SOME QUESTIONS OF EDJVET AND PRIDE ABOUT INFINITE GROUPS

BY

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Dedicated to the memory of Bill Boone

1. The height of Pride

In his paper [9] Stephen Pride describes a pre-order \( \preceq \) on the class of groups. In effect, as modified slightly in [2] the definition is that \( H \preceq G \) if there exist:

\[
\begin{align*}
(\ast) \quad & \text{a subgroup } G_0 \text{ of finite index in } G \\
& \text{and a normal subgroup } G_1 \text{ of } G_0; \\
& \text{a subgroup } H_0 \text{ of finite index in } H \\
& \text{and a finite normal subgroup } H_1 \text{ of } H_0; \\
& \text{an isomorphism } G_0/G_1 \to H_0/H_1.
\end{align*}
\]

If \( H \preceq G \) and \( G \preceq H \) then we write \( G \sim H \), and we use \([G]\) to denote the equivalence class consisting of all such groups \( H \). The relation \( \preceq \) induces a partial order, also denoted \( \preceq \), on the collection of all equivalence classes, with the class \([\{1\}]\) of all finite groups as its unique least member. The ideal \( \text{Id}[G] \) is defined to be the partially ordered set consisting of all equivalence classes \([H] \preceq [G]\). A group \( G \) is said to be atomic if \( \text{Id}[G] \) consists of \([\{1\}]\) and \([G]\); it is said to be of height \( h \), and we write \( \text{ht}[G] = h \), if \( \text{Id}[G] \) is of height \( h \) as partially ordered set. In the papers [2], [9] a number of questions about these concepts are raised. These, and one or two others, are stated in \( \S 2 \) below. Answers are given in \( \S \S 3-8 \). In a final section (\( \S 9 \)) I prove some small results relating the pre-order \( \preceq \) and the property \( \text{max-N} \). The fact that a finitely generated atomic group satisfies \( \text{max-N} \) is typical of these.

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2. The questions

**Question A.** Does there exist a countable group that is SQ-universal but of finite height?

This is part of Problem 8 on page 333 of [9]. Hurley [5, pp. 207, 212] announces an affirmative answer, indeed, the existence of a countable atomic group that is SQ-universal, but his construction does not appear to have been published. I shall give such an example in §3 below.

**Question B.** Does there exist a finitely generated group that is SQ-universal but of finite height?

This is another part of Problem 8 of [9], and it occurs also as Problem 4 in [5]. In §4 I shall produce a finitely generated group of height 3 that is SQ-universal. On the other hand, finitely generated atomic groups satisfy max-N (see §9 below) and therefore cannot be SQ-universal. I do not know whether or not there exist finitely generated SQ-universal groups of height 2. Probably not—but I have no real evidence.

**Question C₁.** Do there exist finitely generated just infinite groups not satisfying max-SN?

A group is said to be just infinite if all its non-trivial normal subgroups are of finite index. The question is Problem 5 on page 323 of [9] and Problem 4' of [2]. It arises in connection with:

**Question C₂.** Is every finitely generated atomic group finite-by-$\mathfrak{D}_2$-by-finite?

Here $\mathfrak{D}_2$ is the class of groups in which every non-trivial subnormal subgroup has finite index. The question is put in the form of a conjecture on page 12 of [2]. In §5 I construct a group that supplies a positive answer to Question C₁ and a negative answer to Question C₂.

**Question D.** If $G$ has finite height, are all maximal chains in $\text{Id}[G]$ of the same length?

This is Problem 1 of [2]. A counterexample is given in §6 below.

**Question E.** Let $G$ be a countable group with normal subgroups $K_1, K_2$ such that $K_1 \cap K_2 = \{1\}$. Is it true that if $G/K_1$ and $G/K_2$ both have finite height then $G$ has finite height (bounded by a function of $\text{ht}(G/K_1)$ and $\text{ht}(G/K_2)$)?
The question was suggested by Theorem 4.5 of [9]. It has a negative answer that will be given in §7 below.

**Question F.** If \( G \) is a finitely generated group of finite height, must \( \text{Id}[G] \) be finite?

This is Problem 5 of [2]. A counterexample is produced in §8 below.

### 3. Answer A

**Example A.** A countable group \( A \) that is SQ-universal but atomic.

**Construction.** Let \( P \) be a countable perfect group that is SQ-universal. For definiteness let us take \( P \) to be the triangle group with presentation

\[
\langle a, b | a^2 = b^3 = (ab)^7 = 1 \rangle
\]

or, with an eye to future developments, the free product \( \text{Alt}(5) \ast \text{Alt}(5) \) (see [8] for a proof of SQ-universality of these groups). Let \( A := \text{wr}^\omega P \) as defined by P. Hall [3] (except that, as in [7], I use \( \text{wr} \) to denote the restricted wreath product and reserve \( \text{Wr} \) for unrestricted wreath products). We think of \( A \) as a direct limit as follows. Define \( A_0 := P \) and thereafter \( A_{i+1} := A_i \text{wr} P \), with \( A_i \) embedded in \( A_{i+1} \) as the first factor of the base group; then \( A = \bigcup A_i \). Let \( P_i \) be the top group in \( A_i \) (with \( P_0 := A_0 \)). Then

\[
A_i = \langle P_0, P_1, P_2, \ldots, P_i \rangle,
\]

and if

\[
B_i := \langle P_{i+1}, P_{i+2}, P_{i+3}, \ldots \rangle = \text{wr}^{(\omega - \{0, 1, \ldots, i\})} P \cong A,
\]

then \( A = A_i \text{wr} B_i \) where here the wreath product is a permutational one. Let \( K_i \) denote the base group in this wreath product. It is the normal closure of \( A_i \) in \( A \) and is isomorphic to a (restricted) direct power of \( A_i \).

