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ATOMIC DECOMPOSITION OF GENERALIZED LIPSCHITZ SPACES

BY

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1. Introduction

In this note we introduce new function spaces denoted by $B(\rho)$ and B_w where ρ is non-negative, non-decreasing, $\rho(0) = 0$ and $\rho(t)/t$ is in $L^1(T)$, the Lebesgue space L^1 with T the perimeter of the unit disk in the complex plane and w a weight which will be in some of the class A_p, $1 \le p \le \infty$.

For w a weight, we say b is a weighted special atom if $b(t) \equiv 1/2\pi$ or if there is an interval $I \subseteq T$ with left and right halves L, R such that

$$b(t) = \frac{1}{w(I)}[\chi_L(t) - \chi_R(t)]$$

where $w(I) = \int_I w(x)\,dx$. Then we say $f \in B_w$ if there are weighted special atoms b_n such that $f(t) = \sum_{n=0}^{\infty} c_n b_n(t)$ with $\sum_{n=0}^{\infty}|c_n| < \infty$. B_w is endowed with the norm $\|f\|_{B_w} = \inf \sum_{n=0}^{\infty}|c_n|$, where the infimum is taken over all possible representations of f, which becomes a Banach space.

In the definition of weighted special atoms above, if we replace $w(I)$ with $\rho(|I|)$, where ρ is as above, then this new space will be denoted by $B(\rho)$.

The spaces $B(\rho)$ and B_w will be called weighted special atom spaces.

Notice that for particular w and ρ, the spaces B_w and $B(\rho)$ coincide with those spaces defined in [2], [3], [4], [5]; for example $\rho(t) = t^{1/p}$ for $\frac{1}{2} < p < \infty$, $B(\rho) = B^p$.

We would like to mention that $B(\rho)$ for some ρ is the real atomic decomposition of some well known Besov-Bergman-Lipschitz spaces; for example for $\rho(t) = t^{1/p}$ and $1 < p < \infty$, $B(\rho)$ is equivalent as a Banach space to the space of those real valued functions for which

$$\int_0^{2\pi}\int_0^{2\pi}\frac{|f(x) - f(y)|}{|x - y|^{2-1/p}}\,dx\,dy < \infty.$$

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This space is known as Besov space. Also for the same ρ, $B(\rho)$ is the boundary value of those analytic functions F for which

$$\int_0^1 \int_0^{2\pi} |F'(re^{i\theta})| (1-r)^{1/p-1} \, d\theta \, dr < \infty;$$

see [4], [5], and [15] for these results.

One of the main results of this paper is that for some ρ, $B(\rho)$ is the real characterization of those analytic functions in the disc for which

$$\int_0^1 \int_0^{2\pi} |F'(re^{i\theta})| \frac{\rho(1-r)}{1-r} \, d\theta \, dr < \infty,$$

which is a generalization of an earlier result (see [4] and [5]).

We point out that for $\rho(t) = t$, $B(\rho)$ is properly contained in the Hardy space H^1 in the disc and that $B(\rho)$ is contained in all the nontrivial H_ϕ, H_ϕ is defined in [17]. Recently in [7], it has been shown that there is an f in $B(\rho)$ so that its Fourier series diverges almost everywhere and $B(\rho) \subseteq H_\phi$. It follows that for all ϕ, H_ϕ has a divergent Fourier series.

In these notes we give some accounts of these spaces; in particular we show some properties which lead to the computation of the dual spaces. We also show an interpolation theorem for operators acting on these spaces into the Lorentz spaces.

To make the presentation reasonably self-contained, we shall include a resume of pertinent results and definitions.

Throughout this paper, the constant C may not be the same in every occurence.

2. Preliminaries

DEFINITION 2.1. Let

$$I(f) = \frac{1}{|I|} \int f(x) \, dx.$$

Then we say that a non-negative function w, which we call weight, is in the class A_p for $1 < p < \infty$ if and only if $I(w) I(w^{1/(1-p)})^{p-1} < M$, where M is an absolute positive constant.

We define A_1 as follows: $w \in A_1$ if $\sup_{x \in I} L(w) \leq Cw(x)$ a.e. where C is an absolute constant. Notice that $A_1 \subset \bigcap_{p>1} A_p$. Define A_∞ as follows: $w \in A_\infty$ if for all measurable sets $E \subset T$, there is a $\delta > 0$ such that

$$\frac{w(E)}{w(T)} \leq C \left(\frac{|E|}{|T|} \right)^\delta$$

where C is an absolute constant and the bars mean the Lebesgue measure. Then $w \in A_\infty$ if and only if $w \in \bigcup_{1<p<\infty} A_p$.

Muckenhoupt introduced A_p weights in [10] and has contributed a lot to the theory of A_p weights.

Suppose $\rho(t)/t$ is in the Lebesgue space $L^1(T)$. Let $\sigma(t) = \int_0^t \rho(s)/s\, ds$. Then we say ρ is Dini if $\sigma(t) \le C\rho(t)$.

DEFINITION 2.2. We define the weighted Lipschitz class by

$$\Lambda_*(\rho) = \{g: T \to R, \text{continuous}, g(x+h) + g(x-h) - 2g(x) = O[\rho(2h)]\}.$$

The $\Lambda_*(\rho)$ norm is given by

$$\|g\|_{\Lambda_*(\rho)} = \sup_{h>0,\, x} \left| \frac{g(x+h) + g(x-h) - 2g(x)}{\rho(2h)} \right|.$$

Similarly we define Λ_{*w} for $w \in A_\infty$ by replacing $\rho(2h)$ with $w([x-h, x+h])$.

Notice that for $\rho(t) = t$, $\Lambda_*(\rho)$ is the Zygmund class and for $\rho(t) = t^\alpha$ we have the Lipschitz class for $0 < \alpha < 2$.

3. Some properties of $B(\rho)$ and B_w

In this section we state and prove some properties of the spaces $B(\rho)$ and B_w.

LEMMA 3.1. *Let I be an interval in T. Then*:

(i) $\|\chi_I\|_{B_w} \le C|I|^\delta$ *for some* $0 < \delta < 1$ *when* $w \in A_\infty$.

(ii) *If* $\rho(t)/t \in L^1$, *let* $\sigma(t) = \int_0^t \rho(s)/s\, ds$. *Then*

$$\|\chi_I\|_{B(\rho)} \le C\left[|I|^{1/2} + \sigma\left(|I|^{1/2}\right)\right].$$

Proof. For simplicity, we will treat $|T|$ as 1, rather than 2π.

(i) Suppose first that $I = [0, 2^{-N}]$. Let

$$I_1 = [0, 2^{1-N}],\ I_2 = [0, 2^{2-N}],\ I_3 = [0, 2^{3-N}], \dots,\ I_n = [0, 2^{n-N}]$$

and $I_N = [0,1] = T$.

Define L_n and R_n as the halves of I_n, and let

$$\phi_n(t) = \chi_{L_n}(t) - \chi_{R_n}(t).$$

Let $g(t) = \sum_{n=1}^{N} 2^{N-n}\phi_n(t)$. Then $1 + g(t) = 2^N\chi_I(t)$, so we have

$$\chi_I = 2^{-N} + \sum_{n=1}^{N} 2^{-n}\phi_n. \tag{3.2}$$

By the A_∞ condition, there exists a $\delta > 0$ with

$$\frac{w(E)}{w(J)} \leq C\left(\frac{|E|}{|J|}\right)^{\delta}$$

for all E measurable sets contained in the interval J. At the expense of some sharpness, but no more, we can take $\delta < 1$. In particular, we have

$$w(I_n) \leq Cw(T)|I_n|^{\delta} \leq C2^{\delta(n-N)}.$$

Let

$$b_0(t) = 1 \quad \text{and} \quad b_n(t) = \frac{1}{w(I_n)}\phi_n(t).$$

Then b_n are weighted special atoms and (3.2) becomes

$$\chi_I(t) = 2^{-N}b_0(t) + \sum_{n=1}^{N} 2^{-n}w(I_n)b_n(t).$$

Hence,

$$\begin{aligned} \|\chi_I\|_{B_w} &\leq 2^{-N} + \sum_{n=1}^{N} 2^{-n}w(I_n) \\ &\leq 2^{-N} + C\sum_{n=1}^{N} 2^{-n}2^{\delta(n-N)} \\ &\leq 2^{-N} + C2^{-N\delta} \\ &\leq C2^{-N\delta} \\ &= C|I|^{\delta}. \end{aligned}$$

