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THE GEOMETRY OF BRS TRANSFORMATIONS

BY

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1. Introduction

In 1976 C. Becchi, A. Rouet and R. Stora [1] noticed that in gauge field theories the effective Lagrangian, which is no longer gauge invariant, is still invariant under a new class of transformations now called BRS transformations

$$sA = d\eta + [A, \eta], \quad s\eta = -1/2[\eta, \eta]$$

where A is the potential field (connection one form) and η is the ghost field.

We show how these BRS transformations can be interpreted as purely differential geometric objects. We define a general BRS cohomology $\mathbf{H}^{q,p}$ of the infinite dimensional Lie algebra $\mathbf{g}$ of infinitesimal gauge transformations with respect to an induced representation. As a special case, namely with respect to the adjoint representation, we obtain the classical BRS transformations as coboundary operator

$$\mathbf{s}: \mathbf{C}^{q,p} \to \mathbf{C}^{q+1,p}$$

of this complex. The Wess-Zumino consistency condition is expressed as $\mathbf{s}^2 = 0$, while the ghost field η is interpreted as the canonical Maurer-Cartan form on the infinite dimensional Lie group $\mathbf{G}$ of gauge transformations.

2. The gauge group G

Let π: $P \to M$ be a principal bundle with structure group G (not necessarily compact), i.e., we have a free right action R: $P \times G \to P$ of G on P, denoted by $p \cdot a = R(p, a)$, $p \in P$, $a \in G$. The *gauge group* $\mathbf{G}$ is the group of gauge transformations of P; i.e., $\mathbf{G}$ consists of all fiber preserving automorphisms ϕ of P

$$\mathbf{G} = \{\phi \in \mathrm{Diff}^\infty(P) | \phi(p \cdot a) = \phi(p) \cdot a, \pi(\phi(p)) = \pi(p), p \in P, a \in G\}.$$

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G is a group under composition, hence a subgroup of $\text{Diff}^\infty(P)$, the diffeomorphism group of P. Since gauge transformations preserve the fibers of P we can realize each $\phi \in \mathbf{G}$ by $\phi(p) = p \cdot \tau(p)$, where τ is a smooth map τ: $P \to G$ satisfying $\tau(p \cdot a) = a^{-1}\tau(p)a$. Let

$$\text{Gau}(P) = \{\tau \in C^\infty(P, G) | \tau(p \cdot a) = a^{-1}\tau(p)a, p \in P, a \in G\}.$$

$\text{Gau}(P)$ is a group under pointwise multiplication, hence a subgroup of the "loop group" $C^\infty(P, G)$.

The first observation is that the relation $\phi(p) = p \cdot \tau(p)$, $\phi \in \mathbf{G}$, $\tau \in \text{Gau}(P)$ defines a group isomorphism

$$\mathbf{G} \cong \text{Gau}(P), \quad \phi_1 \circ \phi_2 \to \tau_1 \cdot \tau_2. \tag{2.1}$$

The gauge group **G** has still another interpretation in terms of associated vector bundles, given by a left action

$$\rho\colon G \times F \to F$$

of G on some manifold F. The twisted bundle $\pi_G = P \times_G F$ is given as follows: G acts on the right on $P \times F$ by $(p, x) \cdot a = (p \cdot a, \rho(a^{-1}, x))$, $x \in F$. The corresponding orbit space $P \times_G F$ is a smooth fiber bundle π_G over M, π_G: $P \times_G F \to M$, $\pi_G[p, x] = \pi(p)$, where $[p, x]$ denotes the orbit through $(p, x) \in P \times F$. Any smooth section s of π_G can be realized by $s(\pi(p)) = [p, \tau(p)]$ where τ is a smooth map τ: $P \to F$ satisfying $\tau(p \cdot a) = \rho(a^{-1}, \tau(p))$.

