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UNIQUENESS OF THE PREDUAL OF THE BLOCH SPACE AND ITS STRONGLY EXPOSED POINTS

BY

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Introduction

Let E be a Banach space. We denote by $\| \quad \|_E$ or $\| \quad \|$ the norm of E and denote by E_1, $\operatorname{ext}(E_1)$ and E^*, the closed unit ball of E, the set of extreme points of E_1 and the dual of E respectively.

Let $\mathbf{D}$ denote the open unit disk in the complex plane and for a in $\mathbf{D}$, let ϕ_a denote the Moebius function $\phi_a(z) = (a - z)/(1 - \bar{a}z)$ on $\mathbf{D}$. Then all Moebius functions are represented by $e^{i\theta}\phi_a$ for $0 \le \theta \le 2\pi$ and a in $\mathbf{D}$. If

$$f(z) = \sum_{k=1}^{\infty} \lambda_k \phi_k(z) + c \tag{1}$$

for some λ_k, c in the complex field $\mathbf{C}$ and some Moebius functions ϕ_k with $\sum_{k=1}^{\infty} |\lambda_k| < \infty$, then f is analytic on $\mathbf{D}$ and continuous on $\overline{\mathbf{D}}$. All such functions form a Banach space, denoted by $\mathbf{M}$, with the norm

$$\|f\|_{\mathbf{M}} = \inf\left\{ \sum_{k=1}^{\infty} |\lambda_k| + |c| : (1) \text{ holds} \right\} \tag{2}$$

and $\|f \circ \phi\|_{\mathbf{M}} = \|f\|_{\mathbf{M}}$ holds for any Moebius function ϕ and f in $\mathbf{M}$. The space $\mathbf{M}$ is called the minimal Moebius invariant space and $\mathbf{M}$ as well as the space $\mathbf{M}/\mathbf{C}$, where functions in $\mathbf{M}$ are identified by the difference of constants, is studied in [2] and [3] by J. Arazy, S.D. Fisher and J. Peetre. For example, they proved that

$$\operatorname{ext}(\mathbf{M}_1) = \{\text{Moebius functions}\} \cup \{\text{unimodular constants}\}$$

(see Corollary 9 in [2]).

In this paper we will prove that Moebius functions are not only extreme points of $\mathbf{M}_1$ but also strongly exposed points of $\mathbf{M}_1$ and that unimodular constant functions are not strongly exposed points of $\mathbf{M}_1$.

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DEFINITION. Let x be an element of the unit ball of a Banach space E. A functional x^* in the unit ball of E^* for which $\langle x, x^*\rangle = 1$ is called a *support functional of* E_1 *at* x. Then x is a *strongly exposed* point of E_1 if there is a support functional x^* of E_1 at x satisfying the following condition: for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\operatorname{Re}\langle y, x^*\rangle > 1 - \delta \text{ for } y \text{ in } E_1 \text{ implies } \|y - x\| < \varepsilon$$

By the definition it is clear that strongly exposed points of E_1 are always extreme points of E_1.

We will prove the following result.

THEOREM. *Strongly exposed points of the unit ball of the minimal Moebius invariant space* **M** *are identical to Moebius functions.*

We can have a similar statement for **M**/**C**.

COROLLARY 1. *Moebius functions are strongly exposed points of the unit balls of* **M**/**C**.

An application of the theorem will allow us to determine the unique predual of the Bloch space.

An analytic function f on **D** is called a Bloch function if

$$\rho(f) = \sup_{|z|<1} |f'(z)|(1 - |z|^2) < \infty.$$

Properties of Bloch functions are well known (see [1]). The set of all Bloch functions is denoted by **B**. Then **B**/**C** becomes a Banach space with respect to the norm ρ and this Banach space is called the Bloch space. It is proved in [2] and [3] that (**M**/**C**)* is isometrically isomorphic to the Bloch space and J. Arazy and S.D. Fisher asked in [3] if **M**/**C** is the *unique predual* of the Bloch space. A Banach space E is called the unique predual of E^* if for any Banach space F, F is isometrically isomorphic to E whenever the dual F^* is isometrically isomorphic to E^*. The following corollary will give an affirmative answer to the above question.

COROLLARY 2. *The Banach space* **M**/**C** *is the unique predual of the Bloch space* **B**/**C**.