It should be clear that \( A \) is countable. Also, \( A \) is SQ-universal because if \( X \) is any countable group then there is a normal subgroup \( Q \) of \( P \) such that \( X \) is embeddable into \( P/Q \), and so \( X \) is embeddable into \( P/Q \text{wr} B_0 \), which is a homomorphic image of \( P_0 \text{wr} B_0 \), that is, of \( A \). It only remains to show that \( A \) is atomic.

If \( x \in A_{i+1} - K_i \) then, since \( A_i \) is perfect and \( A_{i+1} = A_i \text{wr} P \), the normal closure of \( x \) in \( A_{i+1} \) contains the whole of the base group of this wreath product (see, for example, [7, Lemma 8.2]), and so the normal closure of \( x \) in \( A \) contains the whole of \( K_i \). Consequently, if \( N \) is a proper normal subgroup of
A then $K_i \leq N \leq K_{i+1}$ for some value of $i$. Now if $H \leq A$ then there exist subgroups $H_0, H_1$ of $H$ as in (⋆) such that $H_0/H_1$ is isomorphic to a quotient group of a subgroup of finite index in $A$. It follows, since $A$ has no proper subgroups of finite index, that either $H_0/H_1 = \{1\}$, in which case $[H] = [\{1\}]$, or $H_0/H_1 \cong A/N$ for some proper normal subgroup $N$ of $A$, in which case $H_0/H_1$ has a quotient group isomorphic to $A/K_{i+1}$ for some $i$ and, since $A/K_{i+1} \cong A$, we then have $[H] = [A]$. Thus $A$ is atomic.

4. Answer B

**Example B.** A finitely generated group that is SQ-universal and of height 3.

**Construction.** The first ingredient is the free product $P := \text{Alt}(5) \ast \text{Alt}(5)$ which we use to manufacture the group $A := \text{wr}^\omega P$ as in §3. The remaining ingredients are:

- a finitely generated infinite simple group $S$;
- an infinite subset $\Sigma$ of $S$ such that $|\Sigma \cap \Sigma x| \leq 1$ for all $x \in S - \{1\}$;
- an enumeration $\Sigma = \{\sigma_0, \sigma_1, \sigma_2, \ldots\}$ of $\Sigma$;
- an automorphism $\alpha$ of $S$ such that $\sigma_i \alpha = \sigma_{i+1}$ for all $i \geq 0$.

This is not too much to ask: if we embed the group with presentation

$$\langle s, t | t^{-1}st = s^2 \rangle$$

into a finitely generated simple group $S$ (using, for example, the methods of P. Hall [4]) then we can take $\Sigma$ to be $\{s, s^2, s^4, s^8, \ldots\}$ and $\alpha$ to be the inner automorphism consisting of conjugation by $t$.

Now let $W := A \text{Wr} S$, the unrestricted standard wreath product. Elements of the base group in $W$, the cartesian power $A^S$, will be written as sequences $(x_\sigma)_{\sigma \in S}$. Let $P_i$ be the subgroup of $A$ given that name in §3 and let $Q_i, R_i$ be its two free factors $\text{Alt}(5)$. Choose generators $a_i, b_i$ of $Q_i$ and $c_i, d_i$ of $R_i$ such that

$$a_i^2 = b_i^3 = (a_ib_i)^5 = 1 \quad \text{and} \quad c_i^2 = d_i^3 = (c_id_i)^5 = 1.$$

Let $u, v \in A^S$ be the elements $(u_\sigma)_{\sigma \in S}, (v_\sigma)_{\sigma \in S}$ defined by

$$u_\sigma := \begin{cases} 
1 & \text{if } \sigma \notin \Sigma \\
a_i & \text{if } \sigma = \sigma_{2i} \\
c_i & \text{if } \sigma = \sigma_{2i+1},
\end{cases}$$

$$v_\sigma := \begin{cases} 
1 & \text{if } \sigma \notin \Sigma \\
b_i & \text{if } \sigma = \sigma_{2i} \\
d_i & \text{if } \sigma = \sigma_{2i+1}.
\end{cases}$$
The group $B$ that we want is the subgroup $\langle u, v, S \rangle$ of $W$. Obviously $B$ is finitely generated. What has to be proved is that $B$ is sq-universal and of height 3.

Let $M := B \cap A^S$, so that $B/M \cong S$, and let $L := A^{(S)}$, the restricted direct power of $A$ consisting of all sequences of finite support in $A^S$. The crux of the matter is the fact that $L \leq B$ and $B/L \cong \text{Alt}(5) \wr S$ with base group $M/L$.

The idea of the proof that $L \leq B$ is exactly that of [6, pp. 469, 470]. First we observe that $M$ is a subcartesian power, that is, its projection to each factor in $A^S$ is surjective. Therefore if

$$A^* := \{(w_\sigma)_{\sigma \in S} | w_\sigma = 1 \text{ if } \sigma \neq 1\},$$

the "first coordinate subgroup" in $A^S$, then $M \cap A^* \not\subseteq A^*$. Consider the commutator

$$[s_1u_{s_1}^{-1}, s_2u_{s_2}^{-1}],$$

where $s_1, s_2 \in S$ and $s_1 \neq s_2$. It is the sequence $(w_\sigma)_{\sigma \in S}$ where $w_\sigma = [u_{s_1}, u_{s_2}]$. If $w_\sigma \neq 1$ then $u_{s_1} \neq 1$ and $u_{s_2} \neq 1$, and so $s_1 \in \Sigma$ and $s_2 \in \Sigma$, that is, $\sigma \in \Sigma s_1^{-1} \cap \Sigma s_2^{-1}$. But $\Sigma s_1^{-1} \cap \Sigma s_2^{-1} = (\Sigma \cap \Sigma s_2^{-1}s_1)s_1^{-1}$, and this is either empty or a singleton. Therefore $(w_\sigma)_{\sigma \in S}$ has at most one non-identity component and (by definition) is a member of one of the "coordinate subgroups" of $A^S$. If we take $s_1$ to be $s_{2i}$ and $s_2$ to be $s_{2i+1}$ we find that $[s_{2i}u_{s_{2i}}^{-1}, s_{2i+1}u_{s_{2i+1}}^{-1}]$ is the sequence $(w^{(i)}_\sigma)_{\sigma \in S}$ such that