In our case let $F = G$ and ρ be the conjugation action $\rho(a, b) = aba^{-1}$. Then $\text{Ad}(P) \equiv P \times_G G$ is a smooth bundle of groups (not a principal bundle) over M and sections of $\text{Ad}(P)$ can be multiplied pointwise, making the space of sections $C^\infty(\text{Ad } P)$ into a group,

$$C^\infty(\text{Ad } P) \cong \{\tau\colon P \to G | \tau(p \cdot a) = a^{-1}\tau(p)a, p \in P, a \in G\}.$$

Note that $\text{Ad}(P)$ has a trivial subbundle $P \times_G Z$ where Z is the center of G. In general $\text{Ad}(P)$ is not trivial but if G is abelian, then $\text{Ad}(P)$ is a trivial vector bundle over M. With this identification the gauge group **G** is canonically group isomorphic to the group $C^\infty(\text{Ad } P)$;

$$\mathbf{G} \cong C^\infty(\text{Ad } P). \tag{2.2}$$

To put a topology on **G** we complete the space of smooth sections $C^\infty(\text{Ad } P)$ with respect to the Sobolev H_s-norm and give **G** and $\text{Gau}(P)$ the induced topologies; denoting the corresponding spaces by $H_s(\text{Ad } P), \mathbf{G}_s, \text{Gau}_s(P)$. If $s > 1/2 \dim M$, then

$$\mathbf{G}_s \cong H_s(\text{Ad } P) \cong \text{Gau}_s(P) \tag{2.3}$$

are smooth Hilbert manifolds with smooth group operations

$$(\phi_1, \phi_2) \to \phi_1 \circ \phi_2 \colon \mathbf{G}_s \times \mathbf{G}_s \to \mathbf{G}_s, \phi \to \phi^{-1} \colon \mathbf{G}_s \to \mathbf{G}_s,$$

i.e., $\mathbf{G}_s$ is a smooth Hilbert Lie group (e.g., see [9]).

3. The gauge algebra g

The *gauge algebras* **g** is the Lie algebra of the gauge group **G**, i.e. the algebra of infinitesimal gauge transformations on P. Again there are three different interpretations of **g**.

(A) The Lie algebra **g** of the Lie group **G** is the space of all G-invariant, vertical (i.e., tangent to the fibers) smooth vector fields X on P, i.e.,

$$\mathbf{g} = \{ X \in \mathbf{X}^{\infty}(P) | R_a^* X = X, X(p) \in g, a \in G, p \in P \}$$

where g is the Lie algebra of G and $R_a(p) = R(p, a)$. Under the commutator bracket **g** is a Lie subalgebra of $\mathbf{X}^{\infty}(P)$, the Lie algebra of all smooth vector fields on P.

(B) The Lie algebra gau(P) of the Lie group Gau(P) is the space of all Ad-invariant g-valued functions on P, i.e.

$$\mathrm{gau}(P) = \{ \xi \in C^{\infty}(P, g) | \xi(p \cdot a) = \mathrm{Ad}_{a^{-1}} \xi(p), p \in P, a \in g \},$$

where Ad is the adjoint representation of G on g. Under pointwise bracket gau(P) is a Lie subalgebra of the "loop algebra" $C^{\infty}(P, g)$.

(C) Let ad(P) denote the vector bundle associated to the adjoint action of G on g;

$$\mathrm{ad}(P) = P \times_G g \to M.$$

The space of sections $C^{\infty}(\mathrm{ad}P)$ is a Lie algebra under pointwise bracket; it is the Lie algebra of the Lie group $C^{\infty}(\mathrm{Ad}\, P)$.

PROPOSITION 3.1. *The Lie algebras* **g**, gau(P), *and* $C^{\infty}(\mathrm{ad}P)$ *are canonically isomorphic.*

Proof. (1) Any section $s \in C^{\infty}(\mathrm{ad}P)$ can be identified with a map $\xi \colon P \to g$ satisfying $\xi(p \cdot a) = \mathrm{Ad}_{a^{-1}} \xi(p)$ i.e., $\xi \in \mathrm{gau}(P)$. Given any $\xi \in \mathrm{gau}(P)$ we define a section $s \in C^{\infty}(\mathrm{ad}P)$ by $s(\pi(p)) = [p, \xi(p)]$.

