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A COMPLETE CHARACTERIZATION OF STOCHASTIC PROCESSES DEFINED ON REGULAR SEMIGROUPS WITH APPLICATIONS TO STOCHASTIC MODELS

BY

P.B. CERRITO

1. Introduction

It is the purpose of our paper to show that the whole is greater than the sum of its parts. Numerous papers have been written concerning various properties of probability functions defined on semigroups. The simplest structures of discrete and compact semigroups were considered first (see Martin-Lof [7] and Mukherjea and Tserpes [11]). Then the completely simple case was dealt with for properties of random walks (see [11] and [1]). The completely regular case has been shown by the author in [2]. Also, the special case of matrix semigroups has been studied relying primarily on the properties of rank (see Hognas and Mukherjea [5]). We intend to show that these separate studies can be combined into one harmonious concept. By doing so, the properties become far more flexible and can be used to prove some unsolved conjectures.

In order to do so, we first need to define the necessary terminology. Most of this information is available in Clifford and Preston [3], Paalman-deMiranda [12], Petrich [13], and Mukherjea and Tserpes [11]. Let S be a locally compact, Hausdorff, second countable topological semigroup. An element $x \in S$ is regular provided that there exists some $y \in S$ such that $x = xyx$. S is completely regular if $x = xyx$ and $xy = yx$; S is a union of maximal pairwise disjoint subgroups. The properties of S are listed in Clifford and Preston [3].

S is a completely simple semigroup if it contains no proper ideals and contains an idempotent minimal with respect to the partial ordering $e \leq f$ if $ef = fe = e$. If S is completely simple then we can write $S = X \times G \times Y$ where $G = eS \cap Se = eSe$, $X = E(Se)$, and $Y = E(eS)$ where e is a minimal idempotent and the notation $E(A)$ denotes the set

$$\{ f \in A \colon f \text{ is idempotent} \}.$$

Note that $S = \bigcup_{g_x, g_y} \{g_x\} \times G \times \{g_y\}$ is a union of maximal disjoint semigroups and hence it is also completely regular.

We intend to consider several probabilistic concepts in this paper. Therefore it is also necessary to define these terms for semigroups. Let X have law μ. We

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denote the n-fold convolution iterate by μ^n. If $X_1, X_2, \ldots, X_n$ are iid of law μ then $Z_n = X_1 X_2 \ldots X_n$ is a right random walk of law μ. A left random walk $L_n = X_n X_{n-1} \ldots X_1$ is defined similarly. Although any theorem involving Z_n has its analog for L_n, they can have very different properties depending on the algebraic structure of S. As shown in Martin-Lof [7], a Markov chain defined on a semigroup can be defined in the usual way so that we can use the standard notation. In particular, x is recurrent if $P_x(Z_n \in N_x \text{ i.o.}) = 1$ for any neighborhood N_x of x. For $x \in S$, x is essential only if $xS \subset Sx$. Since there exists $f \in E(S)$ such that $xS = fS$, if there exists any $e \in S$ for which $ef = e = fe$ then x cannot be essential. Thus a necessary condition for x to be recurrent is that x be essential and its corresponding idempotent be minimal. The condition is not sufficient.

Finally, we assume that the support of the process generates S. That is if $S^* = \{y$: given a neighborhood N of y there exists some n such that $P\{(Z_n \in N) > 0\}$ then we assume $S = S^*$. If this is not the case then we can get identical results by replacing S by S^*, showing e is minimal with respect to $E(S^*)$, and redefining neighborhoods so that $N = N^* \cap S^*$. The arguments are not anymore difficult but become bulky in terms of added notation.

2. Completely Regular Semigroups

The behavior of random walks defined on a completely regular semigroup S was shown in Cerrito [2]. To get these results, the minimality of the idempotents was exploited to demonstrate that this behavior depends completely on the corresponding groups; the study of which can be found in Revuz [14]. Therefore the extension to continuous time will be readily apparent.