$$w^{(i)}_\sigma = \begin{cases} [a_i, c_i] & \text{if } \sigma = 1 \\ 1 & \text{if } \sigma \neq 1, \end{cases}$$

and so $[s_{2i}u_{s_{2i}}^{-1}, s_{2i+1}u_{s_{2i+1}}^{-1}] \in A^* \cap M$. Obviously $A^* \cong A$, and we observed in §3 that any proper normal subgroup of $A$ is contained in the subgroup $K_r$ for some $r$. Since $w^{(r+1)}_\sigma \in K_r$ we must have $A^* \cap M = A^*$. Thus $A^* \leq M$ and, as $L$ is generated by the conjugates $s^{-1}A^*$ for $s \in S$, also $L \leq M$. The fact that $B/L \cong \text{Alt}(5) \wr S$ now follows easily. For, since $u^2 = v^3 = (w)^5 = 1$ we have $\langle u, v \rangle \cong \text{Alt}(5)$; moreover, if $s \in S - \{1\}$ then $s^{-1}us$ and $s^{-1}vs$ commute with both $u$ and $v$ modulo $L$; and of course the conjugates $s^{-1}\langle u, v \rangle s$ for $s \in S$ are independent modulo $L$.

In §3 we defined normal subgroups $K_i$ of $A$ and we saw that every proper normal subgroup of $A$ lies between $K_i$ and $K_{i+1}$ for some $i$. It follows easily that if $N$ is a proper normal subgroup of $B$ then $N = M$ or $N = L$ or $K_i^{(S)} \leq N \leq K_{i+1}^{(S)}$ for some $i$. We need to prove that $B/K_i^{(S)} \cong B$. Now $B/K_i^{(S)} \cong \langle u', v', S \rangle$, where $u'$ and $v'$ are obtained from $u, v$ respectively by replacing the $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{2i+2}, \sigma_{2i+3}$ coordinates by 1. And the map

$$u \mapsto u', \quad v \mapsto v', \quad s \mapsto sa^{2i+4} \quad \text{if } s \in S$$
gives an isomorphism $B \to \langle u', v', S \rangle$. Thus if $K_{i+1} \leq N \leq K_i$ then $B/N$ has a quotient group isomorphic to $B$. It follows easily that $\text{Id}[B]$ consists of $\{1\}, [S], [\text{Alt}(5) \wr S]$ and $[B]$, and hence that $h[B] = 3$.

Now let $X$ be any countable group. Since $A$ is sq-universal there is a normal subgroup $N$ of $A$ such that $X$ is embeddable into $A/N$. The homomorphism of $A$ onto $A/N$ induces a homomorphism of $W$ onto $(A/N) \wr S$ that maps $L$ to the direct power $(A/N)^{(3)}$ and so $X$ is embeddable into the image of $B$. Thus $B$ is sq-universal, as required.

5. Answers $C_1$ and $C_2$

Before describing the relevant construction here we prove a result that sets the scene. Throughout this section $S$ will be a non-abelian finite simple group and $\Sigma$ a faithful transitive $S$-space. In due course we take $\Sigma$ to be $\{1, 2, 3, 4, 5, 6\}$ and $S$ to be $\text{Alt}(\Sigma)$.

**Theorem 5.1.** Let $G$ be a group such that:

(i) $G$ is perfect;

(ii) $G$ is residually finite;

(iii) $G \cong G \wr_S S$.

Then also:

(iv) every non-trivial normal subgroup has finite index in $G$ (that is, $G$ is just infinite);

(v) every subnormal subgroup is isomorphic to a finite direct power of $G$;

(vi) nevertheless, $G$ does not satisfy max-SN;

(vii) $G$ is atomic.

**Proof.** We have $G = G_1 \wr_\Sigma S$, where $G_1 \cong G$. Consequently $G = G_n \wr_{\Delta_n} W_n$, where $G_n \cong G$,

$$W_n : = S \wr_\Sigma S \wr_\Sigma \cdots \wr_\Sigma S \ (n \text{ factors})$$

and

$$\Delta_n : = \Sigma \times \Sigma \times \cdots \times \Sigma \ (n \text{ factors}).$$

Let $K_n$ be the base group in this wreath product, so that $K_n \cong G^{\Delta_n}$, and $G/K_n \cong W_n$. Put $K_0 : = G$.

**Lemma 5.2.** If $N \trianglelefteq G$ and $N \neq \{1\}$ then $N = K_n$ for some $n$.

**Proof.** First we prove that $K_0, K_1, K_2, \ldots$ are the only normal subgroups of finite index. Let $X$ be a finite group and $f : G \to X$ a homomorphism. If $m$ is large enough there must be two distinct coordinate subgroups $G_{mi}, G_{mj}$ (direct factors isomorphic to $G$) of $K_m$ that have the same image under $f$. 
Since $G_{m_1}, G_{m_2}$ centralise each other it follows that their common image is abelian and since $G$ is perfect that image must be $\{1\}$; then, since $K_m$ is the normal closure of $G_{m_1}$, also $K_m \leq \text{Ker}(f)$. Therefore $\text{Im}(f)$ is a homomorphic image of $G/K_m$ that is, of $W_m$. In $W_m$, however, the base group $S^{2^{m-1}}$ is a minimal (non-trivial) normal subgroup (because it is a direct power of the non-abelian simple group $S$ and its simple direct factors are permuted transitively under conjugation in $W_m$) and its centraliser is trivial. Therefore it is the unique minimal normal subgroup. That is, $K_{m-1}/K_m$ is the unique minimal normal subgroup of $W_m$, and it follows by induction that $K_0, K_1, \ldots, K_{m-1}, K_m$ are the only normal subgroups of $G$ that contain $K_m$. Thus $\text{Ker}(f) = K_n$ for some $n$.