(2) For any $\xi \in \mathrm{gau}(P)$ define $Z_\xi \in \mathbf{g}$ by

$$Z_\xi(p) = \left.\frac{d}{dt}\right|_{t=0} R(p, \exp t\xi(p)), \quad (= \xi(p)^*(p));$$

i.e., Z_ξ is the fundamental vector field on P generated by $\xi \in \mathbf{g}$. It is invariant iff $\xi(p \cdot a) = \mathrm{Ad}_{a^{-1}}\xi(p)$. This defines an isomorphism between gau(P) and $\mathbf{g}$. ■

To topologize $\mathbf{g}$ accordingly, we complete the space of smooth sections $C^\infty(\mathrm{ad}\, P)$ with respect to the H_s-Sobolev norm and give $\mathbf{g}$ and gau(P) the induced topologies; denoting the corresponding spaces by $H_s(\mathrm{ad}\, P)$, $\mathbf{g}_s$, $\mathrm{gau}_s(P)$. If $s > 1/2 \dim M$ then

$$\mathbf{g}_s \cong H_s(\mathrm{ad}\, P) \cong \mathrm{gau}_s(P) \tag{3.1}$$

are Hilbert spaces.

There is a natural exponential map Exp: $\mathrm{gau}_s(P) \to \mathrm{Gau}_s(P)$ defined by

$$(\mathrm{Exp}\, \xi)(p) = \exp(\xi(p)),$$

where exp: $g \to G$ is the exponential map of G. The map Exp is a local diffeomorphism form a neighborhood of zero in $\mathrm{gau}_s(P)$ onto a neighborhood of the identity in $\mathrm{Gau}_s(P)$. Smoothness of Exp follows from the Ω-lemma: $\mathrm{Exp} = \Omega_{\exp}\colon H_s(P, g) \to H_s(P, G)$, $\Omega_{\exp}(\xi) = \exp \circ \xi$. Summarizing we have:

PROPOSITION 3.2. *For $s > 1/2 \dim M$, $\mathbf{G}_s \cong \mathrm{Gau}_s(P) \cong H_s(\mathrm{Ad}\, P)$ are smooth Hilbert Lie groups with Lie algebras $\mathbf{g}_s \cong \mathrm{gau}_s(P) \cong H_s(\mathrm{ad}\, P)$.*

Remark. We will switch between these three interpretations of gauge transformations as elements of either $\mathbf{G}$, Gau(P), or $C^\infty(\mathrm{Ad}\, P)$ and translate important facts from one picture to the others. Typically we will denote elements of $\mathbf{G}$ by ϕ, elements of Gau(P) by τ and elements of $C^\infty(\mathrm{Ad}\, P)$ by s. As example, the corresponding exponential map Exp: $\mathrm{gau}_s(P) \to \mathbf{G}_s$ is given by $(\mathrm{Exp}\, \xi)(p) = p \cdot \exp(\xi(p))$.

4. Representation of G and g on $\Lambda(P, V)$

Let ρ be a representation of G on a finite dimensional vector space V and let $\Lambda^k(P, V)$ be the space of V-valued equivariant k-forms Φ on P, i.e. satisfying $R_a^*\Phi = \rho(a^{-1}) \cdot \Phi$, $a \in G$. Let

$$\Lambda(P, V) = \sum_k \Lambda^k(P, V).$$

For $h \in g$ let Z_h denote the fundamental vector field on P generated by h and denote by i_h and L_h the operators interior product i_{Z_h} and Lie derivative L_{Z_h} respectively, extended to the space $\Lambda(P, V) \cong \Lambda(V) \otimes V$; $i_h = i_{Z_h} \otimes \mathrm{id}$, $L_h = L_{Z_h} \otimes \mathrm{id}$. Consider the derived representations ρ' of g on $\Lambda(V, P)$. For any $\Phi \in \Lambda(P, V)$ we have $\rho'(h)\Phi = -L_h\Phi$, $h \in g$. Then $\mathbf{G}$ is represented on $\Lambda(P, V)$ by $\pi(\phi)\Phi = (\phi^{-1})^*\Phi$, $\phi \in \mathbf{G}$.