By rotation, this holds whenever $|I| = 2^{-N}$. Next, if $I = [0, \beta]$ for $\beta \in T$, then $\beta = \sum_{i=1}^{\infty} c_i/2^i$ where $c_i = 0$ or 1. Therefore $[0, \beta] = \sum_{i=1}^{\infty} I_i$, with I_i an interval of length $c_i/2^i$, so that $\chi_{[0,\beta]}(t) \leq \sum_{i=1}^{\infty}\chi_{I_i}(t)$, and hence

$$\|\chi_I\|_{B_w} \leq \sum_{i=1}^{\infty}\left(\frac{c_i}{2^i}\right)^{\delta}.$$

Let N be the first integer with $C_N \neq 0$. Then $2^{-N} \leq \beta \leq 2^{1-N}$ and

$$\sum_{i=N}^{\infty} \left(\frac{c_i}{2^i}\right)^{\delta} \leq \sum_{i=N}^{\infty} 2^{-i\delta} \leq C\beta^{\delta},$$

or $\|\chi_I\|_{B_w} \leq C|I|^{\delta}$.

Finally, for an arbitrary interval $I = [\alpha, \alpha + \beta]$, simply rotate T, treating T as $[\alpha, \alpha + 1]$, taking the I_n's in (3.2) as $[\alpha, \alpha + 2^{n-N}]$.

(ii) In (3.2), let $b_0(t) = 1$ and $b_n(t) = \phi_n(t)/\rho(2^{n-N})$. The b_n's are weighted special atoms and

$$\chi_I = 2^{-N} b_0(t) + \sum_{n=1}^{N} 2^{-n}\rho(2^{n-N}) b_n(t).$$

Arguing as in (i), we suppose $|I| = \beta = \Sigma_{n=1}^{\infty} c_n 2^{-n}$ where $c_n = 0$ or 1 and $C_N = 1$. Then

$$\chi_I = \sum_{k=N}^{\infty} c_k \left(2^{-k} b_{0k} + \sum_{n=1}^{k} 2^{-n}\rho(2^{k-n}) b_{nk}(t) \right)$$

where the b_{nk}'s are weighted special atoms, and

$$\begin{aligned}\|\chi_I\|_{B(\rho)} &\leq \sum_{k=N}^{\infty} \left[2^{-k} + \sum_{n=1}^{k} 2^{-n}\rho(2^{n-k}) \right] \\ &\leq C\beta + \sum_{k=N}^{\infty} \sum_{n=1}^{k} 2^{-n}\rho(2^{n-k}).\end{aligned}$$

This double sum is

$$\sum_{k=N}^{\infty} \sum_{m=0}^{k-1} 2^{m-k}\rho(2^{-m}) = \sum_{m=0}^{N} 2^{m}\rho(2^{-m}) \sum_{k=N}^{\infty} 2^{-k} + \sum_{m=N+1}^{\infty} 2^{m}\rho(2^{-m}) \sum_{k=m}^{\infty} 2^{-k}$$

so we must estimate

$$2^{-N} \sum_{m=0}^{N} 2^{m}\rho(2^{-m}) \tag{3.3}$$

and

$$\sum_{m=N+1}^{\infty} \rho(2^{-m}). \tag{3.4}$$

It is quite easy to show (3.4):

$$\sum_{m=n+1}^{\infty} \rho(2^{-m}) \le C \sum_{m=N+1}^{\infty} \int_{2^{-m}}^{2^{1-m}} \frac{\rho(t)}{t}\, dt$$
$$\le C \int_0^{2^{-N}} \frac{\rho(t)}{t}\, dt \le C\sigma(\beta).$$

For (3.3),

$$2^{-N} \sum_{n=0}^{N} 2^n \rho(2^{-n}) = 2^{-N} \sum_{n=0}^{N} \int_{2^{-n}}^{2^{1-n}} 2^{2n} \rho(2^{-n})\, dt$$
$$\le C 2^{-N} \int_{2^{-N}}^{2} \frac{\rho(t)}{t^2}\, dt$$
$$= C 2^{-N} \left(\int_{2^{-N}}^{\sqrt{\beta}} \frac{\rho(t)}{t^2}\, dt + \int_{\sqrt{\beta}}^{2} \frac{\rho(t)}{t^2}\, dt \right)$$
$$\le C\sigma(\sqrt{\beta}) + C 2^{-N} \rho(2) \int_{\sqrt{\beta}}^{2} \frac{1}{t^2}\, dt$$
$$\le C\left[\sigma(\sqrt{\beta}) + \sqrt{\beta}\right].$$

Keeping in mind that $\beta < \sqrt{\beta}$, (ii) follows.

The next result is a duality pairing between the weighted Lipschitz spaces and the weighted special atomspaces.

THEOREM 3.2 (Hölder's Inequality). *If $f \in X$ and $g \in Y$ then*

$$\left| \lim_{r \to 1} \int_T f(t) g_r'(t)\, dt \right| \le \|f\|_X \cdot \|g\|_Y$$

where $X = B(\rho)$ or B_w, $Y = \Lambda_*(\rho)$ or Λ_{*w} respectively, $g_r = P_r * g$ is the Poisson integral of g and the prime means derivative.

Proof. Let us restrict ourselves to $X = B(\rho)$ and $Y = \Lambda_*(\rho)$. We have

$$f(t) = \frac{1}{\rho(2h)} \left[\chi_{(x_0+h,\, x_0]}(t) - \chi_{[x_0-h,\, x_0]}(t) \right].$$

In fact, we have

$$\lim_{r \to 1} \int_T f(t) g_r'(t)\, dt = \frac{1}{\rho(2h)} \left[g(x_0 + h) + g(x_0 - h) - 2g(x_0) \right]$$

and thus by definition of the $\Lambda_*(\rho)$-norm we get

$$\left|\lim_{r\to 1}\int_T f(t)g_r'(t)\,dt\right| \le \|g\|_{\Lambda_*(\rho)}.$$

From this it follows that the theorem is true for a finite linear combination of weighted special atoms and consequently the extension for any $f \in B(\rho)$ is trivial.

The proof for $X = B_w$ and $Y = \Lambda_{*w}$ is similar.

COROLLARY 3.3. *If $f \in X$ and $g \in Y$, then*

$$\|g\|_Y = \sup_{\|f\|_X \le 1}\left|\lim_{r\to 1}\int_T f(t)g_r'(t)\,dt\right|.$$

Proof. Again let us take $X = B(\rho)$ and $Y = \Lambda_*(\rho)$. Then for

$$f(t) = \frac{1}{\rho(2h)}\left[\chi_{(x_0+h,\,x_0]}(t) - \chi_{[x_0-h,\,x_0]}(t)\right],$$

notice that $\|f\|_{B(\rho)} \le 1$ and consequently

$$\sup_{\|f\|_{B(\rho)}\le 1}\left|\lim_{r\to 1}\int_T f(t)g_r'(t)\,dt\right| \ge \left|\frac{g(x_0+h)+g(x_0-h)-2g(x_0)}{\rho(2h)}\right|$$

which implies

$$\sup_{\|f\|_B\le 1}\left|\lim_{r\to 1}\int_T f(t)g_r'(t)\,dt\right| \ge \|g\|_{\Lambda_*(\rho)}.$$

Combining this with Theorem 3.2 gives the desired result.

4. Duality

Consider the mapping ϕ_g: $B(\rho) \to \mathbf{R}$ defined by

$$\phi_g(f) = \lim_{r\to 1}\int_T f(t)g_r'(t)\,dt,$$

with g a fixed function in $\Lambda_*(\rho)$ and g_r as before. One can easily see that ϕ_g is a linear functional on $B(\rho)$. Moreover, Theorem 3.2 (Hölder's Inequality) tells us that $|\phi_g(f)| \le \|g\|_{\Lambda_*(\rho)}\|f\|_{B(\rho)}$, and therefore ϕ_g is a bounded linear functional on $B(\rho)$. The same situation holds for g fixed in Λ_{*w} and f in B_w.