Now let $N$ be any non-trivial normal subgroup of $G$. Since $G$ is residually finite we must have $\bigcap K_m = \{1\}$ and so there exists $n$ such that $N \leq K_n$ and $N \not\leq K_{n+1}$. If $x \in N - K_{n+1}$ then, as one sees by a very small modification of the argument used to prove Lemma 8.2 of [7], the normal closure of $x$ in $G$ contains $K_{n+1}$. Thus $K_{n+1} < N \leq K_n$ and, as we have already seen, it follows that $N = K_n$, as required.

This deals with assertion (iv) of Theorem 5.1. Now every non-trivial normal subgroup of $G$ is isomorphic to a finite direct power of $G$ and so to prove (v) we need to show that a normal subgroup of a finite direct power of $G$ is itself isomorphic to a finite direct power of $G$. But if $X_1, X_2, \ldots, X_k$ are groups all of whose normal subgroups are perfect, and if $N \leq X_1 \times X_2 \times \cdots \times X_k$ then, as is very easy to prove, $N = Y_1 \times Y_2 \times \cdots \times Y_k$ where $Y_i \leq X_i$ for $1 \leq i \leq k$.

To prove (vi) we proceed as follows. Suppose, as inductive hypothesis, that $G$ has a subnormal subgroup $X_n \times Y_n$ with $Y_n \cong G$. This is certainly true for $n = 0$ with $X_0 := \{1\}, Y_0 := G$. Now $Y_n \cong G \wr \Sigma S$ and we can take $X_{n+1} := X_n \times Z_1, Y_{n+1} := Z_2$, where $Z_1, Z_2$ are two of the direct factors in the base group of the wreath product. Then $X_{n+1} \times Y_{n+1}$ is subnormal in $G$, so induction supplies a properly increasing sequence $X_0 < X_1 < X_2 < \cdots$ of subnormal subgroups of $G$.

Suppose now that $H \leq G$. There exist subgroups $H_0, H_1$ as in (*), such that $H_0/H_1$ is a homomorphic image of a subgroup $G^*$ of finite index in $G$. Moreover, we can take $G^*$ to be normal in $G$. Then $G^*$ is a finite direct power of $G$ and, by what has already been shown, it follows that $H_0/H_1$ is a direct product of finitely many groups, each of which is finite or isomorphic to $G$. Therefore either $H$ is finite or $H \leq H$, and so $\text{Id}[G]$ consists of $\{1\}$ and $[G]$, that is, $G$ is atomic. This completes the proof of Theorem 5.1.

Assertions (iv) and (vi) applied to the following example give a positive answer to Question $C_1$, and (vii) gives a negative answer to Question $C_2$.

Example C. A finitely generated group $C$ that is perfect, residually finite and isomorphic to $C \wr \Sigma S$. 

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Construction. We take $\Sigma := \{1, 2, 3, 4, 5, 6\}$ and $S := \text{Alt}(\Sigma)$. Define $\Delta_n := \Sigma^n$ (n-fold cartesian power) and $W_n := \text{wr}^n S$ with its natural action as a permutation group on $\Delta_n$. Embed $W_{n-1}$ into $W_n$ as the top group in the representation $W_n = S \text{wr}_{\Delta_{n-1}} W_{n-1}$, and take $S_n$ to be one of the direct factors (coordinate subgroups) of the base group. Now define $W$ to be the direct limit $\bigcup W_n$—so that $W$ is, in fact, P. Hall’s wreath power $\text{wr}^{-N} S$. If $V_n := \langle S_{n+1}, S_{n+2}, \ldots \rangle$ then $V_n \cong W$ and $W = V_n \text{wr}_{\Delta_n} W_n$. And if $L_n$ is the normal closure of $V_n$ in $W$, that is, the base group in this wreath product, then $L_n \cong V_n^{\Delta_n} \cong W$. It is not hard to see directly that $W, L_1, L_2, L_3, \ldots$ are the only non-trivial normal subgroups of $W$—although this also follows from Lemma 5.2.

There is a natural surjective homomorphism $W_n \to W_{n-1}$ for each $n$, and we define $\overline{W}$ to be the inverse limit $\varprojlim W_n$. Elements of $\overline{W}$ can be expressed uniquely in the form $t_n t_{n-1} \cdots t_2 t_1$, where $t_i$ is in the base group of $W_i$ (we take the ‘base group’ of $W_1$ to be $W_1$ itself); then elements of $\overline{W}$ may be uniquely described by left-infinite sequences $\cdots t_n t_{n-1} \cdots t_2 t_1$, where $t_i$ is in the base group of $W_i$. Each factor $t_i$ may in turn be written as a product $\prod_{\delta \in \Delta_{i-1}} s_i(\delta)$, where $s_i(\delta) \in S$, this expression being unique up to the order of its factors (we take $\Delta_0$ to be a singleton set so that $t_i$ is simply a member of $S$). The rule for multiplication in $\overline{W}$ is determined by that in the finite wreath products. Since it is quite complicated, and since we shall need only very special cases, I do not write it down explicitly. The group $W$ may be seen as that subgroup of $\overline{W}$ that consists of sequences in which $t_i = 1$ for all except finitely many values of $i$. In fact, $\overline{W}$ is the completion of $W$ with respect to the topology that has the groups $L_n$ as a base for the neighbourhoods of 1. If $\overline{V}_1$ is the closure of $V_1$ in $\overline{W}$ then $\overline{W} = \overline{V}_1 \text{wr}_S S$ and $\overline{V}_1 \cong \overline{W}$. We can describe $\overline{V}_1$ explicitly as the set of all sequences

