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REAL PARTS OF NORMAL EXTENSIONS OF SUBNORMAL OPERATORS[1]

BY

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1. Introduction and main theorem

A bounded linear operator S on a separable Hilbert space H is said to be subnormal if S has a normal extension N to a Hilbert space $K \supset H$. In case S has no normal part then S is said to be a pure subnormal operator. Further, N is called the (essentially unique) minimal normal extension if the only reducing space of N which contains H is K. (For the basic properties of subnormal operators, see Halmos [3], Chapter 21, and for a detailed exposition of the subject, see Conway [2].) Since H is invariant under N then $H^{\perp} = K \ominus H$ is invariant under N^*. As in Conway [1], the operator $T = N^*|H^{\perp}$, is called the dual of $S = N|H$. Further, one can express N and N^* as operator matrices

$$(1.1)\quad N = \begin{bmatrix} S & X \\ 0 & T^* \end{bmatrix} \quad \text{and} \quad N^* = \begin{bmatrix} S^* & 0 \\ X^* & T \end{bmatrix} \quad \text{on} \quad K = H \oplus H^{\perp}.$$

In Olin [6], p. 228, it is shown that since S is pure with minimal normal extension N then T is also pure with minimal normal extension N^*. Further ([1], p. 196), T is the dual of S with spectrum $\sigma(T) = \{\bar{z} : z \in \sigma(S)\}$. Simple calculations with the matrices of (1.1) show that

$$(1.2)\qquad S^*S - SS^* = XX^*, \quad T^*T - TT^* = X^*X$$

and

$$(1.3)\quad \mathrm{Re}(N) = \tfrac{1}{2}(N + N^*) = \begin{bmatrix} \mathrm{Re}(S) & \tfrac{1}{2}X \\ \tfrac{1}{2}X^* & \mathrm{Re}(T) \end{bmatrix} \quad \text{on} \quad K = H \oplus H^{\perp}.$$

Since S and T are pure subnormal (hence also hyponormal) operators, both $\mathrm{Re}(S)$ and $\mathrm{Re}(T)$ are absolutely continuous operators on H and $H^{\perp}$, respectively; Putnam [8], pp. 42–43.

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THEOREM 1. *Let S be a pure subnormal operator on H with the minimal normal extension N on $K \supset H$, and let T be the dual of S. Suppose that*

(1.4) *$D^{1/2}$ is of trace class, where $S^*S - SS^* = D(\geqq 0)$.*

Then

(1.5) $\mathrm{Re}(N)$, *on K, has an absolutely continuous part, which, on the corresponding absolutely continuous subspace of K, is unitarily equivalent to* $\mathrm{Re}(S) \oplus \mathrm{Re}(T)$ *on $K = H \oplus H^{\perp}$.*

More generally, if a and b are real and $a^2 + b^2 > 0$ then $a\mathrm{Re}(N) + b\mathrm{Im}(N)$ has an absolutely continuous part which is unitarily equivalent to $[a\mathrm{Re}(S) + b\mathrm{Im}(S)] \oplus [a\mathrm{Re}(T) + b\mathrm{Im}(T)]$.

Proof. It follows from (1.3) that $\mathrm{Re}(N)$ is the sum of $\mathrm{Re}(S) \oplus \mathrm{Re}(T)$ and the selfadjoint perturbation

$$\frac{1}{2}\begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}.$$

The square of this last operator is $\frac{1}{4}(XX^* \oplus X^*X)$. In view of (1.4) and (1.2), $(XX^*)^{1/2}$ is of trace class. Thus, X and hence also $(X^*X)^{1/2}$ are of trace class. As was noted above, $\mathrm{Re}(S) \oplus \mathrm{Re}(T)$ is absolutely continuous, and so (1.5) is a consequence of the well-known Rosenblum-Kato theory [10], [4]; for example, see also, [5], p. 540 and [8], p. 101. The last part of Theorem 1 readily follows by replacing S by $e^{it}S$, where t is real, and the proof is complete.

