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A REMARK ON THE SUBGROUPS OF FINITELY GENERATED GROUPS WITH ONE DEFINING RELATION

BY

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Dedicated to the memory of W.W. Boone

1. If G is a finitely generated free group, then G has only countably many non-isomorphic subgroups. Our objective here is to point out that even the simplest one-relator groups can contain continuously many non-isomorphic subgroups. This will follow readily from two simple observations.

LEMMA 1. *Suppose that the group G is the free product of its subgroups G_i $(i \in I)$. Furthermore suppose that each G_i is freely indecomposable and that no G_i is cyclic. Then every non-cyclic freely indecomposable free factor of G is isomorphic to one of the G_i.*

Proof. Let H be such a free factor of G. By the Kurosh subgroup theorem H is a free product of conjugates of subgroups of the G_i and a free group. Thus, replacing H by a conjugate if necessary, it follows that H is a subgroup of some G_i. But, again by the subgroup theorem, H is then a free factor of this G_i. So $H = G_i$, as required.

LEMMA 2. *Let E be any group. Suppose that E contains a countably infinite number of non-isomorphic, freely indecomposable subgroups. Then the free product*

$$P = E * \langle u \rangle$$

of E with the infinite cyclic group $\langle u \rangle$ on u contains continuously many non-isomorphic subgroups.

Proof. Let $E_1, E_2, \ldots$ be an infinite sequence of freely indecomposable, non-cyclic, non-isomorphic subgroups of E. Let

$$E(i) = u^{-i} E_i u^i \quad (i = 1, 2, \ldots).$$

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Then

$$F = gp(E(i))|i = 1,2,\dots)$$

is the free product of its subgroups $E(i)$. Now for each properly ascending sequence $\sigma = \sigma(1), \sigma(2), \dots$ of positive integers we define

$$F_\sigma = gp(E(\sigma(i))|i = 1,2,\dots).$$

Then it follows from Lemma 1 that $F_\sigma \cong F_\tau$ only if $\sigma = \tau$. This proves Lemma 2.

2. Now consider the group

$$E = \langle a, t; t^{-1}at = a^2 \rangle.$$

Let

$$E_i = gp(a, t^{2^i}).$$

Each E_i is solvable and hence freely indecomposable. Moreover the E_i have different factor derived groups; hence $E_i \cong E_j$ only if $i = j$.

Now consider the free product P of E and the infinite cyclic group on u. By Lemma 2 P is a one-relator group with continuously many subgroups, as desired.

In particular, it follows that P contains (countable) subgroups which are not recursively presentable! Since every one-relator group can be embeded in a 2-generator one-relator group (for example, see [1, p. 259]), it follows that there are 2-generator one-relator groups which contain continuously many non-isomorphic subgroups. This helps, in part, to explain why the isomorphism problem for one-relator groups is so difficult.

3. Next, consider the one-relator group

$$G = \langle a, b, u, v; [a, b] = [u, v]^2 \rangle,$$

where as usual, $[x, y] = x^{-1}y^{-1}xy$. Let

$$E = gp(a, b, c^{-1}ac, c^{-1}bc),$$

where $c = [u, v]$. Then E is the fundamental group of a two-dimensional orientable surface of genus two and therefore contains the fundamental groups of all higher genus as subgroups of finite index (for example, see William S. Massey [2]). Now let $P = gp(E, u)$. Then it is not hard to see that $P = E * \langle u \rangle$.

Now E satisfies the hypothesis of Lemma 2 and so it follows that G contains continuously many non-isomorphic subgroups!

REFERENCES

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