Strongly exposed points of the unit ball of M

Let $E = \mathbf{M}$ or $\mathbf{B}$ and f be in E. For simplicity we use the same letter f as the element $f + c$ of **M**/**C** containing f. For a pair $\{f, g\}$ of analytic functions on $\mathbf{D}$ with Taylor expansions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

if $\lim_{r\to 1-0}\sum_{n=0}^{\infty} n a_n b_n r^{2n}$ exists, we write this limit by $\langle f, g\rangle$. Then for any f in $\mathbf{M}$ and any g in $\mathbf{B}$, the above limit exists and

$$\langle f, g\rangle = \langle g, f\rangle = \lim_{r\to 1-0} \frac{1}{\pi}\int_{\gamma\mathbf{D}} f'(z) g'(\bar{z})\, dA(z),$$

where dA is the Lebesgue area measure. Using this, we have the invariance

(3) $$\langle f\circ\phi_a, g\circ\phi_{\bar{a}}\rangle = \langle f, g\rangle \quad \text{for all } a \text{ in } \mathbf{D}.$$

We also have

(4) $$\langle f, \phi_a\rangle = -f'(\bar{a})(1 - |a|^2)$$

for all analytic functions f on $\mathbf{D}$ and hence

(5) $$\langle \phi_a, \phi_{\bar{a}}\rangle = 1 \quad \text{for all } a \text{ in } \mathbf{D}.$$

The Bloch space $\mathbf{B}/\mathbf{C}$ is isometrically isomorphic to $(\mathbf{M}/\mathbf{C})^*$ under the pairing $\langle f, g\rangle$ for f in $\mathbf{M}/\mathbf{C}$ and g in $\mathbf{B}/\mathbf{C}$ (see [2]). Hence we shall consider g in $\mathbf{B}/\mathbf{C}$ as an element of $(\mathbf{M}/\mathbf{C})^*$ and that of $\mathbf{M}^*$ by the paring $\langle f, g\rangle$ for f in $\mathbf{M}$.

LEMMA 1. *Moebius functions are strongly exposed points of* $\mathbf{M}_1$.

Proof. Let ϕ be the Moebius function with the representation $\phi = c\phi_a$ for a unimodular c and a in $\mathbf{D}$. Then the function $\bar{c}\phi_{\bar{a}}$, denoted by $\tilde{\phi}$, is a support functional of $\mathbf{M}_1$ at ϕ by (5) and $\|\tilde{\phi}|_{\mathbf{B}/\mathbf{C}} = \rho(\tilde{\phi}) = 1$.

We will show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that:

(6) if $\mathrm{Re}\langle f, \tilde{\phi}\rangle > 1 - \delta$ for f in $\mathbf{M}_1$, then $\|f - \phi\|_{\mathbf{M}} < \varepsilon$.

First, we will show that for $\varepsilon > 0$ there exists $\delta_0 > 0$ such that:

(7) if $\mathrm{Re}\langle \psi, \tilde{\phi}\rangle > 1 - \delta_0$ for a Moebius function ψ, then $\|\psi - \phi\|_{\mathbf{M}} < \varepsilon/3$.

Let us recall that $\|\phi_a - \phi_0\|_{\mathbf{M}} \to 0$ as $a \to 0$ (see [2]). Then we can choose $\delta_1 > 0$ such that

$$\|t\phi_b - \phi_0\|_{\mathbf{M}} < \varepsilon/3 \quad \text{if } |b| < \delta_1 \text{ and } |t - 1| < \delta_1$$

with a unimodular t in $\mathbf{C}$ and b in $\mathbf{D}$.

Since $1 - |b|^2 \geq \operatorname{Re} t(1 - |b|^2)$ and $1 \geq \operatorname{Re} t \geq \operatorname{Re} t(1 - |b|^2)$, we can choose $\delta_0 > 0$ such that if

$$\operatorname{Re} t\big(1 - |b|^2\big) > 1 - \delta_0,$$

then

$$|b| < \delta_1 \quad \text{and} \quad |t - 1| < \delta_1$$

and hence

$$\|t\phi_b - \phi_0\|_{\mathbf{M}} < \varepsilon/3 \quad \text{if } \operatorname{Re} t\big(1 - |b|^2\big) > 1 - \delta_0$$

with a unimodular t in $\mathbf{C}$ and b in $\mathbf{D}$.