In this section we show that indeed $\Lambda_*(\rho)$ generates all the bounded linear functional on $B(\rho)$, similarly for B_w and Λ_{*w}.

In this paper X^* will denote the dual space of X, that is, the space of all bounded linear functional ϕ on X with the norm

$$\|\phi\| = \sup_{\|f\|_X<1} |\phi(f)|.$$

THEOREM 4.1 (Duality Theorem). *If $\phi \in B^*(\rho)$ (or B_w^*) there is a $g \in \Lambda_*(\rho)$ (or Λ_{*w}) so that $\phi = \phi_g$; that is,*

$$\phi(f) = \lim_{r\to 1}\int_T f(t)g_r'(t)\,dt \quad \textit{for all } f \in B(\rho)\ (\textit{or } B_w),$$

where g_r is as before. Moreover $\|\phi\| = \|g\|_Y$, where $Y = \Lambda_*(\rho)$ or Λ_{*w}. Conversely if

$$\phi(f) = \lim_{r\to 1}\int_T f(t)g_r'(t)\,dt \quad \textit{for } f \in B(\rho)\ (\textit{or } B_w)$$

then $\phi \in B^*(\rho)$ (or B_w^*). Furthermore the mapping $\psi\colon \Lambda_*' \to A$ defined by $\psi(g') = \phi_g$, $A = B^*(\rho)$ (or B_w^*) is an isometric isomorphism, where ρ is increasing, $\rho(0) = 0$, and $\rho(t)/t \in L^1(T)$, $w \in A_\infty$.

Proof. Again we restrict ourselves to the case $B^*(\rho)$. If

$$\phi(f) = \lim_{r\to 1}\int_T f(t)g_r'(t)\,dt \quad \text{for } f \in B(\rho),$$

then we already have seen that Theorem 3.2 implies that ϕ is a bounded linear functional, that is, $\phi \in B^*(\rho)$, so it remains to prove the other direction. In fact, let $\phi \in B^*(\rho)$ and define $g(s) = \phi(\chi_{[0,s]})$ for $s \in [0, 2\pi]$. Observe that

$$g(s+h) - g(s) = \phi(\chi_{(s,s+h]})$$

and thus Lemma 3.1(ii) (in the case of B_w we use (i) and the boundedness of ϕ tells us that g is continuous. On the other hand,

$$\frac{g(s+h) + g(s-h) - 2g(s)}{\rho(2h)} = \phi\left[\frac{1}{\rho(2h)}(\chi_{(s,s+h]} - \chi_{(s-h,s]})\right]$$

Consequently by the boundedness of ϕ we get

$$|g(s+h) + g(s-h) - 2g(s)| \le \|\phi\|\rho(2h),$$

so that $\|g\|_{\Lambda_*} < \infty$ and therefore $g \in \Lambda_*(\rho)$.

Notice that $\lim_{r\to 1} g_r = g$ uniformly where g_r as before. So

$$\phi(\chi_{[0,s]}) = g(s) = \lim_{r\to 1} g_r(s) = \lim_{r\to 1}\int_0^s g_r'(t)\,dt = \lim_{r\to 1}\int_T \chi_{[0,s]}(t) g_r'(t)\,dt.$$

Therefore if I is any interval in T, it follows that

$$\phi(\chi_I) = \lim_{r\to 1}\int_T \chi_I(t) g_r'(t)\,dt.$$

Consequently if b is any weighted special atom we have

$$\phi(b) = \lim_{r\to 1}\int_T b(t) g_r'(t)\,dt,$$

and so the functional representation for $B(\rho)$ is proved for weighted special atoms and therefore for a finite linear combination of them. Thus the extension for any $f \in B(\rho)$ is trivial.

We have proved that given $\phi \in B^*(\rho)$ there is a g in $\Lambda_*(\rho)$ such that $\phi = \phi_g$; moreover, Corollary 3.3 tells us that $\|\phi\| = \|g\|_{\Lambda_*(\rho)}$ and so by definition of $\Lambda'_*(\rho) = \{g' \colon g \in \Lambda_*(\rho)\}$ it follows that the mapping $\psi\colon \Lambda'_*(\rho) \to B^*(\rho)$ defined by $\psi(g) = \phi_g$ is an isometry, and so the duality theorem is proved.

We point out that the concept of derivative that is being used in $\Lambda'_*(\rho)$ is the general notion given to us by the theory of distribution. That is, we say $g' = h$ if

$$\int_T g(t)\psi'(t)\,dt = -\int_T h(t)\psi(t)\,dt$$

for all infinitely differentiable functions ψ on T. Integration by parts shows us that this is indeed the relation that we would expect if g has continuous derivative, and $g' = h$ has the usual meaning.

See [2], [3], [4], [6], for the unweighted case where $\rho(t) = t^{1/p}$ for $\frac{1}{2} < p < \infty$.

5. Interpolation theorem

In this section we present a theorem on the interpolation of operators acting on the weighted special atom spaces into the Lorentz spaces. In order to state it we need some definitions.

Let f be a real valued measurable function on T. For $y > 0$ let

$$m(f, y) = m(|f|, y) = |\{x \in T, |f(x)| > y\}|.$$

$m(f, y)$ is called distribution function of f, $|\cdot|$ means the Lebesgue measure on T. $m(f, y)$ is non-negative, non-increasing and continuous from the right.

By f^* we mean the decreasing rearrangement of f, which is defined as

$$f^*(t) = \inf\{y; m(f, y) \leq t\}.$$

A linear operator T: $X \to Y$ is said to be bounded if

$$\|T\| = \sup\{\|Tf\|_Y; \|f\|_X \leq 1\} < \infty.$$

We shall say that a measurable function f belongs to the Lorentz spaces $L(p, q)$ if

$$\|f\|_{pq} = \left[\frac{q}{p}\int_0^\infty \left(f^*(t)t^{1/p}\right)^q \frac{dt}{t}\right]^{1/q} < \infty$$

for $0 < p < \infty, 0 < q < \infty$ where f^* is the decreasing rearrangement of f and, for $q = \infty$, the space $L(p, \infty)$ is well known as weak L^p-space. Equivalently, f belongs to $L(p, \infty)$, if there exists a positive number A such that

$$m(f, y) \leq \left(\frac{A}{y}\right)^p, \quad 0 < p < \infty.$$

Notice that $L(p, p)$ is the usual Lebesgue space L^p; also $\|f\|_{pq}$ is not a norm, since the triangle inequality may fail. However, one can find a norm equivalent to $\|f\|_{pq}$ under some restrictions on p and q, and thus for those values, $L(p, q)$ becomes a Banach space.

DEFINITION 5.1. We say that an operator is ρ-restricted weak type r if for any interval $I \subset [0, 2\pi]$ we have

$$(T\chi_I)^*(t) \leq M\frac{\rho(2|I|)}{t^{1/r}}$$

where the * means the decreasing rearrangement of $T\chi_I$, M is an absolute constant and ρ is a non-negative function with $\rho(0) = 0$.

Now we are ready to state the following interpolation for operators.

THEOREM 5.2. *Let T be a linear operator such that T is ϕ-restricted weak type p_1 with constant M_1 and also is ψ-restricted weak type p_2 with constant M_2. Then for $\rho(t) = \phi^\alpha(t)\psi^{1-\alpha}(t)$, T: $B(\rho) \to L(p, q)$ boundedly with*

$$\|Tf\|_{L(p,q)} \leq CM_1^t M_2^{1-t}\|f\|_{B(\rho)}$$

where

$$\frac{1}{p} = \frac{t}{p_1} + \frac{1-t}{p_2}, \quad p_2 < p < p_1, \; q \geq 1.$$

C is an absolute constant depending only on p, q, p_1, p_2 and

$$\alpha\left(\frac{1}{p_2} - \frac{1}{p_1}\right) = \frac{1}{p} - \frac{1}{p_1}.$$

Proof. Let f be a weighted special atom, that is

$$f(t) = \frac{1}{\rho(|I|)}[\chi_L(t) - \chi_R(t)],$$

where L and R are the left and right halves of I and $|L| = |R| = |I|/2$.

