ILLINOIS JOURNAL OF MATHEMATICS
Volume 34, Number 1, Spring 1990

DOUBLE POINTS AND THE ORNSTEIN-UHLENBECK PROCESS ON WIENER SPACE

BY

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Introduction

The starting point for this paper is Lyons (1987) which proved that the Ornstein-Uhlenbeck process

$$\{Y(t): -\infty < t < \infty\}$$

on d-dimensional space hits states with double points if $d < 6$ but does not do so if $d \geq 6$.

I wish to offer an alternative proof of the above results using ideas found in works by Kahane and Geman, Horowitz and Rosen. Within their framework I am able to also show:

(i) If $d = 5$ then

$$\dim\{t: Y(t)(\quad) \text{ has double points}\} = \tfrac{1}{2}.$$

(ii) If $d = 4$ then

$$\dim\{t: Y(t)(\quad) \text{ has double points}\} = 1.$$

The proof of these results draws heavily on the papers German, Horowitz and Rosen (1984) and Berman (1970).

Section 1

The Ornstein-Uhlenbeck process was introduced by Malliavin (1982). It is a process

$$\{Y(t)(\quad): t \geq 0\}$$

on the Wiener space of continuous functions $C(R_+, R^d)$ which has Wiener measure as a stationary measure and such that for each $s_1 < s_2 < \cdots < s_n$,

$$\{Y(t)(s_1): t \geq 0\}\{Y(t)(s_2) - Y(t)(s_1): t \geq 0\} \cdots$$
$$\{Y(t)(s_n) - Y(t)(s_{n-1}): t \geq 0\}$$

Received December 7, 1987.

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are independent R^d valued Ornstein-Uhlenbeck processes with respective stationary measures given by

$$N(0, s_1), N(0, s_2 - s_1), \dots, N(0, s_n - s_{n-1}).$$

Meyer (1980) noted that if $\{W(t,s): (t,s) \in \{(R_+)^2\}$ is a standard d-dimensional Brownian sheet then the process

$$Y(t)(\cdot) = e^{-t/2}W(e^t, \cdot), \quad -\infty < t < \infty,$$

is a realization of the process. So every a.s. statement about the Ornstein-Uhlenbeck process can be considered as a statement concerning the d-dimensional Brownian sheet. Thus for instance, the statement that a.s. $Y(t)(\ )$ hits paths in $C(R_+, R^d)$ with double points is equivalent to the statement

$$\text{a.s. } \exists t, s, r \text{ with } e^{-t/2}W(e^t, r) = e^{-t/2}W(e^t, s)$$

which in turn is trivially equivalent to the statement

$$\text{a.s.} \exists\, t, s, r \text{ with } W(t, r) = W(t, s).$$

Section 2

By independence and similarity properties we can see that
If a.s. there does not exist $t \in [1,2]$, $s \in [1,2]$ and $r \in [3,4]$ such that

$$W(t, s) = W(t, r).$$

then a.s. there does not exist $t, s, r \in R_+ \backslash \{0\}$ satisfying

$$W(t, s) = W(t, r).$$

If with positive probability there exists $t \in [1,2]$, $s \in [1,2]$ and $r \in [3,4]$ such that

$$W(t, s) = W(t, r)$$

then a.s. there exists $t, s, r \in R_+ \backslash \{0\}$ satisfying

$$W(t, s) = W(t, r).$$

Therefore to establish whether paths with double points are a.s. hit or not it will be sufficient to consider

$$(t, s, r) \in [1,2] \times [1,2] \times [3,4].$$

Similarly to establish our Hausdorff dimension results it is only necessary to consider times within this domain.

In the following we will denote the domain $[1,2] \times [1,2] \times [3,4]$ by Δ.

Section 3

Consider the Gaussian field $\{X(t, s, r): (t, s, r) \in \Delta\}$ defined by

$$X(t, s, r) = W(t, s) - W(t, r).$$

By the observations of Section 2 we can see that the question of whether $\{Y(t)(\ \ ): t \geq 0\}$ hits paths with double points or not is equivalent to the question of whether

$\{X(t, s, r): (t, s, r) \in \Delta\}$ hits the point $0 \in R^d$ with positive probability.

We now show that this is equivalent to a question about the range of the process.