Suppose that a Moebius function $\psi = t\phi_b$ satisfies the assumption of (7):

$$\operatorname{Re}\langle \psi, \tilde{\phi}\rangle > 1 - \delta_0.$$

Since $\phi_{\bar{a}} \circ \phi_{\bar{a}}$ is equal to the identity function $-\phi_0$, using (3) we have

$$\langle \psi, \tilde{\phi}\rangle = t\bar{c}\langle \phi_b, \phi_{\bar{a}}\rangle = -t\bar{c}\langle \phi_b \circ \phi_a, \phi_0\rangle.$$

Let us set $\phi_b \circ \phi_a = \phi_d$ where $d = \phi_b(a)$. Then $\langle \phi_d, \phi_0\rangle = 1 - |d|^2$ by (4) and $\langle \psi, \tilde{\phi}\rangle = -t\bar{c}(1 - |d|^2)$, so we can get

$$\| - t\bar{c}\phi_d - \phi_0\|_{\mathbf{M}} < \varepsilon/3$$

by the choice of δ_0. Since $\|f \circ \phi_a\|_{\mathbf{M}} = \|f\|_{\mathbf{M}}$ for f in $\mathbf{M}$ and $\phi_a \circ \phi_a = -\phi_0$,

$$\begin{aligned}\|\psi - \phi\|_{\mathbf{M}} &= \|t\phi_b - c\phi_a\|_{\mathbf{M}} \\ &= \|t\bar{c}\phi_b - \phi_a\|_{\mathbf{M}} = \|t\bar{c}\phi_b \circ \phi_a - \phi_a \circ \phi_a\|_{\mathbf{M}} = \|t\bar{c}\phi_d + \phi_0\|_{\mathbf{M}} < \varepsilon/3,\end{aligned}$$

which proves (7).

Next, setting $\delta = \delta_0\varepsilon/12$ with $\delta_0 < 1$, we will show that (6) holds for δ. Suppose that a function f in $\mathbf{M}_1$ satisfies the assumption of (6):

$$\operatorname{Re}\langle f, \tilde{\phi}\rangle > 1 - \delta.$$

Then

$$1 - \delta < |\langle f, \tilde{\phi}\rangle| \leq \|f\|_{\mathbf{M}} \leq 1.$$

By the definition of norm of $\mathbf{M}$, f has a representation

$$f = \sum_{k=1}^{\infty} \lambda_k\phi_k + c \quad \text{for } \lambda_k \text{ and } c \text{ in } \mathbf{C},$$

with

$$1-\delta<|\alpha|+|c|<1+\delta,$$

where ϕ_k $(k=1,2,\ldots)$ are Moebius functions and $\alpha=\sum_{k=1}^{\infty}|\lambda_k|$. Since $\langle c,\tilde{\phi}\rangle=0$ and $|\langle\phi_k,\tilde{\phi}\rangle|\le 1$ for all k, we have $1-\delta<|\langle f,\tilde{\phi}\rangle|\le\alpha$ and hence $1-\delta<\alpha<1+\delta$ and $|c|<2\delta$. Let us choose γ with $0<6(\alpha-1+\delta)/\varepsilon<\gamma<\delta_0$, which is guaranteed by $1-\delta<\alpha<1+\delta$ and $12\delta/\varepsilon=\delta_0$. Define

$$\Lambda=\left\{k\colon \operatorname{Re}\langle c_k\phi_k,\tilde{\phi}\rangle<1-\gamma\right\},$$

where $c_k=\lambda_k/|\lambda_k|$ for $k=1,2,\ldots$ Then

$$\begin{aligned}\operatorname{Re}\langle f,\tilde{\phi}\rangle &= \sum_{k=1}^{\infty}|\lambda_k|\operatorname{Re}\langle c_k\phi_k,\tilde{\phi}\rangle\\ &\le \sum_{k\notin\Lambda}|\lambda_k|+\Big(\sum_{k\in\Lambda}|\lambda_k|\Big)(1-\gamma)\\ &= \sum_{k=1}^{\infty}|\lambda_k|-\Big(\sum_{k\in\Lambda}|\lambda_k|\Big)\gamma\\ &= \alpha-\Big(\sum_{k\in\Lambda}|\lambda_k|\Big)\gamma.\end{aligned}$$