Let $X \in \mathbf{g}$ and $\xi \in \mathrm{gau}(P)$ such that $X(p) = Z_{\xi(p)}(p)$, i.e., $X \cong \xi$ under the identification 3.1. Let $L_\xi = L_{Z_\xi}$ and $i_\xi = i_{Z_\xi}$.

PROPOSITION 4.1. *The derived representation π' of $\mathbf{g}$ on $\Lambda(P, V)$ is given by*

$$\pi'(X)\Phi = L_\xi\Phi, \quad X \in \mathbf{g}, \Phi \in \Lambda(P, V).$$

Proof. The flow of Z_ξ is given by $\exp t\xi$ hence

$$\pi'(X)\Phi = \pi'(\xi)\Phi = \frac{d}{dt}_{|t=0}\pi(\exp t\xi)\Phi$$
$$= \frac{d}{dt}_{|t=0}(\exp(-t\xi))^*\Phi = L_{Z_\xi}\Phi = L_\xi\Phi. \qquad \blacksquare$$

We put a Hilbert space structure on $\Lambda(P, V)$ as follows: For $s > 1/2 \dim P$ let $\mathbf{X}_s(P)$ denote the completion of the space of smooth vector fields $\mathbf{X}^\infty(P)$ under the Sobolev H_s-norm. $\mathbf{X}_s$ is a Hilbert space. Then the space $\Lambda^k_s(P, V)$ is the space of all continuous V-valued and equivariant, skew k-linear maps on $\mathbf{X}_s(P)$. With the induced topologies $\Lambda^k_s(P, V)$ and $\Lambda_s(P, V) = \Sigma_k \Lambda^k_s(P, V)$ are Hilbert spaces.

The representation π induces an action on $\mathbf{G}_s$ of $\Lambda_s(P, V)$: $\phi \cdot \Phi = \pi(\phi)\Phi = (\phi^{-1})^*\Phi$. This action is smooth since $\phi \to \phi^{-1}$ and pull back are both smooth.

Special subrepresentations. Let $V = g$ and $\rho = \mathrm{Ad}$, the adjoint representation of G in g. For $\Phi \in \Lambda^k(P, g)$ and $\Psi \in \Lambda^j(P, g)$ we have

$$[\Phi, \Psi] = \Phi \wedge \Psi - (-1)^{jk}\Psi \wedge \Phi \in \Lambda^{j+k}(P, g);$$

e.g., for $\omega \in \Lambda^1(P, g)$ we get $\frac{1}{2}[\omega, \omega] = \omega \wedge \omega$.

Identifying $\mathbf{g}$ with $\Lambda^0(P, g)$, i.e., with Ad-equivariant g-valued functions on P, we get the next result.

COROLLARY 4.2. *The derived representation π' of $\mathbf{g}$ on $\Lambda^0(P, g) \cong \mathbf{g}$ is the adjoint representation of $\mathbf{g}$:*

$$\pi'(X)(Y) = \mathrm{ad}_X(Y) = [X, Y], \quad X, Y \in \mathbf{g}.$$

Denote by $\mathbf{A}$ the space of connection 1-forms (or gauge potentials) on P; i.e., $\omega \in \mathbf{A}$ iff ω is a g-value 1-form on P satisfying $R^*_a\omega = \mathrm{Ad}_{a^{-1}}\omega$ and $\omega(Z_h) = h \in g$. For fixed $\omega_0 \in \mathbf{A}$ we write $\omega = \omega_0 + \tau$ with $\tau \in \Lambda^1(P, g)$; i.e., we regard $\mathbf{A}$ as affine space $\mathbf{A} = \omega_0 + \Lambda^1(P, g)$ with tangent space $T_{\omega_0}\mathbf{A} = \Lambda^1(P, g)$. With the induced topology from $\Lambda^1_s(P, g)$ we denote $\mathbf{A}$ by $\mathbf{A}_s$.

