv_1)D_{\mathbf{Z}} \cap U = \{0\}\}$$

is unbounded. Consequently, there is a neighborhood W of the identity element e of G such that

$$A = \left\{t \in \mathbb{R}\colon \exp(tv_1)D\exp(tv_1)^{-1} \cap W = \{e\}\right\}$$

is unbounded.

If d is an element of D, writing $d = m_d t_d$ with m_d in M_D and t_d in T_D, then we define $a(d)\colon M \to M$ by $a(d)(m) = m_d t_d(m)$. Denote by $M/a(d)$ the orbit space relative to the above action a of D on M, and by m^* the orbit of m ($m \in M$). If g is an element of G, we define $r(Dg) = (p(g))^*$. Then, we have a homeomorphism $r\colon G/D \to M/a(D)$ (p. 240 in [1]).

On account of the facts that D is discrete, $\exp V$ is closed in M, and every element of M can be expressed uniquely as a product $m \exp v$ with m in

$A_h(M_D)$ and v in V, a standard sequential argument shows that $a(D)\exp V$ is closed in M, $(\exp V)^*$ is closed in $M/a(D)$, and the map s: $V \to (\exp V)^*$ sending v to $(\exp v)^*$ is a homeomorphism.

Since D is an L-subgroup of G, $\overline{\alpha(\exp Av_1)}$ is compact in G/D, where α: $G \to G/D$ is the canonical map. Thus, $r\overline{(\alpha(\exp Av_1))}$ is a compact subset of $(\exp V)^*$; and hence, $s^{-1}(r\overline{(\alpha(\exp Av_1))})$ is a compact subset of V. Clearly, this implies that A must be bounded, a contradiction. And hence, the theorem is proved.

3. Products of closed subgroups and radicals

In this section, we shall consider the product of a closed subgroup H of an analytic group G and the radical $R(G)$ of G. We shall use the Auslander-Wang-Zassenhaus Theorem [6, 8.24] which states that if R is a solvable closed analytic normal subgroup of the Lie group G and H is a closed subgroup of G such that H_0 is solvable, then $(\overline{HR})_0$ is also solvable. Our result is the following.

THEOREM 3. *Let H be a closed subgroup of the analytic group G. If $F = \overline{R(G)H}$, then $F = R(F_0)H$ and $F_0 = R(F_0)H_0$, where $R(F_0)$ is the radical of F_0.*

Proof. Let $B = H \cap F_0$. Since $R(G)$ is contained in F_0, $R(G)$ is contained in $R(F_0)$. Thus, $R(G)H \subseteq R(F_0)H \subseteq F$. It follows from the definition of F that

$$\overline{R(F_0)H} = F. \tag{12}$$

Together with the fact that F_0 is open in F and the equality $(R(F_0)H) \cap F_0 = R(F_0)B$, we may conclude that

$$\overline{R(F_0)B} = F_0. \tag{13}$$

Let f: $F_0 \to F_0/R(F_0)$ be the canonical map. Since $f(B_0)$ is an analytic normal subgroup of the semisimple analytic group $f(F_0)$, $f(B_0)$ is closed in $f(F_0)$. Suppose that $f(B_0) \neq f(F_0)$. Then, there is a non-trivial semisimple closed analytic normal subgroup E of $f(F_0)$ so that $f(B_0) \cap E$ is discrete and $f(B_0)E = f(F_0)$ (cf. [2, XI. 2.2]). From (13), $\overline{f(B)} = f(F_0)$. Together with the discreteness of $f(B_0) \cap E$, and the observation that

$$f(F_0)/(f(B_0) \cap E)$$

is a direct product of

$$f(B_0)/(f(B_0) \cap E) \quad \text{and} \quad E/(f(B_0) \cap E),$$

and $f(B)E = f(B_0)E = f(F_0)$, we have

$$\overline{f(B) \cap E} = E. \tag{14}$$

Let $C = B \cap f^{-1}(E)$. Then $f(C) = f(B) \cap E$; and hence, $\overline{f(C)} = E$ by (14). It follows that $f^{-1}(E) = \overline{R(F_0)C}$. Since $f^{-1}(E)$ is connected, we obtain that

$$f^{-1}(E) = \left(\overline{R(F_0)C}\right)_0 \tag{15}$$

On the other hand, by the definition of C, C_0 is contained in B_0; and hence, $f(B_0) \cap E$ contains $f(C_0)$. The discreteness of $f(B_0) \cap E$ therefore forces the connected group $f(C_0)$ to be trivial; i.e., C_0 is contained in $R(F_0)$. In particular, C_0 is solvable. Applying the Auslander-Wang-Zassenhaus theorem, $(\overline{R(F_0)C})_0$ is solvable. By (15), $f^{-1}(E)$ and hence E are solvable. Thus, E must be trivial since E is semisimple, a contradiction! Therefore, $f(B_0) = f(F_0)$; i.e.,

$$R(F_0)B_0 = F_0. \tag{16}$$

Together with the fact that F_0 is open in F and $R(F_0)B_0$ is contained in $R(F_0)H$, we may conclude that $R(F_0)H$ is an open subgroup of F; and hence, $F = R(F_0)H$ by (12).

By the definition of B, B_0 is contained in H_0; and hence, F_0 is contained in $R(F_0)H_0$ by (16). On the other hand, being a connected subgroup of F, $R(F_0)H_0$ is also contained in F_0. Consequently, $F_0 = R(F_0)H_0$. This completes the proof.

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Case Western Reserve University
Cleveland, Ohio