We define $\{X(t)\colon t \geq 0\}$ to be a stochastic process. In particular, we let $X(0)$ be the initial state of the process and let

$$T_A = \inf_t \left(t > 0\colon X(t) \in A\right),$$

and for any neighborhood N_y of $y \in S$ let $P_{xy}(N_y) = P_y(T_{N_y} < \infty)$. Then x is recurrent if and only if $\rho_x(N_x) = 1$ for every neighborhood of x in G.

Since S is completely regular, we can write $S = \bigcup_{g \in E(S)} G_g$ where G_g is a topological group for all $g \in E(S)$. Let $x \in G_g$. We want to determine the various properties of $X(t)$. As we will show, these properties depend very heavily on the algebraic structure of S. If x is essential then $gS \subset Sg$ and g is a minimal idempotent. It is clear that no non-essential element can be recurrent since

$$P_x\left(X(t) \in (S_g)^c\right) = P_x\left(X(t) \notin gS \cap (S_g)\right) > 0$$

and

$$\rho_{xx}(N_x) = \sum_{y \in S_g} \rho_{xy}(S_g)\rho_{yx}(N_x) < 1.$$

Thus we consider G_g where g is a minimal idempotent and for any neighborhood N_x of x,

$$P_x\big(X(t) \in N_x\big) = P_x\big(X(t) \in N_x \cap G_g\big).$$

Therefore we can define a corresponding process $Y(t)$ on G_g so that for any measurable $A \subset G_g$, $A = A^* \cap G_g$, and $Y_g(t) = gX(t)g$. Then for $y \in G_g$,

$$P_y\big(Y_g(t) \in A\big) = P_y\big(gX(t)g \in A\big) = P\big(ygX(t)g \in A\big) = P_y\big(X(t) \in A^*\big)$$

since $yg = y$. Therefore the properties of $X(t)$ are identical with those of $Y(t)$ if the initial state $X(0)$ of the process lies in G_g. The converse is also true. That is, if for $g \in E(S)$ we define a stochastic process $Y_g(t)$ on G_g then for any random variable $X(0)$ defined on S of law Π_0 we define $X(t)$ on S so that

$$P_x\big(X(t) \in A\big) = P_x\big(Y_g(t) \in A\big) \quad \text{for} \quad x \in G_g.$$

That is, if $Y_g(t)$ has law $\mu_g(t)$ then $\mu(t) = \Pi_0(t)Y_g(t)$ for $X(t)$ of law μ.

If $g \in E(S)$ such that g is not minimal in $E(S)$ then for any element $x \in G_g$, x must be transient with respect to $X(t)$ regardless of the recurrence properties of $Y_g(t)$. If g is minimal then x is recurrent with respect to $X(t)$ if and only if it is recurrent with respect to $Y_g(t)$. Thus

$$S = T \cup R$$

where

$$T = \{G_g\colon g \text{ is not minimal}\}$$

and

$$R = \{G_g\colon g \text{ is minimal}\}$$

and any stochastic process $X(t)$ defined on S is reducible to R, a completely simple and completely regular subsemigroup for S. Also, if $g \in E(T)$ either there exists $e \in R$ such that $e \leq g$ or there exists a sequence (f_n) in $E(t)$ such that $g = f_n$ for some n and $f_1 \geq \cdots \geq f_n \geq \ldots$. In this case, $S_g \subset T$ and $P_g(X(t) \in T) = 1$ so that transience becomes a permanent condition. Thus we can write $S = T_1 \cup T_2 \cup R$ where no element in T_1 is a minimal idempotent and for any $g \in E(T_2)$ there exists $e \in E(R)$ such that $e \leq g$. Therefore we have the following theorem:

THEOREM 2.1. *Let $S = T_1 \cup T_2 \cup R$ be completely regular where T_1, T_2, R are defined as above and let $(X(t): t \geq 0)$ be any stochastic process defined on S. Then for any $g \in E(R)$, $Y_g(t)$ defined by*

$$P(Y_g(t) \in A) = P(X(t) \in A \cap G_g)$$

defines a stochastic process on the topological group G_g such that for any $x \in G_g$, $\rho_{xx} = 1$ with respect to $X(t)$ if and only if $\rho_{xx} = 1$ with respect to $Y_g(t)$. Moreover no element of R^c can be recurrent. However, for any $y \in T_2$, $P_y(T_R < \infty) = 1$ whereas for any $y \in T_1$, $P_y(X(t) \in T_1) = 1$ so that T_2 consists of finite transience and T_1 consists of infinite transience.