$$\cdots t_n t_{n-1} \cdots t_2 t_1$$

in which $t_1 = 1$ and for all $i > 1$, $s_i(\sigma_{i-1}, \ldots, \sigma_2, \sigma_1) = 1$ if $\sigma_1 \neq 1$. And we can define an isomorphism $\overline{W} \to \overline{V}_1$ explicitly:

$$\cdots t_n t_{n-1} \cdots t_2 t_1 \mapsto \cdots v_n v_{n-1} \cdots v_2 v_1$$

where

$$v_1 = 1, \quad v_i = \prod_{\Delta_{i-1}} u_i(\sigma_{i-1}, \ldots, \sigma_2, \sigma_1),$$

and

$$u_i(\sigma_{i-1}, \ldots, \sigma_2, \sigma_1) = \begin{cases} 1 & \text{if } \sigma_1 \neq 1 \\ s_{i-1}(\sigma_{i-1}, \ldots, \sigma_2) & \text{if } \sigma_1 = 1. \end{cases}$$
Similarly, if $\overline{V}_n$ is the closure of $V_n$ then $\overline{V}_n \cong \overline{W}$ and $\overline{W} = \overline{V}_n \wr \Delta_n W_n$. If $\overline{L}_n$ is the closure of $L_n$ then $\overline{L}_n$ is the base group in this wreath product and $\overline{L}_n$ consists of those sequences $\cdots t_{i-1} t_i \cdots t_2 t_1$ such that $t_i = 1$ if $i \leq n$. Since $\overline{L}_n$ has finite index in $\overline{W}$ and $\bigcap_n \overline{L}_n = \{1\}$, $\overline{W}$ is residually finite.

Calculation in $\overline{W}$ may be simplified if we represent it as a permutation group. There is a natural surjective map $\Delta_n \to \Delta_{n-1}$ that is compatible with the actions of $W_n$ on $\Delta_n$ and $W_{n-1}$ on $\Delta_{n-1}$ and with our surjective homomorphism $W_n \to W_{n-1}$. It follows that there is a natural action of $\lim W_n$ on $\lim \Delta_n$; that is, if we define $\overline{\Delta} := \Sigma^{-N} = \lim \Delta_n$, there is a natural action of $\overline{W}$ on $\overline{\Delta}$. Elements of $\overline{\Delta}$ may be thought of as left-infinite sequences

$$(\ldots, \sigma_n, \sigma_{n-1}, \ldots, \sigma_2, \sigma_1),$$

where $\sigma_i \in \Sigma$ for all $i$. The action of $\overline{W}$ on $\overline{\Delta}$ is the following:

$$(\ldots, \sigma_n, \sigma_{n-1}, \ldots, \sigma_2, \sigma_1) \cdots t_{n-1} t_n \cdots t_2 t_1 = (\ldots, \rho_n, \rho_{n-1}, \ldots, \rho_2, \rho_1),$$

where, if $t_i = \prod_{\delta \in \Delta_{n-1}} s_i(\delta)$ as before, then

$$\rho_i = \sigma_i s_i(\sigma_{i-1}, \ldots, \sigma_2, \sigma_1).$$

The set $\overline{\Delta}_1$ of all sequences $(\ldots, \sigma_n, \sigma_{n-1}, \ldots, \sigma_2, 1)$ is a block of imprimitivity for $\overline{W}$ in $\overline{\Delta}$. Its stabiliser is $\overline{V}_1$. The map $\overline{\Delta} \to \overline{\Delta}_1$ given by

$$(\ldots, \sigma_n, \sigma_{n-1}, \ldots, \sigma_2, \sigma_1) \mapsto (\ldots, \sigma_{n-1}, \sigma_{n-2}, \ldots, \sigma_1, 1)$$

induces the isomorphism $\overline{W} \to \overline{V}_1$ described above; and the natural bijection $\overline{\Delta} \to \overline{\Delta}_1 \times \Sigma$ induces our isomorphism $\overline{W} \to \overline{V}_1 \wr \Sigma S$.

We are now ready to define the group $C$. For each permutation $t \in \text{Alt}(\Sigma)$ and each element $\tau \in \Sigma$ define

$$w(t, \tau) := \ldots t_{\tau} t_{\tau} t_{\tau} \cdots t_{\tau} \in \overline{W}.$$  

By this I mean that the $i$-th component of $w(t, \tau)$ is the element $t$ in that coordinate subgroup of the base group of $W_i$ that is indexed by $(\tau, \ldots, \tau, \tau)$ in $\Delta_{i-1}$: that is,

$$t_{\tau} \cdots t_{\tau} = \prod_{\Delta_{i-1}} s_i(\sigma_{i-1}, \ldots, \sigma_2, \sigma_1),$$

where

$$s_i(\sigma_{i-1}, \ldots, \sigma_2, \sigma_1) :=
\begin{cases}
1 & \text{if } (\sigma_{i-1}, \ldots, \sigma_2, \sigma_1) \neq (\tau, \ldots, \tau, \tau) \\
t & \text{if } (\sigma_{i-1}, \ldots, \sigma_2, \sigma_1) = (\tau, \ldots, \tau, \tau).
\end{cases}$$

Now define

\[ C := \langle w(t, \tau) \mid t \in \text{Alt}(\Sigma), \tau \in \text{fix}(t) \rangle. \]

Certainly \( C \) is finitely generated: the given generating set has 360 members but very much smaller ones will suffice. Consider for the moment a fixed element \( \sigma \) of \( \Sigma \) and all generators \( w(t, \sigma) \) with \( \sigma \in \text{fix}(t) \). It is easy to see that

\[ w(t_1, \sigma)w(t_2, \sigma) = w(t_1t_2, \sigma) \]

and so these \( w(t, \sigma) \) form a subgroup of \( \overline{W} \) that is isomorphic to \( \text{Alt}(5) \). Thus \( C \) is generated by six subgroups isomorphic to \( \text{Alt}(5) \) and therefore \( C \) is perfect. Since \( \overline{W} \) is residually finite, also \( C \) is residually finite, and all we still have to prove is that \( C \cong C \wr X S \).