Now since T is ϕ and ψ restricted weak type p_1 and p_2 respectively we get

$$(5.3) \qquad (Tf)^*(t) \le M_1 \frac{\phi(|I|)}{\rho(|I|)t^{1/p_1}} \quad \text{and} \quad (Tf)^*(t) \le M_2 \frac{\psi(|I|)}{\rho(|I|)t^{1/p_2}}.$$

We now evaluate

$$\frac{p}{q}\|Tf\|_{pq}^q = \int_0^\infty [(Tf)^*(t)t^{1/p}]^q \frac{dt}{t}.$$

We have

$$\begin{aligned}
\frac{p}{q}\|Tf\|_{pq}^q &= \int_0^\sigma [(Tf)^*(t)t^{1/p}]^q \frac{dt}{t} + \int_\sigma^\infty [(Tf)^*(t)t^{1/p}]^q \frac{dt}{t} \\
&\le M_1^q \left[\frac{\phi(|I|)}{\rho(|I|)}\right]^q \int_0^\sigma t^{q/p - q/p_1 - 1}\,dt \\
&\quad + M_2^q \left[\frac{\psi(|I|)}{\rho(|I|)}\right]^q \int_\sigma^\infty t^{q/p - q/p_2 - 1}\,dt \quad \text{by (5.3)} \\
&= M_1^q \left[\frac{\phi(|I|)}{\rho(|I|)}\right]^q \frac{pp_1}{q(p_1 - p)} \sigma^{q(p_1 - p)/pp_1} \\
&\quad + M_2^q \left[\frac{\psi(|I|)}{\rho(|I|)}\right]^q \frac{pp_2}{q(p - p_2)} \sigma^{q(p_2 - p)/pp_2}
\end{aligned}$$

As σ is arbitrary we may take

$$\sigma = M\left[\frac{\psi(|I|)}{\phi(|I|)}\right]^{p_1 p_2/(p_1 - p_2)}$$

where M is a constant that we will determine later. Thus we get

$$\frac{p}{q}\|Tf\|_{pq}^q \leq AM_1^q \frac{\psi(|I|)}{\rho(|I|)^q}^{(q/p)p_2(p_1-p)/(p_1-p_2)}$$
$$\cdot[\phi(|I|)]^{q[1-(p_2)/p/(p_1-p)/(p_1-p_2)]} \cdot M^{q(p_1-p)/pp_1}$$
$$+BM_2^q \frac{\phi(|I|)}{\rho(|I|)^q}^{(qp_1/p)(p-p_2)/(p_1-p_2)}$$
$$\cdot[\psi(|I|)^{q[1+(p_1/p)(p_2-p)/(p_1-p_2)]} \cdot M^{q(p_2-p)/pp_2}$$

where

$$A = \frac{pp_1}{q(p_1-p)} \quad \text{and} \quad B = \frac{pp_2}{q(p-p_2)}.$$

Notice that

$$\frac{p_2}{p}\frac{p_1-p}{p_1-p_2} - \frac{p_1}{p}\frac{p_2-p}{p_1-p_2} = 1,$$

so

$$\frac{p}{q}\|Tf\|_{pq}^q \leq \left[AM_1^q M^{q(p_1-p)/pp_1} + BM_2^q M^{q(p_2-p)/pp_2}\right]$$
$$\cdot\left[\frac{\psi(|I|)^{(qp_2/p)(p_1-p)/(p_1-p_2)} \cdot \phi(|I|)^{(qp_1/p)(p-p_2)/(p_1-p_2)}}{\rho(|I|)^q}\right].$$

Since

$$\frac{p_1}{p}\frac{p-p_2}{p_1-p_2} = 1 - \frac{p_2}{p}\frac{p_1-p}{p_1-p_2}$$

and

$$\rho = \phi^\alpha \psi^{1-\alpha} \quad \text{for } \alpha = \frac{p_2}{p}\frac{p_1-p}{p_1-p_2}$$

we get

$$\|Tf\|_{pq} \leq \left[\frac{q}{p}\left(AM_1^q M^{q(p_1-p)/pp_1} + BM_2^q M^{q(p_2-p)/pp_2}\right)\right]^{1/q}. \tag{5.4}$$

Since M is arbitrary we may take

$$M = \left[\frac{M_1}{M_2}\right]^{p_1p_2/(p_2-p_1)};$$

this value of M minimizes the right hand side of (5.4).

Substituting the value of M in (5.4) and noticing that if

$$t = \frac{p_1}{p}\frac{p_2 - p}{p_2 - p_1}$$

then

$$\frac{p_2}{p}\frac{p_1 - p}{p_2 - p_1} = 1 - t,$$

we obtain

$$\|Tf\|_{pq} \le \frac{q}{p}\left[\frac{pp_1}{q(p_1 - p_2)} + \frac{pp_2}{q(p - p_2)}\right]^q M_1^t M_2^{1-t} = C(p, q, p_1, p_2) = C.$$

Then for any $f \in B(\rho)$ and $q \ge 1$, we get $\|Tf\|_{pq} \le CM_1^t M_2^{1-t}\|f\|_{B(\rho)}$. Therefore Theorem 5.2 is proved.

Remark 1. Theorem 5.2 is also true if we replace ρ by a weight w in A_∞; the proof is the same.

Remark 2. If $\phi(t) = t^{1/p_1}$, $\psi(t) = t^{1/p_2}$, $\alpha = (p_2/p)(p_1 - p)/(p_1 - p_2)$ then $\rho(t) = t^{1/p}$. This gives an earlier result in [6, page 153].

6. B_w and S_w

DEFINITION 6.1. A weight w is in B_p if there exists a constant C such that, for any interval I, with center x_I, we have

$$\frac{|I|^p}{w(I)}\int_{x\notin I}\frac{w(x)}{|x - x_I|^p}\,dx \le C.$$

$\bigcup_{p>1} B_p$ is the collection of all absolutely continuous doubling measures ω, that is, $\omega \in B_p$ for some p iff

$$\omega([x - 2h, x + 2h]) \le C\omega(x - h, x + h]).$$

Also $A_p \subset B_p$, for if J is the middle half of I and if $f = M^*\chi_J$, the Hardy-Littlewood maximal function of χ_J, then $f(x) \le C|I|/|x - x_I|$ for all $x \notin I$. By Muckenhoupt's Theorem,

$$\int_{x\notin I} f^p\omega(x)\,dx \le C\int(\chi_J)^p\omega(x)\,dx \le C\omega(I)$$

and this translates to the B_p condition. On the other hand, there exist B_p weights that are not in any A_q. For a good discussion of these classes, see [14].

DEFINITION 6.2. Let F be analytic function in the disk, we say that $F \in S_w$ if and only if

$$\|F\|_{S_w} = |F(0)| + \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} |F'(re^{i\theta})| w(\theta)\, d\theta\, dr < \infty.$$

To each $f \in B_w$ we associate the analytic function

$$F(z) = \frac{1}{2\pi}\int_T \frac{e^{it} + z}{e^{it} - z} f(t)\, dt,$$

and so we have the following result.

THEOREM 6.3. *$B_w \subseteq S_w$ if and only if $w \in B_2$.*

The theorem means that B_w is continuously contained in S_w if and only if w is in the class B_2.

Although A_∞ is assumed in the duality of B_w, A_∞ is not assumed here.