The space $\mathbf{A}_s$ is invariant under the induced action of $\mathbf{G}_s$. Indeed,

$$R_a^*(\pi_\phi\omega) = R_a^*(\phi^{-1})^*\omega = (\phi^{-1}\circ R_a)^*\omega = (R_a\circ\phi^{-1})^*\omega = (\phi^{-1})^*R_a^*\omega$$
$$= (\phi^{-1})^*(\mathrm{Ad}_{a^{-1}}\omega) = \mathrm{Ad}_{a^{-1}}(\phi^{-1})^*\omega = \mathrm{Ad}_{a^{-1}}(\pi_\phi\omega).$$

and

$$(\pi_\phi\omega)(Z_h)(p) = (\phi^{-1})^*\omega(Z_h)(p) = \omega(\phi^*Z_h)(p)$$
$$= \omega\left(\frac{d}{dt}\bigg|_{t=0}\phi^{-1}(p\cdot\exp th)\right)$$
$$= \omega\left(\frac{d}{dt}\bigg|_{t=0}\phi^{-1}(p)\cdot\exp th\right) = \omega(Z_h(p)) = h.$$

Let D_ω denote the exterior covariant derivative with respect to $\omega\in\mathbf{A}$:

$$D_\omega\colon \Lambda^k(P,g)\to\Lambda^{k+1}(P,g),\ D_\omega(\Phi) = d\Phi + \tfrac{1}{2}[\omega,\Phi].$$

The curvature 2-form (or gauge field) Ω of ω, defined by $\Omega = D_\omega\omega\in\Lambda^2(P,g)$ satisfies the structure equation of Maurer-Cartan $\Omega(X,Y) = d\omega(X,Y) + \frac{1}{2}[\omega(X),\omega(Y)]$, $X, Y\in\mathbf{X}(P)$, written compactly as $\Omega = d\omega + \frac{1}{2}[\omega,\omega]$, and the Bianchi identity $D_\omega\Omega = 0$.

PROPOSITION 4.3. *The induced action of the derived representation π' of* $\mathbf{g}$ *on* $\mathbf{A}$ *is given by*

$$\pi'(X)\omega = D_\omega\xi,\quad X\in\mathbf{g},\ \omega\in\mathbf{A},$$

where $X = Z_\xi$, $\xi\in\mathrm{gau}(P)$.

Proof. From Proposition 4.1 we have $\pi'(X)\omega = L_\xi\omega$. But $L_\xi\omega = di_\xi\omega + i_\xi d\omega$ and $i_\xi\omega = \omega(Z_\xi) = \xi$, so

$$\pi'(X)\omega = d\xi + i_\xi d\omega.$$

From the structure equation we get $d\omega(Z_\xi, Y) = -\frac{1}{2}[\omega(Z_\xi),\omega(Y)] + \Omega(Z_\xi, Y)$ for any $Y\in\mathbf{X}(P)$. But $\Omega(Z_\xi, Y) = 0$ since Z_ξ is vertical. So

$$d\omega(Z_\xi,Y) = -\tfrac{1}{2}[\omega(Z_\xi),\omega(Y)] = -\tfrac{1}{2}[\xi,\omega(Y)] = -\tfrac{1}{2}[\xi,\omega](Y).$$

Hence $i_\xi d\omega = d\omega(Z_\xi) = -\frac{1}{2}[\xi,\omega]$; and $L_\xi\omega = d\xi + \frac{1}{2}[\omega,\xi] = D_\omega\xi$. ∎

We want to compute these representations under the identification (2.1). Denote the components of the right action $R\colon P \times G \to P$ by $R_a\colon P \to P$, $R_a(p) = R(p, a)$ and $R_p\colon G \to P$, $R_p(a) = R(p, a)$.