Define \( w^*(t, \tau) := \cdots t \tau t \tau t \tau \), so that \( w(t, \tau) = w^*(t, \tau)^t \). If \( s, t \) are permutations in \( \text{Alt}(\Sigma) \) that fix both 5 and 6 then, as is easy to see, \( w(s, 5)^* \) and \( w(t, 6)^* \) commute with each other and with both \( s \) and \( t \). Computing commutators we therefore have that

\[ [w(s, 5), w(t, 6)] = [s, t] \in W_1 \leq \overline{W}. \]

If we take \( s, t \) to be the 3-cycles \((123), (134)\) respectively then we find that \((12)(34) \in C \). Similarly all other double transpositions lie in \( C \) and so \( W_1 \leq C \). Consequently \( C = (C \cap I_1), W_1 \). Now let \( \tau \) be any element of \( \Sigma \) and take any permutation in \( \text{Alt}(\Sigma) \) fixing \( \tau \). Choose \( s \in \text{Alt}(\Sigma) \) that maps \( \tau \) to 1. Then

\[ s^{-1}w^*(t, \tau)s = \cdots t \tau t \tau \tau \]

and this is the element corresponding to \( w(t, \tau) \) in our isomorphism of \( \overline{W} \) to \( \overline{V}_1 \). Since \( C \) is generated by the members of \( W_1 \) together with the elements \( w^*(t, \tau) \) (with \( \tau \in \text{fix}(t) \)) it is generated by \( W_1 \) together with these conjugates \( s^{-1}w^*(t, \tau)s \). Clearly the latter generate a copy \( C_1 \) of \( C \) inside \( \overline{V}_1 \). Therefore \( C = C_1 \wr X S \) with \( C_1 \cong C \) : thus \( C \) is a finitely generated group satisfying conditions (i), (ii), (iii) of Theorem 5.1, as required.

**Comment 5.3.** Let \( \Delta \) be the \( C \)-orbit in \( \overline{\Delta} \) that contains the sequence \((\ldots, 1, 1, \ldots, 1, 1)\) and let \( \Delta_1 \times \{1\} \) be the \( C_1 \)-orbit of this sequence. If \( t \in W_1 \) maps 1 to \( \tau \) then

\[ (\ldots, 1, 1, \ldots, 1, 1)t = (\ldots, 1, 1, \ldots, 1, \tau) \]

and the \( t^{-1}C_1t \)-orbit of this sequence is \( \Delta_1 \times \{\tau\} \). Consequently the obvious bijection \( \Delta \rightarrow \Delta_1 \times \Sigma \) induces our isomorphism

\[ C \rightarrow C_1 \wr X \text{Alt}(\Sigma), \]
and the obvious bijection $\Delta \to \Delta_1$ induces our isomorphism $C \to C_1$. We shall need the permutation representation of $C$ on $\Delta$ as an ingredient in our next construction.

**Comment 5.4.** The construction of $C$ can be varied in many ways. Here is one. Let $S_1, S_2, S_3, \ldots$ be non-abelian finite simple groups acting faithfully and transitively on sets $\Sigma_1, \Sigma_2, \Sigma_3, \ldots$. Suppose that for some integer $d$ every group $S_i$ can be generated by $d$ subgroups, each of which is isomorphic to $\text{Alt}(5)$ and each of which fixes at least two members of $\Sigma_i$ (if $(S_i, \Sigma_i)$ is $\text{Alt}(n_i)$ in its natural action then this can be achieved with $d = 3$ provided that $n_i \geq 7$ for all $i$). Define $\Delta_1 := \Sigma_1$, $W_1 := S_1$, and thereafter

$$\Delta_n := \Sigma_n \times \Delta_{n-1}, \quad W_n := S_n \text{wr}_{\Delta_{n-1}} W_{n-1}.$$ 

As before we can find a finitely generated subgroup $C$ of the inverse limit $\bar{W} := \lim W_n$ which has the property that all non-trivial normal subgroups have finite index in $C$. But if the sequence $(S_1, \Sigma_1), (S_2, \Sigma_2), (S_3, \Sigma_3), \ldots$ is not periodic then the proper subnormal subgroups are direct products of groups of the same kind of structure as $C$, but none of which is isomorphic to $C$. Under these circumstances $C$ is a finitely generated just infinite group of infinite height.

### 6. Answer D

**Example D.** A group $D$ of height 4 such that $\text{Id}[D]$ has a maximal chain of length 3.

**Construction.** We begin with the group $C$ of §5 and with two faithful transitive $C$-spaces. The first is the $C$-space $\Delta$ described in Comment 5.3, the second $\Gamma$ is the coset space $(C : W)$. A subgroup of finite index in $C$ contains the normal subgroup $K_n$, the kernel of the homomorphism of $C$ onto $W_n$, for some $n$, and $W_nK_n = C$. Therefore every subgroup of finite index in $C$ is transitive on $\Gamma$.