Proof. Suppose $w \in B_2$. It will suffice to show that w-special atoms are in S_w. Indeed, we will simply look at $I = [-h, h]$ and

$$b(t) = \frac{1}{w(I)}\left[\chi_{[0,h)}(t) - \chi_{[-h,0]}(t)\right].$$

Let

$$F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} b(t)\, dt.$$

Then

$$\begin{aligned} F'(z) &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{e^{it}}{(e^{it} - z)^2} b(t)\, dt \\ &= \frac{1}{\pi w(I)}\left[\int_0^h \frac{e^{it}}{(e^{it} - z)^2}\, dt - \int_{-h}^0 \frac{e^{it}}{(e^{it} - z)^2}\, dt\right] \\ &= \frac{1}{i\pi w(I)}\left[\frac{1}{z - e^{-ih}} + \frac{1}{2 - e^{ih}} + \frac{2}{1 - z}\right]. \end{aligned}$$

Let D be the unit disk and $D_1 = \{z \in D: |1 - z| \ge 2h\}$. Now

$$F'(z) = \frac{2}{i\pi w(I)} \cdot \frac{(1 - \cos h)(1 + z)}{(z - e^{ih})(z - e^{-ih})(1 - z)}.$$

On D_1 this denominator has absolute value

$$|(1-z)^2 + 2z(1-\cos h)||1-z| \geq \left(|1-z|^2 - h^2\right)|1-z| \geq \tfrac{3}{4}|1-z|^3$$

So on D_1,

$$|F'(z)| \leq C\frac{h^2}{w(I)|1-z|^3}.$$

Thus

$$\int_{D_1}\!\!\int |F'(re^{i\theta})|w(\theta)\,d\theta\,dr \leq C\frac{h^2}{w(I)}\int\!\!\int_{D_1}\frac{1}{|1-z|^3}w(\theta)\,d\theta\,dr.$$

Let N be the smallest integer with $2^N h \geq 1$. Then

$$\begin{aligned}
&\int\!\!\int_{D_1}|F'(re^{i\theta})|w(\theta)\,d\theta\,dr \\
&\quad\leq C\frac{h^2}{w(I)}\sum_{n=0}^{N}\int\!\!\int_{2^n h\leq|1-z|\leq 2^{n+1}h}\frac{1}{|1-z|^3}w(\theta)\,dr \\
&\quad\leq C\frac{h^2}{w(I)}\sum_{n=0}^{N}(2^n h)^{-3}\int_{1-2^{n+1}h}^{1}\int_{-2^{n+1}h}^{2^{n+1}h}w(\theta)\,d\theta\,dr \\
&\quad\leq C\frac{h^2}{w(I)}\sum_{n=0}^{N}(2^n h)^{-2}\int_{-2^{n+1}h}^{2^{n+1}h}w(\theta)\,d\theta.
\end{aligned}$$

Now since $w \in B_2$, w is a doubling measure; as a result,

$$\int_{-2^{n+1}h}^{2^{n+1}h}w(\theta)\,d\theta \leq C\int_{2^n h<|\theta|<2^{n+1}h}w(\theta)\,d\theta$$

So

$$\begin{aligned}
\int_{D_1}\!\!\int |F'(re^{i\theta})|w(\theta)\,d\theta &\leq C\frac{h^2}{w(I)}\sum_{n=0}^{N}(2^n h)^{-2}\int_{2^n h\leq|\theta|\leq 2^{n+1}h}w(\theta)\,d\theta \\
&\leq C\frac{h^2}{w(I)}\sum_{n=0}^{N}\int_{2^n h\leq|\theta|\leq 2^{n+1}h}\frac{w(\theta)}{\theta^2}\,d\theta \\
&\leq C\frac{|I|^2}{w(I)}\int_{\theta\notin I}\frac{w(\theta)}{\theta^2}\,d\theta \leq C \quad \text{by the } B_2 \text{ condition.}
\end{aligned}$$

On $D \setminus D_1$, the complement of D_1 relative to D, we have

$$|F'(z)| \leq \frac{1}{\pi w(I)}\left[\frac{1}{|z-e^{ih}|} + \frac{1}{|z-e^{-ih}|} + \frac{2}{|1-z|}\right]$$

and

$$|z - e^{ih}|, |z - e^{-ih}|, |1 - z| \le 4h.$$

By simple rotation, it will suffice to bound

$$\iint_{D,\, |1-z|\le 4h} |F'(re^{i\theta})| w(\theta)\, d\theta\, dr$$

or indeed, to bound

$$\begin{aligned}
&\frac{1}{w(I)} \iint_{D,\, |1-z|\le 4h} \frac{w(\theta)}{|1-z|}\, d\theta\, dr \\
&\le \frac{1}{w(I)} \sum_{n=0}^{\infty} \iint_{2^{-n-1}(4h) \le |1-z| \le 2^{-n}(4h)} \frac{w(\theta)}{|1-z|}\, d\theta\, dr \\
&\le \frac{C}{w(I)} \sum_{n=0}^{\infty} \frac{2^n}{h} \iint_{|1-z|\le 2^{2-n}h} w(\theta)\, d\theta\, dr \\
&\le \frac{C}{w(I)} \sum_{n=0}^{\infty} \frac{2^n}{h} \int_{1-2^{2-n}h}^{1} \int_{-2^{2-n}h}^{2^{2-n}h} w(\theta)\, d\theta \\
&\le \frac{C}{w(I)} \sum_{n=0}^{\infty} \int_{2^{2-n-1}h \le |\theta| \le 2^{2-n}} w(\theta)\, d\theta \quad \text{as } w \text{ is a doubling measure} \\
&\le \frac{C}{w(I)} \int_{-4h}^{4h} w(\theta)\, d\theta \le C \quad \text{again, because } w \text{ is doubling.}
\end{aligned}$$

Conversely, suppose $\|F\|_{S_w} \le C$ for all F associated to a w-special atoms.
We must show that

$$\frac{|I|^2}{w(I)} \int_{w \notin I} \frac{w(x)}{|x - x_I|^2}\, dx \le C.$$

For this, we may simply investigate $I = [-h, h]$. Let

$$b(t) = \frac{1}{w(I)} \left[\chi_{[0,h]}(t) - \chi_{[-h,0)}(t) \right]$$

and

$$F(z) = \frac{1}{2\pi} \int_T \frac{e^{it} + z}{e^{it} - z} b(t)\, dt.$$

Now for $h \le |1 - z| \le \frac{1}{2}$, we have

$$|F'(z) \ge C \cdot \left[\frac{h^2}{w(I)} \frac{1}{|1-z|\,|(1-z)^2 + 2(1-\cos h)|} \right] \ge \frac{ch^2}{w(I)|1-z|^3}.$$

So

$$\begin{aligned}
C &\geq \int_{1/2}^{1}\int_{0}^{2\pi} |F'(re^{i\theta})| w(\theta)\, d\theta\, dr \\
&\geq C\frac{h^2}{w(I)}\int_{|\theta|>h}\int_{\substack{1-r<|\theta|,\\ 1-r<1/2}} \frac{1}{\left[(1-r)^2+2r(1-\cos\theta)\right]^{3/2}} w(\theta)\, dr\, d\theta \\
&\geq C\frac{h^2}{w(I)}\int_{h\leq|\theta|\leq 1/2}\int_{(1-r)\leq|\theta|} \frac{1}{\left[(1-r)^2+2r(1-\cos\theta)\right]^{3/2}} w(\theta)\, dr\, d\theta \\
&\quad + C\frac{h^2}{w(I)}\int_{1/2\leq|\theta|}\int_{1/2}^{1} \frac{1}{\left[(1-r)^2+2r(1-\cos\theta)\right]^{3/2}} w(\theta)\, dr\, d\theta \\
&= C\frac{h^2}{w(I)}[\mathrm{I}+\mathrm{II}]
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I} &\geq \int_{h\leq|\theta|\leq 1/2}\frac{1}{|\theta|}\int_{1-r\leq|\theta|} \frac{(1-r)\, dr}{\left[(1-r)^2+2r(1-\cos\theta)\right]^{3/2}} w(\theta)\, d\theta \\
&\geq C\int_{h\leq|\theta|\leq 1/2}\frac{1}{|\theta|}\int_{1-r\leq|\theta|} \frac{(1-r)\, dr}{\left[(1-r)^2+\theta^2\right]^{3/2}} w(\theta)\, d\theta \\
&= C\int_{h\leq|\theta|\leq 1/2}\frac{1}{|\theta|}\left.\frac{1}{\left[(1-r)^2+\theta^2\right]^{1/2}}\right|_{1-r=0}^{1-r=|\theta|} w(\theta)\, d\theta \\
&= C\int_{h\leq|\theta|\leq 1/2}\frac{w(\theta)}{\theta^2}\, d\theta
\end{aligned}$$

and similarly,

$$\mathrm{II} \geq C\int_{1/2\leq|\theta|}\int_{1/2}^{1}\frac{1}{|\theta|^3} w(\theta)\, dr\, d\theta \geq C\int_{1/2\leq|\theta|}\frac{w(\theta)}{|\theta|^2}\, d\theta.$$

Combining I and I gives

$$\frac{h^2}{w(I)}\int_{|\theta|\geq h}\frac{w(\theta)}{\theta^2}\, d\theta \leq C.$$

The theorem is proved.