LEMMA 1. *Consider a continuous R^d-valued process $\{Z(i): i \in I\}$ where I is a compact metric space. Suppose Z can be rewritten in the form*

$$Z(i) = Y(i) + N$$

where N is an R^d-valued random variable with strictly positive density and N and $\{Y(i): i \in I\}$ are independent. Then

$$P[\mathbf{0} \in Z(I)] > 0$$

if and only if

$$P[\lambda_d(Z(I)) > 0] > 0.$$

Proof. Let us condition on the process $\{Y(i): i \in I\}$:

$$P[\mathbf{0} \in Z(I) | Y(i) i \in I] = \int_{R^d} f_N(u) I\{-u \in Y(I)\}\, du$$

where $f_N(\cdot)$ is the density of N. Since $f_N(\cdot)$ is strictly positive the above integral is greater than zero if and only if

$$\int_{R^d} I\{-u \in Y(I)\}\, du$$

is strictly positive. But the latter integral is equal $\lambda_d(Y(I))$. When we integrate over the process Y we see that

$$P[\mathbf{0} \in Z(I)] > 0$$

if and only if $\lambda_d(Y(I)) > 0$ with positive probability. But this occurs if and only if $\lambda_d(Z(I)) > 0$ with positive probability. This concludes the proof of the lemma.

The process $\{X(t, s, r): (t, s, r) \in \Delta\}$ can be written as

$$X(t, s, r) = W(1,2) - W(1,3) + Y(t, s, r)$$

where $W(1,2) - W(1,3)$ is independent of the process $\{Y(t, s, r): (t, s, r) \in \Delta\}$. Because $W(1,2) - W(1,3)$ has strictly positive density we can apply Lemma 1 to show

$\{X(t, s, r): (t, s, r) \in \Delta\}$ hits the point $0 \in R^d$ with positive probability.

if and only if

$X(\Delta)$ has positive Lebesgue measure in R^d with positive probability.

Case 1. $d \geq 6$.
It follows from the argument for Theorem 4.1 of Orey and Pruitt (1973) that a.s. $X(\Delta)$ has zero Lebesgue measure. By the above observations this takes care of this case.

Case 2. $d \leq 5$.
Following Kahane (1968) we define a random measure on R^d by

$$\mu(A) = \lambda_3(X^{-1}(A))$$

where λ_3 is Lebesgue measure on Δ.
The Fourier transform of this measure is

$$\bar{\mu}(u) = \int_\Delta e^{i\langle u, X(T)\rangle}\lambda_3(dT).$$

Obviously if $\mu \ll \lambda_d$ a.s. then we will have shown the desired result. Kahane (1968) shows that a sufficient condition for this is

$$E\left(\int_{R^d} |\bar{\mu}(u)|^2 \lambda_d(du)\right) < \infty$$

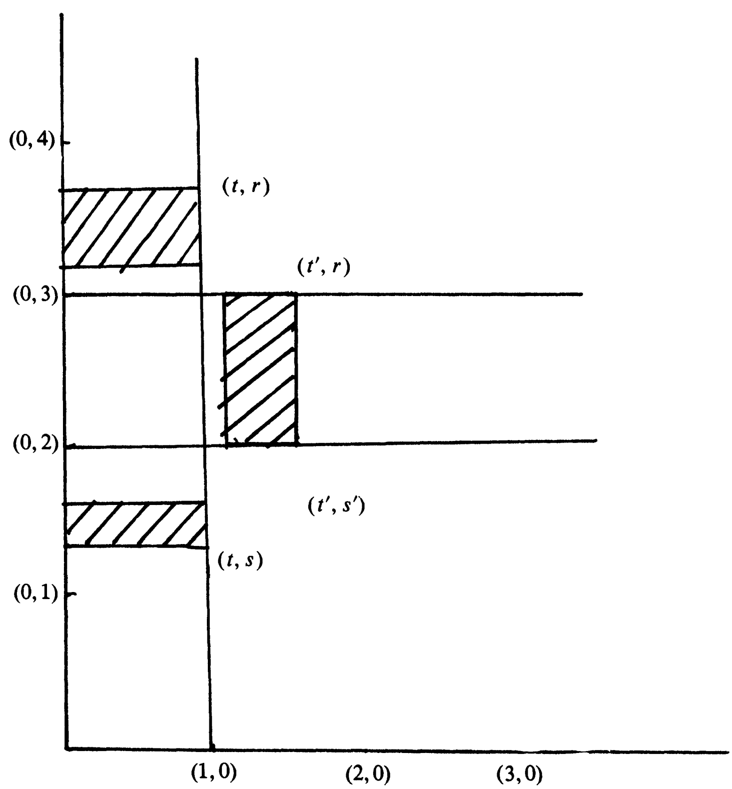
which in turn is equivalent to

$$\int_{R^d}\left(\int_{\Delta^2} e^{-VAR(\langle u,\, X(T)-X(T')\rangle)}\lambda_3(dT)\lambda_3(dT')\right)\lambda_d(du) < \infty$$