Let $S$ be a non-abelian simple group and define $X := S \text{ wr}_\Gamma C$, $Y := S \text{ wr}_\delta C$, $Z := X \times C$ and $D := X \times Y$. I shall prove that the partially ordered set $\text{Id}[D]$ is

$$\begin{align*}
[D] & \quad \downarrow \quad [Z] \\
[1] \quad \downarrow \quad [X] & \quad \downarrow \quad [Y] \\
[C] & \quad \downarrow \quad \left( \{1\} \right)
\end{align*}$$
A subgroup of finite index in $D$ contains a subgroup $X_0 \times Y_0$ where $X_0$ is normal and of finite index in $X$, and $Y_0$ is normal and of finite index in $Y$. Since the base groups $S^{(1)}$, $S^{(\Delta)}$ (restricted direct powers of $S$) are the unique minimal normal subgroups of $X$ and $Y$ respectively, we have $S^{(1)} \leq X_0$, $S^{(\Delta)} \leq Y_0$ and

$$X_0 = S \text{wr}_\Gamma K_m, \quad Y_0 = S \text{wr}_\Delta K_n$$

for some $m, n$. Since $K_m$ is transitive on $\Gamma$, $S^{(1)}$ is the unique minimal normal subgroup of $X_0$. Since $K_n$ has $6^n$ (as it happens) orbits on $\Delta$, on each of which it acts like $C$ on $\Delta$, $Y_0 \cong Y^{6^n}$. Every normal subgroup of $X_0$ and of $Y_0$ is perfect, so a normal subgroup of $X_0 \times Y_0$ is of the form $X_1 \times Y_1$ where $X_1 \leq X_0$ and $Y_1 \leq Y_0$. If $X_1 \neq \{1\}$ then $X_0/X_1$ is isomorphic to a quotient group of $X_0/S^{(1)}$, that is of $K_m$, and therefore $X_0/X_1 \cong C^k \times Q$ for some $k$ and some finite group $Q$; similarly, $Y_0/Y_1 \cong Y^k \times C^l \times R$ for some integers $k, l$ and some finite group $R$. Clearly therefore $Y \leq Y^k \times C^l \leq Y$ and so $Y \sim Y \times C \sim Y^k$ for any positive integer $k$. It follows immediately that Id[$D$] consists of $\{(1), [C], [X], [Y], [X \times C], [D]\}$, and that it is ordered as shown in the diagram.

7. Answer E

Example E. A group $E$ of infinite height that is a subdirect product of two atomic groups.

Construction. Let $S$ be a non-abelian finite simple group, let $R$ be the countable restricted direct power $S^{(\aleph_0)}$, let $R_1 := R$ and $R_2 := R \text{ wr } R$. We define $Q$ to be the wreath power $\text{wr}_\omega R$ and $\Omega$ to be the set on which it naturally acts, that is, since $R$ is to be thought of as acting on itself regularly (by right multiplication), $\Omega$ is the countable restricted direct power $R^{(\omega)}$ (see [3]). Define $E := (R_1 \times R_2) \text{ wr}_\Omega Q$.

If $K_1 := R_1^{(\Delta)}$ and $K_2 := R_2^{(\Delta)}$, so that $K_1 \times K_2$ is the base group in the wreath product, then

$$E/K_1 \cong R_2 \text{ wr}_\Omega Q \cong Q \quad \text{and} \quad E/K_2 \cong R_1 \text{ wr}_\Omega Q \cong Q,$$

and so $E$ is a subdirect product in $Q \times Q$. But $Q$ is atomic (compare §3). So $E$ is a subdirect product of two atomic groups.

On the other hand $E$ has quotient groups of the form

$$E_{m, n} \cong (S^m \times (S^n \text{ wr } R)) \text{ wr}_\Omega Q.$$

If $m = 0$ or $n = 0$ then $E_{m, n} \sim Q$, but if $m \neq 0$ and $n \neq 0$ then it is easy to see that $E_{m, n} \leq E_{m', n'}$ if and only if $m \leq m'$ and $n \leq n'$. Thus $E$ has infinite height.
8. Answer F

Recall that a soluble minimax group is a group $G$ with a subnormal series $\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$ in which every factor $G_i/G_{i-1}$ either is cyclic or is a quasi-cyclic group $\mathbb{Z}_{p^\infty}$ for some prime number $p$. The number of infinite factors in such a series is an invariant $m(G)$, the minimax length. As preparation for our final example we require:

**Lemma 8.1.** If $G$ is a soluble minimax group then

(i) $\text{ht}[G] \leq m(G)$,

and

(ii) $[G]$ consists of only countably many isomorphism classes of groups.

**Proof.** We use induction on $m(G)$ to prove (i). Let $H$ be a group that is strictly smaller than $G$ in Pride's sense. Then exist $G_0, G_1, H_0$ and $H_1$ as in $\ast$ and an isomorphism $G_0/G_1 \to H_0/H_1$, and $G_1$ must be infinite. Therefore $m(G_0/G_1) < m(G)$ and, by inductive hypothesis, $\text{ht}[G_0/G_1] \leq m(G_0/G_1)$. Consequently

$$\text{ht}[H] = \text{ht}[G_0/G_1] \leq m(G) - 1,$$

and so

$$\text{ht}[G] = 1 + \sup\{\text{ht}[H]|H < G\} \leq m(G),$$

as required.

To prove (ii) we first show that if $G_1 \not\leq G_0 \leq G$, $|G: G_0|$ is finite and $G_1$ is infinite, then $G_0/G_1 < G$. Let $Y := G_0/G_1$ and suppose, if possible, that $Y \sim G$. Then there exist subgroups $Y_0, Y_1$ of $Y$ with $|Y: Y_0|$ finite and $Y_1 \not\leq Y$, and there exist subgroups $X_0, X_1$ of $G$ with $|G: X_0|$ finite, $X_1 \not\leq X_0$ and $X_1$ finite, such that $Y_0/Y_1 \cong X_0/X_1$. But

$$m(X_0/X_1) = m(G) = m(G_0) = m(Y) + m(G_1) > m(Y) \geq m(Y_0/Y_1).$$

This contradiction shows that $G_0/G_1 < G$.