In general, a pure subnormal operator for which (1.4) holds does not have a minimal normal extension N for which $\mathrm{Re}(N)$ is absolutely continuous. That is, in general, the absolutely continuous subspace of $\mathrm{Re}(N)$ may be a proper subspace of K. Perhaps the simplest example showing this is that of Sarason cited in [3], p. 307, where S is a unilateral weighted shift with weights $\{2^{-1/2}, 1, 1, \dots\}$. Here the selfcommutator $S^*S - SS^*$ even has finite rank and $\sigma(N)$ consists of the unit circle together with the origin. In particular, 0 is in the point spectrum of N and hence also in that of $\mathrm{Re}(N)$.

Earlier, Wermer [11] (Theorems 1 and 2) gave an example of a pure subnormal S having a minimal normal extension N possessing a pure point spectrum (as has been noted also by Olin [7] and Radjabalipour [9]), so that the eigenvectors of N span K. In particular, $\mathrm{Re}(N)$ must also have a pure point spectrum. In this example, of course, the condition (1.4) cannot be satisfied.

It will be shown in Section 2 below that under the hypothesis (1.4) of Theorem 1, $\mathrm{Re}(N)$ may have, in addition to the absolutely continuous part claimed in (1.5), not only a point spectrum as in the example of Sarason above, but also a purely singular continuous spectrum. Finally, it will be shown in Section 3 that if (1.4) is relaxed to the requirement that $D^{1/2}$ only be of Schmidt class, or equivalently, that D is of trace class, then it is possible that $\mathrm{Re}(N)$ has a purely singular spectrum, so that its absolutely continuous component is missing.

2. An example

It will be shown that there exists a pure subnormal operator S, in fact, an analytic Toeplitz operator, having a selfcommutator D satisfying (1.4) and a minimal normal extension N for which $\mathrm{Re}(N)$ has both an absolutely continuous part and a purely singular continuous part.

Let $f \not\equiv$ const belong to H^∞, so that

$$(2.1) \qquad f(t) \sim \sum_{n=0}^{\infty} c_n e^{int} \not\equiv c_0 \quad \text{and} \quad |f(t)| \leqq \text{const (a.e.)} < \infty,$$

and let $S = T_f$ denote the corresponding Toeplitz operator. See [2], p. 272, [3], p. 136 or [8], pp. 128–132. Relative to the basis $\{e_n\}$, $e_n = e^{int}$ $(n = 0, 1, 2, \dots)$, for H^2, with normalized Lebesgue measure on the unit circle, T_f has the representation as a bounded matrix

$$(2.2) \quad A = (c_{i-j}),\, i, j = 1,2,\dots, \quad \text{and} \quad c_n = 0 \quad \text{for} \quad n = -1, -2, \dots .$$

With respect to the standard orthonormal basis $\{\phi_n\}$ in l^2, where $\phi_1 = (1, 0, 0, \dots)$, $\phi_2 = (0,1, 0, 0, \dots), \dots$, it is seen from a straightforward calculation (for example, see [8], p. 131), that

$$\|A\phi_n\|^2 - \|A^*\phi_n\|^2 = |c_n|^2 + |c_{n+1}|^2 + \dots,$$

so that

$$(2.3) \qquad A^*A - AA^* = B^*B, \quad \text{where} \quad B = (c_{i+j-1}),\, i, j = 1, 2, \dots .$$

Thus, in order that S satisfy (1.4), $(B^*B)^{1/2}$ must be of trace class. However,

$$\begin{aligned} \mathrm{tr}(B^*B)^{1/2} &= \sum_{n=1}^{\infty} \left((B^*B^{1/2}\phi_n, \phi_n\right) \leqq \sum_{n=1}^{\infty} \|(B^*B)^{1/2}\phi_n\| \\ &= \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} |c_k|^2\right)^{1/2} \leqq \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} |c_k| = \sum_{n=1}^{\infty} n|c_n|. \end{aligned}$$

Consequently, the condition

$$(2.4) \qquad \sum_{n=1}^{\infty} n|c_n| < \infty$$

is sufficient in order that (1.4) be satisfied. Since (2.4) implies that $\Sigma|c_n| < \infty$, it is seen that, in particular, (2.4) assures that $f(t)$ of (2.1) is bounded, and even continuous, on $[0, 2\pi]$.