3. Completely simple semigroups

Everything that is known concerning the properties of random walks defined on completely simple semigroups can be found in Mukherjea and Tserpes [11]. However, since S is also completely regular, all the properties of random walks found in Cerrito [1] remain valid. There is one conjecture involving $S = X \times G \times Y$ that remains unsolved although several attempts have been made for G compact (see Mukherjea, Sun, and Tserpes [10]) and G discrete (see Larisse [6]). By using the results for completely regular semigroups, the conjecture can now be proven:

THEOREM 3.1. *Let $S = X \times G \times Y$ be a completely simple semigroup. Then S is recurrent if and only if G is recurrent.*

Proof. Let $Z_n = X_1 X_2 \ldots X_n$ be a recurrent random walk of law μ defined on S. Then there exists $x \in S$ such that $x \to x$ i.o. There exists f idempotent in S such that $xS = fS \subset Sf$ since a recurrent element must be essential. Therefore fS is a group such that $P_x(Z_n \in fS) = 1$ for all n. For any $y \in S$, $xy \in fS$ so that $xyf = fxy = xy$. Define $Y_i = X_i f$. Let N_x be a neighborhood of x in fS, $N_x = N_x^* \cap fS$. Then

$$\begin{aligned} P_x(Y_1 Y_2 \ldots Y_n \in N_x \text{ i.o.}) &= P_x(X_1 f X_2 f \ldots X_n \in N_x^* \cap fS) \\ &= P_x(X_1 X_2 \ldots X_n \in N_x^* \text{ i.o.}) = 1. \end{aligned}$$

Also for f to be essential, f must be minimal with respect to the partial ordering on the idempotents of S. Therefore $fS \cong G$ and G is recurrent.

Now consider a recurrent measure σ defined on G. Write

$$S = \bigcup_{x, y} \{g_x\} \times G \times \{h_y\}.$$

For now assume (g_x, h_y) is a countable set. Let x be a recurrent element in G but $x \notin eS \cap Se$. Then for any measurable set contained in

$$G_{xy} = \{g_x\} \times G \times \{h_y\}$$

we define $\sigma_{xy}(A) = \delta(eAe)$. Note that since e is a minimal idempotent, $eAe \subset eSe = G$. Let (ε_{xy}) be a sequence of constants such that $\sum_{x,y}\varepsilon_{xy} = 1$. Then we define a measure μ on S such that

$$\mu(A) = \sum_{x,y} \sigma_{xy}(A_{xy})\varepsilon_{xy}$$

for any measurable set A in S, $A = \bigcup_{x,y}(g_x) \times A_{xy} \times (h_y)$. Clearly, $\mu(A) > 0$ for all $A \subset S$ and

$$\mu(S) = \sum_{x,y} \sigma_{xy}(A_{xy})\varepsilon_{xy} = \sum_{x,y} 1\varepsilon_{xy} = 1$$

so that μ is well defined. Note also that if the support of σ generates G, then the support of μ generates S. Let $A \subset S$ be a measurable set. Define $Y = eXe$. Then $P(Y \in G) = 1$ so that for $A^* = eAe$,

$$P(Y \in A^*) = P(eXe \in A^*) = P(X \in A) = \mu(A).$$

Since $A = \bigcup_{x,y}(g_x) \times A_{xy} \times (h_y)$,

$$\mu(A) = \sum_{x,y} \sigma_{xy}(A_{xy})\varepsilon_{xy} = \sum_{x,y} \sigma(eA_{xy}e)\varepsilon_{xy} = \sum_{x,y} \sigma(A^*)\varepsilon_{xy} = \sigma(A^*).$$

Therefore $P_x(X_1 \dots X_n \in A) = P_x(Y_1 \dots Y_n \in A^*)$ implies that μ is recurrent on S. Note that the recurrent states of S are equal to $\{fS\colon f \text{ is a minimal idempotent}\}$. But every idempotent is minimal so that every element is recurrent.