7. $B(\rho)$ and $S(\rho)$

By analogy with B_2, we define a class of functions b_2 as follows.

DEFINITION 7.1. A function ρ: $[0, \infty) \to \mathbf{R}$ is said to be in the class b_2 if $\rho(0) = 0$, ρ is increasing and

$$\int_h^1 \frac{\rho(t)}{t^3}\, dt \le C\frac{\rho(h)}{h^2}$$

with C independent of h. Suppose $\rho(t)/t$ is in the Lebesgue space $L^1(T)$. Let $\sigma(t) = \int_0^t \rho(s)/s\, ds$. Then ρ is Dini iff $\sigma(t) \le C\rho(t)$.

LEMMA 7.2. *Let $\rho(t)/t \in L^1(T)$ with $\rho \in b_2$. Then σ satisfies the doubling condition $\sigma(2h) \le \sigma(h) + C\rho(h)$ where C is an absolute constant.*

Proof.

$$\int_h^{2h} \frac{\rho(t)}{t}\, dt = h^2 \int_h^{2h} \frac{\rho(t)}{th^2}\, dt \le 4h^2 \int_h^{2h} \frac{\rho(t)}{t^3}\, dt \le C\rho(h)$$

since $\rho \in b_2$. Hence,

$$\sigma(2h) = \sigma(h) + \int_h^{2h} \frac{\rho(t)}{t}\, dt \le \sigma(h) + C\rho(h).$$

DEFINITION 7.3. An analytic function F on the unit disk D is in the class $S(\rho)$ if and only if

$$\|F\|_{S(\rho)} = |F(0)| + \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} |F'(re^{i\theta})| \frac{\rho(1-r)}{1-r}\, d\theta\, dr < \infty.$$

We have the following theorem which is analogous to Theorem 6.3 above.

THEOREM 7.4. *$B(p) \subseteq S(\rho)$ if and only if $\rho \in b_2$ and ρ is Dini.*
This means that if $f \in B(\rho)$ and

$$F(z) = \frac{1}{2\pi}\int_T \frac{e^{it}+z}{e^{it}-z} f(t)\, dt$$

then $F \in S(\rho)$, and this inclusion is continuous.

Proof. First suppose $\rho \in b_2$ and ρ is Dini. We follow the proof of Theorem 6.3. Look at

$$b(t) = \frac{1}{\rho(h)}\left[\chi_{[0,h]}(t) - \chi_{[-h,0)}(t)\right]$$

and

$$F(z) = \frac{1}{2\pi}\int_T \frac{e^{it}+z}{e^{it}-z} b(t)\,dt.$$

Then

$$F'(z) = \frac{1}{i\pi\rho(h)}\left[\frac{1}{z-e^{ih}} + \frac{1}{z-e^{-ih}} + \frac{2}{1-z}\right].$$

Again let $D_1 = \{z \in D; |1-z| \geq 2h\}$. On D_1,

$$|F'(z)| \leq C\frac{h^2}{\rho(h)|1-z|^3}.$$

Let N be the smallest integer with $2^N h \geq 1$. Then

$$\begin{aligned}
&\iint_{D_1} |F'(re^{i\theta})| \frac{\rho(1-r)}{1-r}\,dr\,d\theta \\
&\leq C\frac{h^2}{\rho(h)} \sum_{n=0}^{N} \iint_{2^n h \leq |1-z| \leq 2^{n+1}h} \frac{1}{|1-z|^3} \cdot \frac{\rho(1-r)}{1-r}\,dr\,d\theta \\
&\leq C\frac{h^2}{\rho(h)} \sum_{n=0}^{N} (2^n h)^{-3} \int_{1-2^{n+1}h}^{1} \int_{-2^{n+1}h}^{2^{n+1}h} \frac{\rho(1-r)}{1-r}\,d\theta\,dr \\
&\leq C\frac{h^2}{\rho(h)} \sum_{n=0}^{N} (2^n h)^{-2} \int_0^{2^{n+1}h} \frac{\rho(t)}{t}\,dt \\
&= C\frac{h^2}{\rho(h)} \sum_{n=0}^{N} (2^n h)^{-2} \sigma(2^{n+1}h) \\
&\leq C\frac{h^2}{\rho(h)} \sum_{n=0}^{N} (2^n h)^{-2} \rho(2^n h) \quad \text{by Lemma 7.2 and Dini} \\
&\leq C\frac{h^2}{\rho(h)} \sum_{n=0}^{N} (2^n h)^{-3} \int_{2^n h}^{2^{n+1}h} \rho(2^n h)\,dt \\
&\leq C\frac{h^2}{\rho(h)} \int_h^2 \frac{\rho(t)}{t^3}\,dt \\
&\leq C \quad \text{by the } b_2 \text{ condition.}
\end{aligned}$$

On $D \setminus D_1$, the complement of D_1 relative to D, as in Theorem 6.3, it will suffice to bound

$$\int\int_{D,\,|1-z|\le 4h} \frac{\rho(1-r)}{\rho(h)(1-r)|1-z|}\,d\theta\,dr$$

$$\le \frac{1}{\rho(h)} \sum_{n=0}^{\infty} \int\int_{2^{-n-1}(4h)<|1-z|<2^{-n}(4h)} \frac{1}{|1-z|} \cdot \frac{\rho(1-r)}{1-r}\,d\theta\,dr$$

$$\le \frac{C}{\rho(h)} \sum_{n=0}^{\infty} \frac{2^n}{h} \int_{1-2^{-n}h}^{1} \int_{-2^{-n}h}^{2^{-n}h} \frac{\rho(1-r)}{1-r}\,d\theta\,dr$$

$$\le \frac{C}{\rho(h)} \sum_{n=0}^{\infty} \int_0^{2^{-n}h} \frac{\rho(t)}{t}\,dt$$

$$\le \frac{C}{\rho(h)} \sum_{n=0}^{\infty} \rho(2^{-n}h) \quad \text{by Dini's condition.}$$

$$\le \frac{C}{\rho(h)} \sum_{2^{-n}h}^{\infty} \int_{2^{-n}h}^{2^{1-n}h} \rho(2^{-n}h)\frac{dt}{t}$$

$$\le \frac{C}{\rho(h)} \int_0^{2h} \frac{\rho(t)}{t}\,dt$$

$$= \frac{C}{\rho(h)}\sigma(2h)$$

$$\le \frac{C}{\rho(h)}[\sigma(h) + C\rho(h)] \quad \text{by Lemma 7.2}$$

$$\le C \quad \text{by Dini's condition.}$$

For the converse, again let

$$b(t) = \frac{1}{\rho(h)}\left[\chi_{[0,h]}(t) - \chi_{[-h,0)}(t)\right]$$

and

$$F(z) = \frac{1}{2\pi}\int_T \frac{e^{it}+z}{e^{it}-z}\,b(t)\,dt.$$

So we have

$$\int\int_D |F'(re^{i\theta})| \frac{\rho(1-r)}{1-r}\, dr\, d\theta < C.$$

In fact, for $|1-z| \geq h$,

$$|F'(z)| \geq \frac{Ch^2}{\rho(h)|1-z|^3}.$$

Hence

$$\int_{1-r>h}\int_{|\theta|\geq 1-r} \frac{h^2}{\rho(h)|1-z|^3} \frac{\rho(1-r)}{1-r}\, d\theta\, dr \leq C.$$

For $|\theta| \geq 1-r$,

$$1-\cos\theta \leq \theta^2/2 \quad \text{and} \quad |1-z| = \left[(1-r)^2 + 2r(1-\cos\theta)\right]^{1/2} \leq \sqrt{3}\,\theta.$$