Hence by the assumption $\operatorname{Re}\langle f,\tilde{\phi}\rangle>1-\delta$ and the choice of γ, we have

$$\sum_{k\in\Lambda}|\lambda_k|<(\alpha-1+\delta)/\gamma<\varepsilon/6 \tag{8}$$

Now, since $\phi=\alpha\phi+(1-\alpha)\phi=(\sum_{k=1}^{\infty}|\lambda_k|)\phi+(1-\alpha)\phi$, we have

$$\begin{aligned}\|f-\phi\|_{\mathbf{M}} &= \Big\|\sum_{k=1}^{\infty}\lambda_k\phi_k+c-\phi\Big\|_{\mathbf{M}}\\ &\le \Big\|\sum_{k=1}^{\infty}|\lambda_k|(c_k\phi_k-\phi)\Big\|_{\mathbf{M}}+\|(1-\alpha)\phi\|_{\mathbf{M}}+|c|\\ &\le \sum_{k\in\Lambda}|\lambda_k|\,\|c_k\phi_k-\phi\|_{\mathbf{M}}+\sum_{k\notin\Lambda}|\lambda_k|\|c_k\phi_k-\phi\|_{\mathbf{M}}+|1-\alpha|+|c|.\end{aligned}$$

Since $\operatorname{Re}\langle c_k\phi_k,\tilde{\phi}\rangle\ge 1-\gamma>1-\delta_0$ for $k\notin\Lambda$, we have $\|c_k\phi_k-\phi\|_{\mathbf{M}}<\varepsilon/3$ for $k\notin\Lambda$ by (7) and we have $\|\phi_k-\phi\|_{\mathbf{M}}\le 2$ for all k by the definition of norm of $\mathbf{M}$. Therefore using (8), since $|1-\alpha|<\delta$, $|c|<2\delta$ and $\delta=\delta_0\varepsilon/12<$

$\varepsilon/12$, we have

$$\|f - \phi\|_{\mathbf{M}} < \varepsilon/6 \cdot 2 + \varepsilon/3 \cdot \alpha + 3\delta < \varepsilon/3 + \varepsilon(1+\delta)/3 + 3\delta$$
$$< \varepsilon/3 + \varepsilon/3 + 4\delta < \varepsilon,$$

which proves (6). Thus ϕ is a strongly exposed point of $\mathbf{M}_1$. The lemma is proved.

LEMMA 2. *Unimodular constant functions are not strongly exposed points of* $\mathbf{M}_1$.

Proof. Let us suppose, to the contrary, that a unimodular constant function c is a strongly exposed point of $\mathbf{M}_1$. Then there exists a support functional l of $(\mathbf{M}^*)_1$ at c satisfying the following conditions: for any $\varepsilon > 0$ there is $\delta > 0$ such that for f in $\mathbf{M}_1$ we have:

(9) if $\operatorname{Re} l(f) > 1 - \delta$ then $\|f - c\|_{\mathbf{M}} < \varepsilon$.

Define

$$L(f) = l(f) - f(0), \quad f \text{ in } \mathbf{M}.$$

Then since $|f(0)| \leq \|f\|_\infty \leq \|f\|_{\mathbf{M}}$ for f in $\mathbf{M}$, L is a bounded linear functional of $\mathbf{M}/\mathbf{C}$ and hence there is g in $\mathbf{B}/\mathbf{C}$ satisfying

$$L(f) = \langle f, g\rangle, \quad f \text{ in } \mathbf{M}.$$

Setting $f = \phi_a$ for a in $\mathbf{D}$, we have $L(\phi_a) = \langle \phi_a, g\rangle = -g'(\bar{a})(1 - |a|^2)$ by (4) and this yields

$$l(\phi_a) = a - g'(\bar{a})(1 - |a|^2), \quad a \text{ in } \mathbf{D}.$$

Now we will show that for any analytic function f in $\mathbf{D}$,

$$\sup_{|z|<1} \left| z - f'(\bar{z})(1 - |z|^2) \right| \geq 1. \tag{10}$$

To see this it suffices to show that there is a sequence $\{a_n\}$ in $\mathbf{D}$ such that $|a_n| \to 1$ and $|f'(\bar{a}_n)|$ are bounded, because $|f'(\bar{a}_n)| < M$ for some M implies

$$\sup\left| a_k - f'(\bar{a}_k)(1 - |a_k|^2) \right| \geq \sup\left\{ |a_k| - M(1 - |a_k|^2) \right\} \geq 1.$$

Suppose that there is no such sequence in $\mathbf{D}$. Then $|f'(a_k)| \to \infty$ holds for any sequence $\{a_k\}$ in $\mathbf{D}$ with $|a_k| \to 1$. Dividing f' by a suitable finite Blashke product B, we have that $h = f'/B$ has no zeros in $\mathbf{D}$ and $|h(z)| \to \infty$ as

$|z| \to 1$. Hence $1/h$ is analytic on $\mathbf{D}$ and $|1/h(z)| \to 0$ as $|z| \to 1$, which leads to a contradiction by the maximum principle. Thus (10) holds.