which in turn is equivalent to

$$\int_{\Delta^2}\frac{1}{[\mathrm{VAR}(\langle(1,0,\ldots 0),\, X(T)-X(T')\rangle)]^{d/2}}\lambda_3(dT)\lambda_3(dT') < \infty.$$

So it therefore only remains to estimate the integrand and bound the integration. Given $T(=(t, s, r))$ and $T'(=(t', s', r'))$ we can put a grid on the time quadrant for the Brownian sheet W which has horizontal and vertical lines through the points $\{(t, r), (t, s), (t, 2), (t, 3)(t', r')\ldots\text{etc}\}$ and through the points with integer components.



These lines and the axes divide up the time quadrant into blocks $\{B_i\}$. If a block has vertices b_1, b_2, b_3, b_4 then the random variable

$$W(b_3) - W(b_2) - W(b_4) + W(b_1)$$

is an R^d normal random variable with mean 0 and componentwise variance equal to the area of the block independent of the other random variables corresponding to other blocks.

The value of W at a corner of the grid (m, n) is equal to the sum of the random variables corresponding to blocks contained in the block with vertices $\{(0,0), (0, n), (m, 0), (m, n)\}$.

Given the above we can see that

$$\begin{aligned} X(T) - X(T') = {} & \{W(1, s) - W(1, s')\} \\ & + \{W(t, 3) - W(t', 3) - W(t, 2) + W(t', 2)\} \\ & + \{W(r, 1) - W(r', 1)\} \\ & + \text{other independent normal random variables.} \end{aligned}$$

Thus componentwise,

$$\mathrm{VAR}(X(T) - X(T')) \geq (|r - r'| + |s - s'|).$$

Given this inequality,

$$\int_{\Delta^2} \frac{1}{[\mathrm{VAR}(\langle (1, 0, \dots 0), X(T) - X(T') \rangle)]^{d/2}} \lambda_3(dT) \lambda_3(dT') \leq K \int_0^1 \frac{R^2}{R^{d/2}} dR$$

which is less than infinity if $d < 6$.

Section 4

Consider time points $T^1, T^2, \dots, T^k$ where $T^i (= (t^i, s^i, r^i))$ is in Δ for each i and

$$\begin{gathered} t^{\pi_t(1)} \leq t^{\pi_t(2)} \leq \cdots \leq t^{\pi_t(k)} \\ s^{\pi_s(1)} \leq s^{\pi_s(2)} \leq \cdots \leq s^{\pi_s(k)} \\ r^{\pi_r(1)} \leq r^{\pi_r(2)} \leq \cdots \leq r^{\pi_r(k)} \end{gathered}$$

for permutations π_t, π_s, π_4 of $\{1, 2, \dots, k\}$.

In this section we wish to show that the component-wise variance of

$$\sum_1^k \langle u_i, X(T^i)\rangle \geq \sum_{i=1}^k \left\{ \left| \sum_{j=1}^k u_{\pi_t(j)} \right|^2 \left(t^{\pi_t(i)} - t^{\pi_t(i-1)}\right) + \left| \sum_{j=1}^i u_{\pi_s(j)} \right|^2 \left(s^{\pi_s(i+1)} - s^{\pi_s(i)}\right) + \left| \sum_{j=i}^k u_{\pi_r(j)} \right|^2 \left(r^{\pi_r(i)} - r^{\pi_r(i-1)}\right) \right\}.$$