The minimal normal extension N of $S = T_f$ on $H^2(0, 2\pi)$ is multiplication by $f(t)$ on $L^2(0, 2\pi)$. For convenience, suppose that all c_n are real, so that $\operatorname{Re}(N)$ is the operator on $L^2(0, 2\pi)$ of multiplication by $g(t)$, where

$$g(t) = \sum_{n=0}^{\infty} c_n \cos nt \not\equiv c_0, \quad c_n \text{ real.} \tag{2.5}$$

It will be shown that $g(t)$ of (2.5) can be chosen so that, in addition to (2.4),

(2.6) $g(t) = g(2\pi - t)$, $0 \leqq t \leqq \pi$, and $g(t)$ is strictly increasing on $[0, \pi]$,

and, further,

(2.7) $g''(t)$ is continuous on $[0, 2\pi]$, $g'(t) \geqq 0$ on $[0, \pi]$ and $g'(t) = 0$ on a subset of $[0, \pi]$ of positive Lebesgue measure.

First, let C be a Cantor set on $[0, \pi]$ of positive measure. If the sequence of removed open intervals of $[0, \pi] \setminus C$ is denoted by $I_1, I_2, \ldots,$ then $\sum |I_n| < \pi$. Next, for each $n = 1, 2, \ldots,$ let $f_n(t)$ on $[0, \pi]$ satisfy:

(2.8) $f_n'(t)$ is continuous, $0 \leqq f_n(t) \leqq 1$ and $|f_n'(t)| \leqq 1$ on $[0, \pi]$; $f_n(t) > 0$ on I_n and $f_n(t) = 0$ on $[0, \pi] \setminus I_n$.

That such functions exist is clear. Next, let

$$h(t) = \sum_{n=1}^{\infty} f_n(t)/n^2, \tag{2.9}$$

so that $h(t) = 0$ on C and $h(t) > 0$ on $[0, \pi] \setminus C$. Also, h' is continuous and can be obtained from term by term differentiation of (2.9). If $g(t)$ is defined by

$$g(t) = \int_0^t h(s)\, ds, \quad 0 \leqq t \leqq \pi, \tag{2.10}$$

then $g \in C^2[0, \pi]$ and $g'(t) = h(t)$ on $[0, \pi]$. Extend the domain of g to $[0, 2\pi]$ by putting $g(2\pi - t) = g(t)$ for $0 \leqq t \leqq \pi$. Clearly,

$$g''(t) = \sum_{n=1}^{\infty} f_n'(t)/n^2$$

and $g''(\pi) = 0$, as a left hand derivative of g' at $t = \pi$. Consequently, the extension of g to $[0, 2\pi]$ has a continuous second derivative there. Further, it is seen that (2.6) and (2.7) are satisfied. Clearly, $g(t)$ has a Fourier series of the form (2.5) and, since $g \in C^2[0,2\pi]$, $|c_n| \leqq |b_n|/n^2$ (b_n real, $n = 1,2,\ldots$), where $\sum b_n^2 < \infty$. In particular,

$$\sum_{n=1}^{\infty} n|c_n| \leqq \sum |b_n|/n \leqq \left(\sum 1/n^2\right)^{1/2}\left(\sum b_n^2\right)^{1/2} < \infty,$$

so that (2.4) holds.