Therefore we have

$$\begin{aligned}
&\sup\{\operatorname{Re} l(\phi)\colon \phi \text{ is a Moebius function}\}\\
&\quad = \sup\left\{\operatorname{Re} l\left(e^{i\theta}\phi_a\colon 0 < \theta \le 2\pi,\ a \text{ in } \mathbf{D}\right\}\right.\\
&\quad = \sup\{|l(\phi_a)|\colon a \text{ in } \mathbf{D}\}\\
&\quad = \sup_{|a|<1} \left|a - g'(\bar{a})(1 - |a|^2)\right|\\
&\quad \ge 1,
\end{aligned}$$

which contradicts (9) by $\|\phi - c\|_{\mathbf{M}} = \|\phi - c\|_\infty \ge 2$ for all Moebius functions ϕ. The proof is complete.

Proof of the theorem. Since strongly exposed points of $\mathbf{M}_1$ are extreme points of $\mathbf{M}_1$ and $\operatorname{ext}\{\mathbf{M}_1) = \{\text{Moebius functions}\} \cup \{\text{unimodular constants}\}$, the theorem is an immediate consequence of Lemma 1 and Lemma 2.

Proof of Corollary 1. Note that for any element $f \in \mathbf{M}/\mathbf{C}$ there is some complex number c such that $\|f + c\|_{\mathbf{M}} = \|f\|_{\mathbf{M}/\mathbf{C}}$.

Now let ϕ be a Moebius function. By Theorem ϕ is a strongly exposed point of $\mathbf{M}_1$ with a support functional $\tilde{\phi}$ of $\mathbf{M}_1$ at ϕ. Namely, for any $\varepsilon > 0$ there is $\delta > 0$ such that $\operatorname{Re}\langle f, \tilde{\phi}\rangle > 1 - \delta$ with $f \in \mathbf{M}_1$ implies $\|f - \phi\|_{\mathbf{M}} < \varepsilon$. Suppose that $\operatorname{Re}\langle f, \phi\rangle > 1 - \delta$ for $f \in (\mathbf{M}/\mathbf{C})_1$. Choose c such that $\|f + c\|_{\mathbf{M}} = \|f\|_{\mathbf{M}/\mathbf{C}} \le 1$. Then

$$\operatorname{Re}\langle f + c, \phi\rangle = \operatorname{Re}\langle f, \phi\rangle > 1 - \delta$$

implies

$$\|f - \phi\|_{\mathbf{M}/c} \le \|f + c - \phi\|_{\mathbf{M}} < \varepsilon.$$

Thus Moebius functions are also strongly exposed points of $(\mathbf{M}/\mathbf{C})_1$.

Uniqueness of the predual of the Bloch space

There are several sufficient conditions for a Banach space to have the unique predual; see [5]. One of the results in [5] will be discussed in the final remark. The following proposition, which will be used later, was announced by Prof. Takashi Ito at a seminar and I do not know a published proof of the result. The author would like to express her hearty thanks to Prof. Takashi Ito for his kindness to agree to including his result with a proof in this paper.

An element of x in E_1 of a Banach space is called a weak-norm continuous point of E_1 if for every net $\{x_\alpha\}$ in E_1, x_α converges to x in norm wherever it converges to x in the weak topology in E. We denote the set of all weak-norm continuous points of E_1 by $C(E_1)$.

PROPOSITION. *A Banach space E is the unique predual of the dual E^* if the closed convex hull of $C(E_1)$ has an interior point in E_1.*