So

$$C \geq \frac{h^2}{\rho(h)} \int_{1-r\geq h} \frac{\rho(1-r)}{1-r} \int_{1-r}^{\pi} \frac{d\theta}{\theta^3}\, dr$$

so that

$$\frac{h^2}{\rho(h)} \int_h^1 \frac{\rho(t)}{t^3}\, dt$$

is bounded and hence $\rho \in b_2$. Now if $|1-z| \leq h/4$,

$$|F'(z)| \geq \frac{C}{\rho(h)|1-z|}.$$

Hence

$$C \geq \frac{1}{\rho(h)} \int\int_{D,\,|1-z|\leq h/4} \frac{1}{|1-z|} \cdot \frac{\rho(1-r)}{1-r}\, d\theta\, dr.$$

Here we consider $|\theta| \leq 1-r$. So

$$1-\cos\theta \leq \theta^2/2 \leq (1-r)^2/2 \quad \text{and} \quad |1-z| \leq \sqrt{2}\,(1-r).$$

So $|1 - z| \leq h/4$ provided $1 - r \leq h/4\sqrt{2}$, and so

$$
\begin{aligned}
C &\geq \frac{1}{\rho(h)} \int_{1-r\leq h/4\sqrt{2}} \int_{|\theta|\leq 1-r} \frac{\rho(1-r)}{(1-r)|1-z|}\, d\theta\, dr \\
&\geq \frac{1}{\sqrt{2}\,\rho(h)} \int \frac{\rho(1-r)}{(1-r)^2} \int_{1-r\geq h/4\sqrt{2}} \int_{|\theta|\leq 1-r} d\theta\, dr \\
&= \frac{\sqrt{2}}{\rho(h)} \int_0^{h/4\sqrt{2}} \frac{\rho(t)}{t}\, dt.
\end{aligned}
$$

So as $\rho(t)/t \in L^1(T)$ and $\sigma(h/4\sqrt{2}) \leq C\rho(h)$. By Lemma 7.2, slightly modified,

$$
\begin{aligned}
\sigma(h) &\leq \sigma\left(\frac{h}{4\sqrt{2}}\right) + C\rho\left(\frac{h}{4\sqrt{2}}\right) \\
&\leq \sigma\left(\frac{h}{4\sqrt{2}}\right) + C\rho(h) \\
&\leq C\rho(h)
\end{aligned}
$$

So ρ satisfies Dini's condition. The theorem is proved.

8. Facts about $\Lambda_*(\rho)$

LEMMA 8.1. *Let $\rho \in b_2$ and $u \in \Lambda_*(\rho)$. Let $P(r, t)$ denote the Poisson kernel, and let*

$$
f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, t) u(\theta - t)\, dt, \quad z = re^{i\theta}.
$$

Then

$$
|f_{\theta\theta}(re^{i\theta})| \leq C \frac{\rho(1-r)}{(1-r)^2},
$$

where $f_{\theta\theta}$ is the second derivative with respect to θ.

Proof. Consider the Poisson kernel

$$
P(r, t) = \frac{1-r}{1 - 2r\cos t + r^2} \quad \text{on} \quad [0, \pi].
$$

Now P_{tt} is an even function of t and changes sign exactly once on the interval $[0, \pi]$, at a point α. We can choose r sufficiently near 1 to force

$\alpha < 1 - r$ [18, p. 109]. So

$$\begin{aligned} f_{\theta\theta}(re^{i\theta}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{tt}(r, \theta - t)u(t)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{tt}(r, t)u(\theta - t)\,dt \\ &= \frac{1}{2\pi}\int_{0}^{\pi} P_{tt}(r, t)[u(\theta + t) - u(\theta - t)]\,dt \end{aligned}$$

by the evenness of P_{tt}. Also

$$\int_0^{\pi} P_{tt}(r, t)\,dt = P_t(r, \pi) - P_t(r, 0) = 0.$$

Hence

$$f_{\theta\theta}(re^{i\theta}) = \frac{1}{2\pi}\int_0^{\pi} P_{rr}(r, t)[u(\theta + t) + u(\theta - t) - 2u(\theta)]\,dt.$$

By the choice of α and by the fact that $u \in \Lambda_*(\rho)$ we have

$$|f_{\theta\theta}(re^{i\theta})| \leq C\int_0^{\alpha} \rho(t)[-P_{tt}(r, t)]\,dt + C\int_{\alpha}^{\pi} \rho(t)P_{tt}(r, t)\,dt.$$

Now,

$$\begin{aligned} \int_0^{\alpha} -\rho(t)P_{tt}(r, t)\,dt &= -\rho(\alpha)P_t(r, \alpha) + \int_0^{\alpha} P_t(r, t)\,d\rho(t) \\ &\leq -\rho(\alpha)P_t(r, \alpha), \end{aligned}$$

since $P_t < 0$ and $d\rho > 0$. But,

$$-\rho(\alpha)P_t(r, \alpha) = \rho(\alpha)\frac{2r\sin\alpha(1 - r^2)}{[1 - 2r\cos\alpha + r^2]^2} \leq C\frac{\rho(1 - r)}{(1 - r)^2}$$

using $\alpha < 1 - r$. Hence

$$\begin{aligned} \int_{\alpha}^{\pi} \rho(t)P_{tt}(r, t)\,dt &= -\rho(\alpha)P_t(r, t) - \int_{\alpha}^{\pi} P_t(r, t)\,d\rho(t) \\ &\leq C\frac{\rho(1 - r)}{(1 - r)^2} + \int_{\alpha}^{\pi} [-P_t(r, t)]\,d\rho(t). \end{aligned}$$

So we must estimate this last integral. Let $\beta = 1 - r$, for $t \leq \beta$, $-P_t(r, t) \leq C\beta/(1 - r)^3$ and so $\int_\alpha^\beta - P_t(r, t)\,d\rho(t) \leq C\beta/(1 - 3)^3[\rho(\beta) - \rho(\alpha)] \leq C\rho(1 - r)/(1 - r)^2$. For $t \geq \beta, 1 - \cos t \geq t^2/\pi$, so that $-P_t(r, t) \leq C/t^3$, and

$$\begin{aligned}
\int_\beta^\pi - P_t(r, t)\,d\rho(t) &\leq C\int_\beta^\pi \frac{d\rho(t)}{t^3} \\
&\leq C\left[\frac{\rho(t)}{t^2}\bigg|_\beta^\pi + 2\int_\beta^\pi \frac{\rho(t)}{t^3}\,dt\right] \\
&\leq C\frac{\rho(\beta)}{\beta^2} \quad \text{since } \rho \text{ satisfies the } b_2 \text{ condition} \\
&= C\frac{\rho(1 - r)}{(1 - r)^2}.
\end{aligned}$$

Thus the lemma is proved.

LEMMA 8.2. *Suppose $\rho \in b_2$ and f is analytic in D with*

$$|f'(re^{i\theta})| \leq C\frac{\rho(1 - r)}{(1 - r)^3}.$$

Then

$$|f(re^{i\theta})| \leq C\frac{\rho(1 - r)}{(1 - r)^2}.$$

Proof. Notice $f(re^{i\theta}) = f(0) + \int_0^r f'(te^{i\theta})e^{i\theta}\,dt$, so that

$$\begin{aligned}
|f(re^{i\theta})| &\leq |f(0)| + C\int_0^r \frac{\rho(1 - t)}{(1 - t)^3}\,dt = |f(0)| + C\int_{1-r}^1 \frac{\rho(s)}{s^3}\,ds \\
&\leq |f(0)| + C\frac{\rho(1 - r)}{(1 - r)^2} \quad \text{by } b_2.
\end{aligned}$$

Notice that the b_2 condition also implies that $\rho(1 - r)/(1 - r)^2$ is bounded below and so the lemma follows.