Since $g(t) = g(2\pi - t)$, it is seen that g is strictly increasing on $[0, \pi]$ and strictly decreasing on $[\pi, 2\pi]$. In addition, it is clear that the operator $\mathrm{Re}(N)$, multiplication by $g(t)$ on $L^2(0, 2\pi)$, is (unitarily equivalent to) the direct sum of multiplication by g on $L^2(0, \pi)$ with itself. Also, if $u, v \in L^2(0, \pi)$, it is seen that

$$\int_0^{\pi} g(t)u(t)\bar{v}(t)\,dt = \int_0^M xu\bar{v}\,d\mu, \quad M = g(\pi),$$

where the strictly increasing continuous function $\mu = \mu(x)$ on $[0, M]$ is the inverse of $g(t)$ on $[0, \pi]$. Consequently, $\mathrm{Re}(N)$ is unitarily equivalent to $Q \oplus Q$, where Q is multiplication by x on $L^2(\mu)$. Since g' is continuous on $[0,\pi]$ and is 0 on the set $C \subset [0, \pi]$, then $\int_C |dg| = \int_C g'dt = 0$. If $Z = g(C)$, then $|Z| = 0$ and $\mu(Z) = |C| > 0$, and so the operator Q has a purely singular continuous component, as was to be shown.

3. Another example

There will be given a pure subnormal analytic Toeplitz operator S for which

(3.1) $S^*S - SS^* = D$ is of trace class

and for which

(3.2) $\mathrm{Im}(N)$ is purely singular,

where, as before, N is the minimal normal extension of S. (It is convenient here to consider $\mathrm{Im}(N)$ rather than $\mathrm{Re}(N)$. If $S_1 = -iS$ has the minimal normal extension N_1 then, of course, $\mathrm{Re}(N_1) = \mathrm{Im}(N)$.)

If A again denotes the matrix corresponding to S as in the beginning of Section 2 it is seen from (2.3) that

$$\mathrm{tr}(B^*B) = \sum_{n=1}^{\infty} \|(B^*B)^{1/2}\phi_n\|^2 = \sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}|c_k|^2\right) = \sum_{n=1}^{\infty} n|c_n|^2,$$

so that relation (3.1) above becomes

$$\sum_{n=1}^{\infty} n|c_n|^2 < \infty. \tag{3.3}$$

It will be shown that there exists a real-valued function $g(t)$ having a Fourier series

$$(3.4)\qquad g(t) = \sum_{n=1}^{\infty} c_n \sin nt \quad (\text{with } \Sigma|c_n| < \infty)$$

satisfying (3.3) and such that the operator of multiplication by $g(t)$ on $L^2(0, 2\pi)$ is purely singular.

The series (3.4) will be obtained as an adaptation of a certain lacunary series arising from Riesz products of the form

$$(3.5)\qquad \prod_{i=1}^{\infty} (1 + \alpha_i \cos n_i t),$$

where, for $i = 1, 2, \ldots,$

$$(3.6)\qquad n_{i+1}/n_i \geqq q > 3, \ -1 \leqq \alpha_i \leqq 1, \ \alpha_i \neq 0 \quad \text{and} \quad \sum \alpha_i^2 = \infty;$$

see Zygmund [12], pp. 208–209. For use below, it may be noted that the first condition of (3.6) assures that

$$(3.7)\qquad n_{i+1} - n_i - n_{i-1} - \cdots - n_1 > n_i;$$

[12], p. 208. Also, if $p_k(t)$ is the (nonnegative) k-th partial product of (3.5), so that

$$p_k(t) = \prod_{i=1}^{k} (1 + \alpha_i \cos n_i t) = 1 + \sum_{n=1}^{\mu_k} \gamma_n \cos nt,$$

then

$$(3.8)\qquad \gamma_n = 0 \text{ if } n \neq n_i \pm n_{i'} \pm n_{i''} \ldots, \quad \text{where } i > i' > i'' \ldots .$$

In addition, the series

$$(3.9)\qquad \lim_{k \to \infty} p_k(t) = 1 + \sum_{n=1}^{\infty} \gamma_n \cos nt$$

is the Fourier-Stieltjes series of the nondecreasing continuous function

$$(3.10)\qquad F(t) = \lim_{k \to \infty} \int_0^t p_k(s)\, ds = t + \sum_{n=1}^{\infty} (\gamma_n/n) \sin nt;$$

that is, if $\gamma_0/2 = 1$,

$$(3.11)\qquad \gamma_n = \pi^{-1} \int_0^{2\pi} \cos nt\, dF(t), \quad n = 0, 1, 2, \ldots .$$