Finally, if the set $\{(g_z, h_y)\}$ is uncountable we use a density function defined on (g_x, h_y) for which $\iint f(g_x, h_y)\,dx\,dy = 1$ and let

$$\sigma(A) = \iint \sigma_{xy}(A) f(g_x, h_y)\,dx\,dy$$

and proceed as before. QED

If $S = T_1 \cup T_2 \cup R$ is completely regular, then we have already shown that for any stochastic process $X(t)$ defined on S, $X(t)$ is recurrent if and only if for some $g \in E(R)$, the corresponding process $Y_g(t)$ is recurrent on G_g. We want to generalize Theorem 3.1 above so that if $Y_g(x)$ is defined in G_g then we can construct a recurrent process on S. If $g \in E(R)$ then we can employ the same argument used in Theorem 3.1 to get

$$X_R(t) = \iint \sigma_{xy,t}(A) f(g_x, h_y)\, dx\, dy$$

where

$$\sigma_{xy,t}(A) = \sigma_t(eAe).$$

However, since no element in $T_1 \cup T_2$ can be recurrent we can consider any measures μ_1, μ_2 defined on S for which $\mu_1(T_1) = \mu_2(T_2) = 1$. Then by defining Π_0 so that $\Pi_0(R)$, $\Pi_0(T_1)$, $\Pi_0(T_2) > 0$ we have for any $A \subset S$,

$$P(X(t) \in A) = \Pi_0(R) P(X_R(t) \in A \cap R) + \Pi_0(T_1)\mu_1(A \cap T_1) + \Pi_0(T_2)\mu_2(A \cap T_2)$$

recurrent on S with $R = \{\text{recurrent states}\}$ and we have:

THEOREM 3.2. *Let $S = T_1 \cup T_2 \cup R$ be completely regular and let $R = X \times G \times Y$ be nonempty. Then S is recurrent if and only if G is recurrent.*

We will now generalize Theorem 3.2 to S where S is an arbitrary regular semigroup. If there exists no element $x \in S$ such that $xS \subset Sx$ then S has no essential element and hence no possible recurrence. Therefore we consider $x \in S$ for which $xS \subset Sx$, so that there exists a minimal idempotent $e \in S$ for which $xS = eS \subset Se$. In this case, eS is a two-sided ideal of S. From Clifford and Preston [3], eS is a completely simple semigroup and $eS = Se$.

If $x \in S$ is not essential, it must be transient. If x is essential then we can define the process (Y_n) on eS so that for $A \subset eS$, $A = A \cap eS$, we have $P_x(Y_n \in A) = P_x(X_n \in A^*)$ and (X_n) is recurrent on eS if and only if (Y_n) is recurrent.

If there exists $0 \in S$ such that $0 \cdot x = 0$ for all $x \in S$ then it is clear that $P_0(X = 0) \equiv 1$ for any x so that 0 must be recurrent and absorbing and any process defined on S must be recurrent but not of very much interest. Therefore we only need to consider semigroups for which the zero element has measure zero. In particular, if

$$P\{(x, y): x, y \in S \quad \text{and} \quad xy = 0\} > 0$$

then no element of S can be recurrent. If the above set has zero probability then the zero element can be removed from the discussion by defining $S' = S - \{0\}$. For any regular semigroups, we have the following result:

THEOREM 3.3. *Let S be an arbitrary regular semigroup and let $x \in S$. If $xS \not\subset Sx$ then x cannot be recurrent for any stochastic process defined on S. If*

$xS \subset Sx$ and for all $y \in S$ for which $xy \in E(S)$, xy is not minimal then x is not recurrent. If S does not have a removable zero then zero is the only recurrent element for any process. If the zero is removable then $xS = X_x^* \times G \times Y_x^*$ is completely simple such that $P_x(X_n \in xS) \equiv 1$ for all n and xS is recurrent with respect to a process only if the corresponding process defined on G_x is recurrent.