Proof. Let F be any Banach space whose dual F^* is isometrically isomorphic to E^*. Suppose that σ is an isometric isomorphism from F^* onto E^*. Then the dual mapping σ^* is an isometric isomorphism from the second dual E^{**} onto F^{**}. Take x in $C(E_1)$ and set $y^{**} = \sigma^*(x)$. Since y^{**} is in $(F^{**})_1$, there is a net $\{y_\alpha\}_{\alpha\in\Lambda}$ in F_1 converging to y^{**} in the weak* topology in F^{**}. Since σ^* is onto, we can choose a net $\{x_\alpha^*\}_{\alpha\in\Lambda}$ in $(E^{**})_1$ such that $\sigma^*(x_\alpha^{**}) = y_\alpha$. Then since σ^* is onto, x_α^{**} converges to x in the weak* topology in E^{**}. Let Γ be an ordered set,

$$\Gamma = \{(A, n) \colon A \text{ is a finite subset of } E^* \text{ and } n \text{ in } N\},$$

with the order

$$(A, n) < (A', n') \quad \text{if } A \subseteq A' \text{ and } n \leq n'.$$

Then for each x_α^{**} and each (A, n) in Γ there is a $x_{\alpha, A, n}$ in E_1 such that

$$|\langle x_\alpha^{**}, x^*\rangle - \langle x_{\alpha, A, n}, x^*\rangle| < 1/n \quad \text{for all } x^* \text{ in } A,$$

and hence the net $\{x_{\alpha, A, n}\}_{(A, n)\in\Gamma}$ converges to x_α^{**} in the weak* topology in E^{**}. Now consider the net $\{x_{\alpha, A, n}\}_{(\alpha, A, n) \text{ in } \Lambda\times\Gamma}$. This net converges to x in the weak* topology in E^{**} and hence in the weak topology in E. Since x is in $C(E_1)$, the net $\{x_{\alpha, A, n}\}$ converges to x in norm. Therefore taking the limit with respect to (Λ, n) in Γ for each fixed α in Λ we can get $\|x_\alpha^{**} - x\| \to 0$ and $\|y_\alpha - y^{**}\| \to 0$ since σ is isometric.

Thus $y^{**} = \sigma^*(x)$ must be in F. Since x is arbitrary in $C(E_1)$, we have

$$\sigma^*[C(E_1)] \subseteq F.$$

Using the assumption that the closure of convex hull of $C(E_1)$ has an interior point of E_1, we have

$$\sigma^*(E) \subseteq F.$$

This together with $\sigma(F^*) = E^*$ implies $\sigma^*(E) = F$, which completes the proof.

Now we are ready to prove Corollary 2.

Proof of Corollary 2. It is clear that strongly exposed points of E_1 for a Banach space E are *necessary weak-norm continuous points* of E_1. Hence Lemma 1 and the proposition imply that **M**/**C** is the unique predual of **B**/**C**.

Remark

It is well known that **B**/**C** is isometrically isomorphic to l^∞ (see [7]) and l^∞ has the unique predual l^1 (see [6]). However **B**/**C** is not isometrically isomorphic to l^∞. For otherwise, there is an isometric isomorphism from **M**/**C** onto l^1 which maps

$$\text{ext}(\mathbf{M}/\mathbf{C})_1 \supseteq \{c\phi_a\colon c \in \mathbf{C} \text{ with } |c| = 1 \text{ and } a \in \mathbf{D}\}$$

onto

$$\text{ext}(1^1)_1 = \{ce_n\colon c \text{ in } C \text{ with } |c| = 1 \text{ and } e_n = (0,\ldots,0,1,0,\ldots), \text{ for } n = 1,2,\ldots\}.$$

The cardinality of $\text{ext}(l^1)_1$ is countable and that of $\text{ext}(\mathbf{M}/\mathbf{C})_1$ is uncountable up to unimodular constant multiplications, which is a contradiction.

The closed subspace $\mathbf{B}_0/\mathbf{C}$ of **B**/**C** is called the little Bloch space, where

$$\mathbf{B}_0/\mathbf{C} = \{f \text{ in } \mathbf{B}\colon f'(z)(1 - |z|^2) \text{ converges to zero as } |z| \to 1\}.$$

Since $\mathbf{B}_0/\mathbf{C}$ is a predual of **M**/**C** (see [2]), **M**/**C** is a separable dual space, so that **M**/**C** has the Radon-Nikodym property, for example see Theorem 4.1.3 in [4]. Using this fact and the result by G. Godefroy (Theorem 11 in [5]), we can also see that **M**/**C** is the unique predual of **B**/**C**. In this paper we proved it by characterizing the strongly exposed points of the unit ball of **M**/**C**.

We do not know if $\mathbf{B}_0/\mathbf{C}$ is the unique predual of **M**/**C** (see [3]). The argument used in this paper is not available to this problem. For, we can show that Moebius functions are neither strongly exposed points nor extreme points of the unit ball of $\mathbf{B}_0/\mathbf{C}$.

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