PROPOSITION 4.4. *Let $\phi \in \mathbf{G}$ and $\tau \in \mathrm{Gau}(P)$ such that $\phi(p) = p \cdot \tau(p)$ (i.e., $\phi \cong \tau$ under the isomorphism (2.1)). For any $\Phi \in \Lambda^1(P, V)$ we have*

$$\pi(\phi)\Phi(p) = (\phi^{-1})^*\Phi(p) = R^*_{\tau^{-1}(p)}\Phi(p) + \left(R_p \circ \tau^{-1}\right)\Phi(p).$$

Proof. We have $\phi = R \circ (\mathrm{id}_p, \tau)$ and $\phi^{-1} = R \circ (\mathrm{id}_p, \tau^{-1})$. Let $\Phi \in \Lambda^1(P, V)$ and $v \in T_pP$. Then

$$\begin{aligned}(\phi^{-1})^*\Phi(p)(v) &= \left(R \circ \left(\mathrm{id}_p, \tau^{-1}\right)\right)^*\Phi(p)(v)\\ &= \Phi\Big(R\left(p, \tau^{-1}(p)\right)\Big(T_p\left(R \circ \left(\mathrm{id}_p, \tau^{-1}\right)\right)(v)\Big).\end{aligned}$$

But

$$\begin{aligned}T_p\left(R \circ \left(\mathrm{id}_p, \tau^{-1}\right)\right)(v) &= T_{(p, \tau^{-1}(p))}R\left(v, T_p\tau^{-1}(v)\right)\\ &= T_{1(p, \tau^{-1}(p))}R(v) + T_{2(p, \tau^{-1}(p))}R\left(T_p\tau^{-1}(v)\right)\\ &= T_pR_{\tau^{-1}(p)}(v) + \left(T_{\tau^{-1}(p)}R_p\right)\left(T_p\tau^{-1}(v)\right)\\ &= T_pR_{\tau^{-1}(p)}(v) + T_p\left(R_p \circ \tau^{-1}\right)(v).\end{aligned}$$

Hence

$$\begin{aligned}(\phi^{-1})^*\Phi(p)(v) &= \Phi\left(p \cdot \tau^{-1}(p)\right)\left(T_pR_{\tau^{-1}(p)}(v)\right)\\ &\quad + \Phi\left(p \cdot \tau^{-1}(p)\right)\left(T_p\left(R_p \circ \tau^{-1}\right)(v)\right)\\ &= \Phi\left(R_{\tau^{-1}(p)}(p)\right)\left(T_pR_{\tau^{-1}(p)}(v)\right)\\ &\quad + \Phi\Big(R_p\left(\tau^{-1}(p)\right)\Big(T_p\left(R_p \circ \tau^{-1}\right)\Big)(v)\Big)\\ &= \left(R^*_{\tau^{-1}(p)}\Phi\right)(p)(v) + \left(R_p \circ \tau^{-1}\right)^*\Phi(p)(v). \qquad \blacksquare\end{aligned}$$

As a special case we get:

COROLLARY 4.5. *The gauge group* $\mathrm{Gau}(P)$ *acts on* $\mathbf{A} \subset \Lambda^1(P, g)$ *by*

$$\tau \cdot \omega(p) = \mathrm{Ad}_{\tau^{-1}} \circ \omega(p) + (\tau^{-1})^*\Theta, \quad \tau \in \mathrm{Gau}(P), \omega \in \mathbf{A},$$

where Θ is the canonical Maurer-Cartan form on $\mathbf{G}$; *i.e.,* Θ *is the left invariant* $\mathbf{g}$*-valued* 1*-form on* $\mathbf{G}$ *determined by* $\Theta(X) = X$, *for all* $X \in \mathbf{g}$.

Proof. (1) For any $\tau \in \mathrm{Gau}(P)$ let ϕ be the corresponding element in $\mathbf{G}$, so $\tau \cdot \omega = (\phi^{-1})^*\omega$. Since $\omega \in \mathbf{A}$ we have $R^*_{\tau^{-1}(p)}\omega(p) = \mathrm{Ad}_{\tau^{-1}(p)}\omega(p)$.