Again, we can write $S = T_1 \cup T_2 \cup R$ where $E(T_1) = \{g \in E(S)$: there exists $\{f_n\} \in S$ for which $g \geq f_1 \geq f_2 \geq \dots\}$, $E(T_2) = \{g \in E(S)$: there exists $e \in S(R)$ for which $g \geq e\}$, and R is a union of completely simple semigroups. Thus we can extend Theorem 3.2 as follows:

THEOREM 3.4. *Let $S = T_1 \cup T_2 \cup R$ be regular with T_1, T_2, R defined above. Then $x \in S$ is recurrent with respect to $X(t)$ if and only if there exists $E \times G \times F \subset R$ completely simple such that $xS = E \times G \times F$ and x is recurrent with respect to $Y(t)$ where*

$$P(Y(t) \in A) = P(X(t) \in eAe) \quad \text{for} \quad e \in xS.$$

Conversely if there exists some $E \times G \times F \subset R$ which is recurrent with respect to a process defined on G then a recurrent process can be constructed for S.

4. Limits of convolution iterates

In addition to the recurrence properties of random walks, considerable interest in semigroups has centered on the limit properties of convolutions (see Mukherjea and Nakassis [7] and [8] and Larisse [6]). In particular, Csiszar [4] determined just when a limit exists when S is a group. We extend his result to semigroups using the techniques of the previous section.

THEOREM 4.1. *Let $\{\mu_n\}_1^\infty$ be a sequence of probability measures defined on the completely regular semigroup S. Then if X_n has law μ_n, either*

$$\alpha(x) = \lim_{n \to \infty} \sup_{x \in S} P_x(X_1 X_2 \dots X_n \in K) = 0$$

for every compact set $K \subset S$ or there exists a consequence $(a_n)_1^\infty$ of constants in S such that for any $k > 0$, $X_k X_{k+1} \dots X_n a_n$ has a limiting distribution as $n \to \infty$.

Proof. We can write $S = \bigcup_{g \in E(S)} G_g = \bigcup_g (gS \cap Sg)$. Since Gg is closed with respect to the topology on S, for any $x \in G_g$,

$$P_x(X_1 \dots X_n \in K) = P_x(X_1 \dots X_n \in K \cap G_g).$$

Therefore

$$P_x(X_1 \dots X_n \in K) = \sup_{G} \sup_{x \in G_g} P_x(X_1 \dots X_n \in G_g \cap K).$$

Clearly then if for every $g \in E(S)$,

$$\lim_{n \to \infty} \sup_{s \in G_g} P_x(X_1 \dots X_n \in G_g \cap K) = 0$$

for every compact set $K \subset S$ then $\alpha(K) = 0$ for all K. Therefore we assume there exists $g \in E(S)$ for which

$$\lim_{n \to \infty} \sup_{K \in G_g} P_x(X_1 \dots X_n \in G_g \cap K) > 0$$

for some compact $K \subset S$. $Y_n = gX_ng$ for all n. Then for any $K \in G_g$,

$$P_x(X_1 \dots X_n \in G_g \cap K) = P_x(Y_1 \dots Y_n \in G_g \cap K).$$

However (Y_n) is a sequence of random variables defined entirely in G_g such that

$$\alpha_g(K) = \lim_{n \to \infty} \sup_{x \in G_g} P_x(Y_1 \dots Y_n \in K) > 0$$

for some compact $K \subset G_g$. By Csiszar [4], there exists a sequence $\{a_n\} \subset G_g$ such that $Y_k Y_{k+1} \dots Y_n a_n$ has a limiting distribution as $n \to \infty$. Since $Y_k = gX_kg$ and g commutes with every element in G_g we have

$$gX_kgX_{k+1}g \dots gX_nga_n = X_kX_{k+1} \dots X_nga_n$$

and for every K, $X_kX_{k+1} \dots X_nga_n$ has a limiting distribution as $n \to \infty$. QED