Now if $H \sim G$ then there exist $G_0, G_1 \leq G$ and $H_0, H_1 \leq H$ as in $\ast$. We may suppose moreover that $H_0 \not\leq H$. Since

$$G_0/G_1 \sim H_0/H_1 \sim H \sim G$$

we must have that $G_1$ is finite. Therefore there are only countably many possibilities for the group $G_0/G_1$, that is, for $H_0/H_1$ up to isomorphism. Using the Lyndon-Hochschild-Serre spectral sequence one may show that if $X$ is a soluble minimax group and $Y$ is a finite $\mathbb{Z}X$-module then all cohomology groups $H^n(X, Y)$ are finite (see [10]). From the finiteness of the second
cohomology groups it follows easily that there are only countably many extensions of a finite group by a given soluble minimax group: thus there are only countably many possibilities for $H_0$. And it is easy to see that there are only countably many extensions of a given countable group by a finite group. Thus there are (up to isomorphism) only countably many possibilities for $H$.

**Example F.** A finitely generated group $F$ such that $ht[F] = 9$ and $Id[F]$ has $2^{\aleph_0}$ members.

**Construction.** Let $N$ be the group that is generated by elements $u_n, v_n, w_n, y_n, z_n (n \geq 0)$ subject to the relations (for all relevant $m, n$):

- $y_n, z_n$ are central;
- $u_{n+1}^2 = u_n; w_{n+1}^2 = w_n; y_{n+1}^2 = y_n; z_{n+1}^2 = z_n; \\
y_0 = z_0 = 1; [v_m, w_n] = 1; [u_m, v_n] = y_{m+n}; [u_m, w_n] = z_{m+n}.$

This group is nilpotent of class 2, its centre $Z(N)$ is isomorphic to $Z_{2^\infty} \times Z_{2^\infty}$ generated by the elements $y_n, z_n$, and $N/Z(N)$ is a direct product of three copies of $2^{-\infty}Z$. There is an automorphism that fixes all $y_n$ and $z_n$, and maps $u_n$ to $u_n^2$, $v_n$ to $v_{n+1}$, $w_n$ to $w_{n+1}$ for all $n$. We take $F$ to be the semi-direct product of $N$ with an infinite cyclic group inducing this automorphism: thus

$$F := \langle N, x | xu_n x^{-1} = u_{n+1}, x^{-1}v_n x = v_{n+1}, x^{-1}w_n x = w_{n+1} \rangle.$$

Clearly $F$ is generated by $\{x, u_1, v_1, w_1\}$, so $F$ is a finitely generated group. Also, $F$ is a soluble minimax group built from four infinite cyclic groups and five copies of $Z_{2^\infty}$, so $m(F) = 9$ and therefore, by Lemma 8.1(i), $ht[F] \leq 9$. In fact it is quite easy to see that $ht[F] = 9$. Now

$$Z(F) = \langle y_n, z_n (n \geq 0) \rangle \cong Z_{2^\infty} \times Z_{2^\infty}.$$

Thus $Z(F)$ has $2^{\aleph_0}$ subgroups (see [1]), that is, $F$ has $2^{\aleph_0}$ normal subgroups. Since $F$ is finitely generated there are only countably many homomorphisms of $F$ to a given countable group and so $F$ must have $2^{\aleph_0}$ non-isomorphic quotient groups. From Lemma 8.1(ii) it follows that these must fall into $2^{\aleph_0}$ equivalence classes, and so $Id[F]$ has $2^{\aleph_0}$ members, as claimed.

### 9. Finitely generated atomic groups

The constructions that I have described in this paper mostly seem to have slightly negative consequences for Pride's theory. Therefore it is a pleasure to report some small positive results.
**Lemma 9.1.** If \( G \) satisfies max-N and \( H \leq G \) then \( H \) satisfies max-N.

**Proof.** Given that \( H \leq G \) there exist subgroups \( G_0, G_1 \) of \( G \) and \( H_0, H_1 \) of \( H \) as in (\( \ast \)). By a theorem of John S. Wilson [11], \( G_0 \) satisfies max-N. Then \( G_0/G_1 \) and therefore also \( H_0/H_1 \) satisfies max-N. Since \( H_1 \) is finite \( H_0 \) satisfies max-N and now by Wilson's theorem again \( H \) satisfies max-N.

**Theorem 9.2.** A finitely generated atomic group satisfies max-N.

**Proof.** Let \( G \) be a finitely generated atomic group. Since \( G \) is finitely generated and infinite it has a just-infinite quotient group \( H \). Then \( H \leq G \) and since \( G \) is atomic \( H \sim G \), whence \( G \leq H \). It follows from Lemma 9.1 with the roles of \( G \) and \( H \) reversed that \( G \) satisfies max-N, as required.

There is a slightly more general version of this theorem.

**Theorem 9.3.** Let \( G \) be a finitely generated group of height \( n \). If there are \( n \) inequivalent atomic groups \( H_1, \ldots, H_n \) such that \( H_i \leq G \) for all \( i \) then \( G \) satisfies max-N.

**Proof.** By Theorem 2 of [2], \( H_1 \times \cdots \times H_n \leq G \), by Theorem 1(ii) of [2], \( \text{ht}[H_1 \times \cdots \times H_n] = n \), and so \( G \sim H_1 \times \cdots \times H_n \). It follows that this direct product is finitely generated, so each group \( H_i \) is finitely generated and, by Theorem 9.2, satisfies max-N. Then \( H_1 \times \cdots \times H_n \) satisfies max-N and so \( G \) satisfies max-N.

**References**


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