THEOREM 8.3. *Let $\rho \in b_2$, $g \in \Lambda_*(\rho)$. Let*

$$f(z) = \frac{1}{2\pi}\int_T \frac{e^{it} + z}{e^{it} - z}g(t)\,dt.$$

Then

$$|f_{\theta\theta}(re^{i\theta})| \le C\frac{\rho(1-r)}{(1-r)^2}.$$

Proof. Write $f = u + iv$, where u is the harmonic extension of g into D. Then a simple series comparison shows that $f_{\theta\theta} = u_{\theta\theta} + iv_{\theta\theta}$ with $f_{\theta\theta}$ analytic. Let $s = \frac{1}{2}(1 - r)$. Then

$$f_{\theta\theta}(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{se^{it}+z}{se^{it}-z}u_{\theta\theta}(se^{it})\,dt$$

so

$$\begin{aligned}|f'_{\theta\theta}(re^{i\theta})| &\le \frac{1}{\pi}\int_0^{2\pi}\frac{|u_{\theta\theta}(se^{it})|}{|se^{it}-z|^2}\,dt\\
&\le \frac{C}{2\pi}\int_0^{2\pi}\frac{\rho(1-s)}{(1-s)^2}\frac{1}{s^2-2sr\cos(\theta-t)+r^2}\,dt \quad \text{by Lemma 8.1}\\
&= C\frac{\rho(1-s)}{(1-s)^2(s^2-r^2)}\frac{1}{2\pi}\int_0^{2\pi}\frac{s^2-r^2}{s^2-2rs\cos(\theta-t)+r^2}\,dt\\
&= C\frac{\rho(1-s)}{(1-s)^2(s^2-r^2)}\\
&\le C\frac{\rho(1-r/2)}{(1-r)^3}\\
&\le C\frac{\rho(1-r)}{(1-r)^3}.\end{aligned}$$

This lemma now follows from Lemma 8.2.

9. The isomorphism between $B(\rho)$ and $S(\rho)$

In this section we shall prove that $B(\rho)$ is identifiable with $S(\rho)$ in the following sense. If $f \in B(\rho)$ then the function F defined by

$$F(z) = \frac{1}{2\pi}\int_T\frac{e^{it}+z}{e^{it}-z}f(t)\,dt$$

belongs to $S(\rho)$, and moreover $\|F\|_{S(\rho)} \le M\|f\|_{B(\rho)}$, where M is an absolute constant. Conversely if a function f belongs to $S(\rho)$ and we let

$$\lim_{r\to 1}\operatorname{Re}F(re^{i\theta}) = f(\theta)$$

then $f(\theta)$ belongs to $B(\rho)$ and moreover $\|f\|_{B(\rho)} \leq N\|F\|_{S(\rho)}$ where N is an absolute constant. Therefore the operator A: $B(\rho) \to S(\rho)$ defined by $A(f) = F$, where F is as above is a Banach space isomorphism. Namely we have;

THEOREM 9.1. *Let ρ be in the class b_2 and also be Dini. Then $f \in B(\rho)$ if and only if $F \in S(\rho)$ where*

$$F(z) = \frac{1}{2\pi}\int_T \frac{e^{it}+z}{e^{it}-z} f(t)\, dt.$$

Moreover there exist positive absolute constants M and N such that

$$M\|f\|_{B(\rho)} \leq \|F\|_{S(\rho)} \leq N\|f\|_{B(\rho)}.$$

Proof. Let $F \in S(\rho)$ with power series $F(z) = \sum_{n=0}^{\infty} a_n z^n$. Let $G(z) = \sum_{n=0}^{\infty} b_n z^n$ be the analytic extension of a function g in $\Lambda_*(\rho)$. Define a linear functional on $S(\rho)$ by

$$\Lambda F = \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} F(re^{i\theta}) G'(re^{-i\theta})\, d\theta.$$

We are going to show that Λ belongs to $S^*(\rho)$.

By Theorem 8.3, we have

$$\left|G_{\theta\theta}(re^{i\theta})\right| \leq C\frac{\rho(1-r)}{(1-r)^2}$$

where $C = K\|G\|_{\Lambda_*(\rho)}$. Now

$$\frac{1}{2\pi}\int_0^{2\pi} F(re^{i\theta})G'(re^{-i\theta})\, d\theta = \sum_{n=0}^{\infty}(n+1)a_n b_{n+1} r^{2n}.$$

So $\Lambda(F) = \sum_{n=0}^{\infty}(n+1)a_n b_{n+1}$, and a power series computation shows that

$$\Lambda(F) = a_0 b_1 + \frac{1}{\pi}\iint_D F'(re^{i\theta})G_{\theta\theta}(re^{-i\theta})\frac{r^2-1}{r}e^{2i\theta}\, d\theta\, dr$$

and

$$|\Lambda(F)| \leq |a_0|\cdot|b_1| + C\iint_D |F'(re^{i\theta})|\,|G_{\theta\theta}(re^{-i\theta})|\frac{1-r}{r}\, d\theta\, dr.$$

Now

$$\frac{1}{r} G_{\theta\theta}(re^{-i\theta}) = \sum_{n=1}^{\infty} n^2 b_n r^{n-1} e^{-in\theta} \to b_1 \quad \text{as} \quad r \to 0.$$

So,

$$|\Lambda(F)| \le |a_0| \cdot |b_1| + C(|b_1|, \|G\|_{\Lambda_*(\rho)}) \int\int_D |F'(re^{i\theta})| \frac{\rho(1-r)}{1-r} \, d\theta \, dr$$

or

$$|\Lambda(F)| \le C(|b_1|, \|G\|_{\Lambda_*(\rho)}) \|F\|_{S(\rho)}.$$

where $C(|b_1|, \|G\|_{\Lambda_*(\rho)})$ is a constant which depends on $|b_1|$ and $\|G\|_{\Lambda_*(\rho)}$.

Now suppose $h \to \phi(h)$, $\phi \in B^*(\rho)$. Then there exists a $g \in \Lambda_*(\rho)$ with Poisson extension $g = P_r * g$ and with

$$\phi(h) = \lim_{r \to 1} \int_T h(x) g_r'(x) \, dx.$$

Since $h(x) = 1/2\pi \in B(\rho)$,

$$\left| \phi\left(\frac{1}{2\pi} \right) \right| = \left| \lim_{r \to 1} \frac{1}{2\pi} \int_T g_r'(x) \, dx \right| \le C \|\phi\|_{B^*(\rho)}.$$

But notice that

$$b_1 = \lim_{r \to 1} \frac{1}{2\pi} \int_T g_r'(x) \, dx,$$

b_1 as in the discussion above. Therefore if $\phi \in B^*(\rho)$ with associated g, the linear functional Λ above is in $S^*(\rho)$ with $\|\Lambda\|_{S^*(\rho)} \le C\|\phi\|_{B^*(\rho)}$. Therefore we have a continuous embedding $B^*(\rho) \subseteq S^*(\rho)$. Since $B(\rho) \subseteq S(\rho)$ continuously, we have the following result.

THEOREM 9.2. *$B(\rho)$ is isomorphic as a Banach space to $S(\rho)$.*

Notice that we have the following situation; the spaces $B(\rho)$ and $S(\rho)$ have the same duals and moreover the mapping A: $B(\rho) \to S(\rho)$ defined by $A(f) = F$ is one-to-one so $B(\rho)$ is regarded as a dense subset of $S(\rho)$, so that classic theorem in functional analysis ensures us that $B(\rho)$ and $S(\rho)$ are equivalent as Banach spaces.

The Hilbert transform of a real valued function on T is defined as the Cauchy principal value of the integral

$$\tilde{f}(x) = PV\frac{1}{\pi}\int_T \frac{f(t)}{2\tan(t-x)/2}\,dt$$

whenever it exists. The function $\tilde{f}$ is also often called the conjugate function of the function f, or the conjugate operator.

One consequence of Theorem 9.1 is that $B(\rho)$ spaces are invariant under conjugation. This can be precisely stated as follows.

COROLLARY 9.3. *If $f \in B(\rho)$, then $\tilde{f} \in B(\rho)$. Moreover $\|\tilde{f}\|_{B(\rho)} \leq M\|f\|_{B(\rho)}$ where M is an absolute constant.*

Proof. If $f \in B(\rho)$ then

$$F(z) = \frac{1}{2\pi}\int_T \frac{e^{it}+z}{e^{it}-z} f(t)\,dt$$

belongs to $S(\rho)$, and $\lim_{r\to 1} iF(re^{i\theta}) = \tilde{f}(\theta)$. So by Theorem 9.1, $\tilde{f} \in B(\rho)$ and $\|\tilde{f}\|_{B(\rho)} \leq C\|iF\|_{S(\rho)}$ so we can conclude that $\|\tilde{f}\|_{B(\rho)} \leq C\|f\|_{B(\rho)}$.

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