Remark 1. From Csiszar [4], either $\alpha(K) = 0$ or $\alpha(K) = 1$. Let X_k have law μ_k. Then we can write

$$\lim_{n \to \infty} \mu_k \mu_{k+1} \dots \mu_n \delta_{a_n} = \sigma_k$$

where δ_x represents point mass at x. Then $\lim_{k \to \infty} \sigma_k = \sigma_\infty$ exists. Moreover, σ_∞ is idempotent so that $\sigma_\infty^2 = \sigma_\infty$ and for any k, $\sigma_k \sigma_\infty = \sigma_\infty$. Therefore σ_∞ is a Haar measure defined on some compact subgroup of $H \subset G_g$ so that this limit is unique given the choice of $g \in E(S)$ and also of H. In addition, the σ_k's are H-uniform and $X_1X_2 \dots X_n$ converges with probability one mod H.

Remark 2. It is possible to extend Theorem 4.1 to the more general regular semigroups. However, more care must be taken since for any $g \in E(S)$, gSg remains a subgroup of S but it ceases to be an ideal. However, for any $g \in E(S)$, we can define $Y_i = gX_ig$. Then for $x \in gSg$,

$$P_x(Y_1 \dots Y_n \in gAg) = P_x(X_1 \dots X_n \in A)$$

for measurable $A \subset S$. Therefore if for all g,

$$\lim_{n \to \infty} \sup_{x \in gSg} P_x(X_1 \dots X_n \in K) = 0$$

for any compact K then we are finished. If not, then

$$1 = \lim_{n \to \infty} \sup_{x \in gSg} P_x(Y_1 \dots Y_n \in gKg) = \lim_{n \to \infty} \sup_{x \in gSg} P_x(X_1 \dots X_n \in K)$$

so that there exists $\{a_n\} \subset gSg$ such that for any k, $Y_kY_{k+1} \dots Y_na_n$ has a limiting distribution as $n \to \infty$. In particular, for any $A \subset S$,

$$P(X_kX_{k+1} \dots X_nga_n \in A) = P(Y_kY_{k+1} \dots Y_na_n \in gAg) \to \sigma_k(gAg) \quad \text{as } n \to \infty.$$

Therefore for every K, $X_kX_{k+1} \dots X_nga_n$ also has a limiting distribution and Theorem 4.1 is also true if completely regular is replaced by regular. These limits have the same properties listed in Remark 1.

We now apply the above result to the special case of stochastic matrices. Let $\{P_n\}$ be a sequence of matrices and define

$$P_{k,n} = P_kP_{k+1} \dots P_n.$$

We want to determine under what conditions $P_{k,n}$ has a limit as $n \to \infty$ and also the behavior of that limit. As we will show, these limits depend primarily on rank. We also provide an algorithm for finding this limit.

For any element P_n in the above sequence, we can define a random variable X_n such that $P(X_n = P_n) = 1$. Then

$$P(X_{k,n} = X_kX_{k+1} \dots X_n = P_{k,n}) = 1.$$

First we dispose of the trivial case. If there exists some k and some n such that $P_{k,n} = 0$ then it is clear that $\lim_{n \to \infty} P_{k \cdot n} = 0$. Therefore we can assume no such k and n exist so that rank $P_{k,n} \geq 1$. We define Δ_r to be the idempotent matrix of rank r which has r ones along the main diagonal and zeros elsewhere. Let S be the semigroup of $m \times m$ matrices. Then for any r, $\Delta_rS\Delta_r$ is a subgroup of S consisting of all matrices with only r nonzero entries along the main diagonal. Note that $\Delta_rS\Delta_r$ is not an ideal of S and S is a regular semigroup that is not completely regular.