Finally, and what is crucial here relation (3.6) implies that

$$F'(t) = 0 \quad \text{a.e.} \qquad ([12], \text{p. } 209). \tag{3.12}$$

Note that for any sequence $n_1 < n_2 < \ldots$ and for any fixed positive integer i, the number of sums of the form $n_i \pm n_{i'} \pm n_{i''} \pm \ldots$, where $i > i' > i'' > \ldots$, is not greater than 3^{i-1}. Next, choose the n_i so sparse that $n_1 < n_2 < \ldots$, $n_{i+1}/n_i \geqq q > 3$, and so that, in addition,

$$\sum_{i=1}^{\infty} 3^i/n_i < \infty. \tag{3.13}$$

Then, choose the α_i so as to satisfy (3.6). By (3.11), $|\gamma_n| \leqq$ const, and hence by (3.7), (3.8) and (3.13),

$$\sum_{n=1}^{\infty} |\gamma_n|/n \leqq (\text{const}) \sum_{i=1}^{\infty} 3^i/n_i < \infty. \tag{3.14}$$

In particular, the series of (3.10) is absolutely convergent. Moreover, by (3.14),

$$\sum n(\gamma_n/n)^2 \leqq (\text{const}) \sum |\gamma_n|/n < \infty. \tag{3.15}$$

Now, choose a second sequence analogous to $\{\alpha_i\}$, say $\{\alpha_i^*\}$, in such a way that the corresponding sequence $\{\gamma_n^*\}$ is not identical with $\{\gamma_n\}$. (Since $\gamma_{n_i} = \alpha_i$ (see [12], p. 209), this can be done in many ways.) If $F^*(t)$ denotes the function corresponding to $F(t)$ let $g(t) = F(t) - F^*(t)$, so that, by (3.10), $g(t)$ has the form (3.4) with

$$c_n = (\gamma_n - \gamma_n^*)/n \quad \text{for } n = 1, 2, \ldots. \tag{3.16}$$

Clearly, $g(t) \not\equiv$ const. Also, since $(\gamma_n - \gamma_n^*)^2 \leqq 2(\gamma_n^2 + \gamma_n^{*2})$, relation (3.15) implies (3.3).

Since $g(t)$ is the difference of continuous monotone functions, $g(t)$ is continuous and of bounded variation on $[0, 2\pi]$. In addition, by (3.12), $g'(t) = 0$ a.e. Consequently, the operator of multiplication by $g(t)$ on $L^2(0, 2\pi)$ has no absolutely continuous part. Since $g(t) = \operatorname{Im}(f(t))$, where $f(t)$ is given by (2.1) with the c_n defined by (3.16) (and $c_0 = 0$), then the above operator is just $\operatorname{Im}(N)$.

4. Remarks

The following result is similar to Theorem 1.

THEOREM 2. *Under the hypotheses of Theorem 1, the absolutely continuous part of $N^*N(=NN^*)$ is unitarily equivalent to the absolutely continuous part of $S^*S \oplus T^*T$.*

The proof is similar to that of Theorem 1 and will be omitted. It may be noted that the absolutely continuous parts of S^*S and of T^*T may be absent as, for instance, is the case when S is an isometry.

Added in proof. Necessary and sufficient conditions in order that the Hankel matrix $B = (c_{i+j-1})$ considered above be of trace class (i.e., that $\operatorname{tr}(B^*B)^{1/2} < \infty$) have been obtained by V. V. Peller, *Nuclearity of Hankel operators*, Steklov Institute of Mathematics (LOMI Preprint E-I-79), Leningrad, 1979. See also the survey by S. C. Power, *Hankel operators on Hilbert space*, Research notes in mathematics, vol. 64, Pitman Adv. Pub. Program, Boston-London-Melbourne, 1982.

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