(2) For any $X \in \mathbf{g}$ let $\xi \in \mathrm{gau}(P)$ such that $X = Z_\xi$. Then

$$R^*_p\omega(X(e)) = \omega\big(R_{p*}X(e)\big) = \omega\big(Z_\xi(p)\big) = Z_\xi(p) = X(p),$$

where $e = \mathrm{id} \in \mathbf{G}$. Hence $R^*_p\omega = \Theta$ and $(R_p \circ \tau^{-1})^*\omega = (\tau^{-1})^*\Theta$. ■

We express 4.5 in local coordinates $\{U_\alpha\}$ of M. Let $g_{\alpha\beta}$: $U_\alpha \cap U_\beta \to G$ be the corresponding transition functions. For each α let $s_\alpha : U_\alpha \to P$ be a local section defined by $s_\alpha(x) = g_\alpha^{-1}(x, e)$, $e \in G$ the identity. Let $\Theta_{\alpha\beta} = g^*_{\alpha\beta}\Theta$ and $\omega_\alpha = s^*_\alpha\omega$ be the induced g-valued 1-forms on $U_\alpha \cap U_\beta$ and U_α respectively. Then

$$\omega_\beta = \mathrm{Ad}_{g^{-1}_{\alpha\beta}}\omega_\alpha + g^*_{\alpha\beta}\Theta = \mathrm{Ad}_{g^{-1}_{\alpha\beta}}\omega_\alpha + \Theta_{\alpha\beta}.$$

If in addition $G = Gl(n, \mathbf{R})$ then the change under gauge transformations becomes

$$\omega_\beta = g^{-1}_{\alpha\beta}\omega_\alpha g_{\alpha\beta} + g^{-1}_{\alpha\beta}\,dg_{\alpha\beta}.$$

This action can be written as

$$g \cdot A = g^{-1}Ag + g^{-1}\,dg$$

where A denotes the vector potential and g the gauge transformation.

5. The BRS cohomology

Recall the Chevalley-Eilenberg cohomology of a Lie algebra with respect to a representation [4]: Let

$$\sigma: \mathbf{g} \to \mathrm{Hom}(W)$$

be a representation of the gauge algebra $\mathbf{g}$ in a not necessarily finite dimensional W and denote by $C^q(\mathbf{g}, W)$ the space of W-valued q-cochaines, $C^0(\mathbf{g}, W) \equiv W$ and $C(\mathbf{g}, W) = \Sigma_q C^q(\mathbf{g}, W)$. The coboundary operator

$$\delta: C^q(\mathbf{g}, W) \to C^{q+1}(\mathbf{g}, W)$$

is given by

$$(\delta\Phi)(X_0,\dots,X_q) = \sum_{i=0}^{q}(-1)^i\sigma(X_i)\Phi(X_0,\dots,\hat{X}_i,\dots,X_q) + \sum_{i<j}(-1)^{i+j}\Phi([X_i,X_j],\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_q),$$

for $q = 0$, $\Phi \in C^0(g, W) = W$, $\delta\Phi$ is defined by $(\delta\Phi)(X) = \sigma(X)\Phi$.

PROPOSITION 5.1. $\delta^2 = 0$.

The proof is analog to the one in finite dimensions (e.g., see [4] p. 115). The cohomology of this complex is the Lie algebra cohomology of g with respect to the representation (σ, W). We define a representation θ of g on $W \otimes \Lambda g^*$ by

$$\theta(X) = \sigma(X) + \mathrm{ad}_X^*, \quad X \in g,$$

i.e., for $\Phi \in C^q(g, W)$ and $X_0,\dots,X_{q-1} \in g$ we have

$$\theta(X)\Phi(X_0,\dots,X_{q-1}) = \sigma(X)\Phi(X_0,\dots,X_{q-1}) + \sum_i(-1)^{i+1}\Phi(\mathrm{ad}_X X_i, X_0,\dots,\hat{X}_i,\dots,X_{q-1}).$$

Furthermore for $X \in g$ let i_X: $C^{q+1}(g, W) \to C^q(g, W)$ be given by

$$i_X\Phi(X_0,\dots,X_q) = \Phi(X, X_0,\dots,X_q).$$

A straightforward calculation gives $i_X \circ \delta + \delta \circ i_X = \theta(X)$, which implies $\delta \cdot \theta(X) = \theta(X) \cdot \delta$.