Given any r, we can define a product on $\Delta_r S \Delta_r$ so that $Y_{n,r} = \Delta_r X_n \Delta_r$. Then if a_n is the element in the first row, first column of P_n, for $a_{k,n} = a_k a_{k+1} \dots a_n$,

$$P(Y_{k,1} Y_{k+1,1} \dots Y_{n,1} = a_{k,n} \Delta_1) = 1$$

for all k and n. The extension to arbitrary r is immediate. If $a_n \neq 0$ for all n then $P_{\Delta_1 a_{k,n}^{-1}}(Y_{k,1} \dots Y_{n,1} = \Delta_1) = 1$ and we can apply Theorem 4.1. If $a_n = 0$ then we can use elementary row-column operations to achieve the same result at the expense of complications in notation. Therefore, for convenience we assume $a_n \neq 0$. By Theorem 4.1,

$$P_{\Delta_1 a_{k,n}^{-1}}(X_k X_{k+1} \dots X_n \Delta_1 = \Delta_1) \to 1 \quad \text{as } n \to \infty.$$

By using the extension to r, if rank $P_{k,n} \geq r$ for all k and n then

$$P_{\Delta_r K}(X_k X_{k+1} \dots X_n \Delta_r = \Delta_r) \to 1 \quad \text{as } n \to \infty$$

for some matrix K of rank r with the only nonzero elements of K on the main diagonal. The matrix $K = K(k, n)$ depends on both k and n. By Theorem 4.1, $P_{k,n} \Delta_r A_n$ converges to a matrix $M_{k,r}$ such that $A_n \in \Delta_r S \Delta_r$. We want to show that $P_{K,n}$ itself has a limit. We have assumed with no loss of generality that the first r rows and columns of $P_{k,n}$ are independent. Define X' to be the $r \times r$ matrix consisting of the $r \times r$ partition of X for any matrix X. Then

$$P'_{k,n} A'_n \to M'_{k,r}$$

where A'_n, M'_k are diagonal matrices. Note that A'_n depends only on the product of the entries in P_n and not on $P_{k,n}$. Therefore if

$$a^i_{k,n} = (P_1)_{ii} (P_2)_{ii} \dots (P_n)_{ii}$$

converges then we have

$$a^i_{k,n} \to A'_{k,r} M'_{k,r} \quad \text{as } n \to \infty$$

where

$$A'_{k,r} = \lim_{n \to \infty} \left(a^i_{k,n} \right).$$

If P_n is a sequence of stochastic matrices then $P'_{k,n}$ consists of the recurrent states of the corresponding Markov chain thus duplicating the results of [8] as to the properties of the limits. To consider the entire matrix $P_{k,n}$, there exist invertible matrices$Q_{k,n}$, $R_{k,n}$ such that

$$Q_{k,n} P_{k,n} R_{k,n} = \begin{pmatrix} P'_{k,n} & 0 \\ 0 & 0 \end{pmatrix}$$

so that

$$\lim P_{k,n} = \lim Q_{k,n}^{-1}\begin{pmatrix} P'_{k,n} & 0 \\ 0 & 0 \end{pmatrix} R_{k,n}^{-1}$$

which exists provided $\lim Q_{k,n}$ and $\lim R_{k,n}$ exist. Note that the existence of these limits can proceed as outlined above. That is, to find the limit of $Q_{k,n}$ we need only consider the product of the diagonal elements.

Finally, we extend our results to infinite matrices. If $P_{k,n}$ has finite rank, the argument is identical to the above. Therefore assume that $P_{k,n}$ is infinitely dimensional. However, we can also write

$$P_{k,n} = Q_{k,n}^{-1}\begin{pmatrix} P'_{k,n} & 0 \\ 0 & 0 \end{pmatrix} R_{k,n}^{-1}$$

so that the existence of a limit for $P_{k,n}$ depends completely on the existence of limits for $Q_{k,n}$, $P'_{k,n}$, $R_{k,n}^{-1}$ and the generalization to infinite dimensional matrices is identical.

In any case, if $\lim_{n\to\infty} P_{k,n} = Q_k$ exists then by Remark 1,

$$\lim_{k\to\infty} a_k = Q_\infty$$

exists and must equal a Haar measure defined on some compact subgroup of S. However since we are using strictly point mass, this subgroup must consist of exactly one element. Thus

$$Q_\infty = \Delta_r \quad \text{where } r = \min_k(\operatorname{rank} Q_k) = \min_k(r_k).$$

What we have shown in this paper is a complete characterization of recurrence for regular semigroups and also a general means of determining the existence of limits of infinite convolutions.

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UNIVERSITY OF SOUTH FLORIDA
TAMPA, FLORIDA