For the BRS transformation we consider a special case. Let $W = \Lambda(P, V)$ and $\sigma = \pi'$ as described in Section 4. Furthermore let $V = g$ and $\rho = \mathrm{Ad}$ the adjoint representation of G on g. Denote $C^{q,p} = C^q(g, \Lambda^p(P, g))$. We define the *BRS transformations* s by

$$\mathbf{s}: \mathbf{C}^{q,p} \to \mathbf{C}^{q+1,p}, \quad \mathbf{s} \equiv \frac{(-1)^{p+1}}{q+1}\delta.$$

From 5.1 we get:

PROPOSITION 5.2. $\mathbf{s}^2 = 0$.

The cohomology of the complex $\{\mathbf{C}^{q,p}, \mathbf{s}\}$ will be called *BRS cohomology* of the gauge algebra g and will be denoted by $H^*_{\mathrm{BRS}}(g)$.

THEOREM 5.3. *Let A be a vector potential and η a ghost field on P; i.e., $A \in \mathbf{A}$ and $\eta \in \mathbf{g}^*$ such that $\eta(X) = X$ for all $X \in \mathbf{g}$. Then*
(1) $\mathbf{s}A = d\eta + [A, \eta]$
(2) $\mathbf{s}\eta = -\frac{1}{2}[\eta, \eta]$.

Proof. (1) For $q = 0$ and $p = 1$ we have $\mathbf{C}^{q,p}C^0(\mathbf{g}, \Lambda^1(P, g)) \cong \Lambda^1(P, g)$ and $\mathbf{A} \subset \Lambda^1(P, g)$. Then $\mathbf{s} = \delta$ and for $A \in \mathbf{A}$, $X \in \mathbf{g}$ we get

$$(\mathbf{s}A)(X) = (\delta A)(X) = \sigma(X) \cdot A = \pi'(X) \cdot A = D_A X = dX + \tfrac{1}{2}[A, X].$$

Also

$$(d\eta)(X) = d(\eta(X)) = dX \quad \text{and} \quad [X, \eta](A) = [A, \eta(X)] = [A, X].$$

Hence $\mathbf{s}A(X) = (d\eta)(X) + \frac{1}{2}[A, \eta](X)$.

(2) For $q = 1$ and $p = 0$ we have $\mathbf{C}^{q,p} = C^1(\mathbf{g}, \Lambda^0(P, g))$. So for $\eta \in \mathbf{g}^*$, i.e., $\eta(X)$: $P \to g$, $X \in \mathbf{g}$ we have $\eta \in \mathbf{C}^{1,0}$. Then $\mathbf{s} = -\frac{1}{2}\delta$, and for X_0, $X_1 \in \mathbf{g}$ we get

$$\begin{aligned}(\mathbf{s}\eta)(X_0, X_1) &= -\tfrac{1}{2}\big(\pi'(X_0)\eta(X_1) - \pi'(X_1)\eta(X_0) - \eta([X_0, X_1])\big) \\ &= -\tfrac{1}{2}\big(L_{X_0}X_1 - L_{X_1}X_0 - [X_0, X_1]\big) \\ &= -\tfrac{1}{2}[X_0, X_1] \\ &= -\tfrac{1}{2}[\eta(X_0), \eta(X_1)] \\ &= -\tfrac{1}{2}[\eta, \eta](X_0, X_1).\end{aligned}$$

■

Remarks. $\mathbf{s}^2 = 0$ is the Wess-Zumino consistency condition. The ghost field η is an anticommuting vector field with values in the Lie algebra g. The equations (1) and (2) in Theorem 5.3 are the BRS transformations.

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