As in the previous section we shall split up the time domain of W into blocks with vertices at the integer tuples and at the points $\{(t^{\pi_t i}, s^{\pi_s(j)}), (t^{\pi_t(i)}, r^{\pi_r(l)})\}$ where i and 1 vary over $\{0, 1, 2, \ldots, k\}$ and j varies over $\{1, 2, \ldots, k+1\}$ and $t^{\pi_t(0)} = 0$, $s^{\pi_s(k+1)} = 2$, $r^{\pi_r(0)} = 3$. By the remarks made in the previous section the normal random variables

$$W(t^{\pi_t(i)}, 3) - W(t^{\pi_t(i-1)}, 3) - W(t^{\pi_t(i)}, 2) + W(t^{\pi_t(i-1)}, 2) \quad \text{for } i = 1 \text{ to } k,$$
$$W(s^{\pi_s(i)}, 1) - W(s^{\pi_s(i-1)}, 1) \quad \text{for } i = 2 \text{ to } k+1,$$

and

$$W(r^{\pi_r(i)}, 1) - W(r^{\pi_r(i-1)}, 1) \quad \text{for } i = 1 \text{ to } k$$

are all independent and have component-wise means equal to zero and variances equal to the areas of the blocks corresponding to them.

Now let us consider the random variable

$$W(s^{\pi_s(i)}, 1) - W(s^{\pi_s(i-1)}, 1);$$

this random variable "contributes" to $X(T^{\pi_s(j)})$ for $j < i$. Similarly the random variable

$$W(r^{\pi_r(i)}, 1) - W(t^{\pi_r(i-1)}, 1)$$

"contributes" to $X(T^{\pi_s(j)})$ for $j \geq i$ and the random variable

$$W(t^{\pi_t(i)}, 3) - W(t^{\pi_t(i-1)}, 3) - W(t^{\pi_t(i)}, 2) + W(t^{\pi_t(i-1)}, 2)$$

"contributes" to $X(T^{\pi_s(j)})$ for $j \geq i$. From this we obtain our desired inequality.

Section 5

Given the result of Section 3 we can apply the work of Rosen (1983) and simply read off the following result

$$\dim\{T(=(t,r,s)) \in \Delta\colon X(T)=0\} \le 3 - d/2 \quad \text{a.s.}$$

Therefore projecting the above random time set onto the axis of the first co-ordinate, we obtain

$$\dim\{t \in [1,2]\colon W(t,s) = W(t,r) \quad \text{for some } (s,r) \in [1,2]\times[3,4]\}$$
$$\le 3 - d/2 \quad \text{a.s.}$$

In particular, for the case $d = 5$ this gives

$$\dim\{t \in [1,2]\colon W(t,s) = W(t,r) \text{ for some } (s,r) \in [1,2]\times[3,4]\}$$
$$\le \tfrac{1}{2} \quad \text{a.s.}$$

The result for $d = 4$ is vacuous.

Section 6

For the other sides of the desired inequalities we use a technique from Berman (1970). In this section we will again make use of the Gaussian process X defined in Section 3. We will assume that R^d is either R^4 or R^5.

From the inequality of Section 4 it follows that for each set $B \subset \Delta$ we can define $\alpha(x, B)$, the local time for $X\colon B \to R^d$. Let us make the following definition:

$$\phi(x,t) = \alpha(x,[1,t]\times[1,2]\times[3,4]) \quad \text{for } x \in R^d \text{ and } t \in [1,2].$$

It follows from standard methods that we can choose versions of $\phi(x,t)$ such that:

(a) $\phi(x,t)$ is continuous in both x and t.

(b) $\phi(x,\ )$ is a distribution function on $[1,2]$.

(c) For a.e. $x \in R^d$ the measure with distribution function given by $\phi(x,\ )$ has support contained in

$$\{t\colon X(t,s,r) = x \text{ for some } (r,s) \in [1,2]\times[3,4]\}$$

In the following we shall work with such a version.

For $(p,t) \in [1,2]^2$ we define the function

$$H(p,t) = \int_{R^d} \phi(x,t)\phi(x,p)\,dx$$

By considering step functions and using monotone class arguments we see that for positive $g(\ ,\ )$,

$$\int_{[1,2]^2} H(dp, dt) g(p, t) = \int_{R^d} \left(\int_{[1,2]^2} g(p, t) \phi(x, dp) \phi(x, dt) \right) dx.$$

Now since $\phi(x,\ )$ is positive and increasing and $\int_{R^d} \phi^2(x, 1)\, dx < \infty$ we obtain from Parseval's formula that

$$\int_{R^d} \phi(x, p) \phi(x, t)\, dx = \int_{R^d} f(u, p) \overline{f(u, t)}\, du$$

where $f(u, p)$ ($f(u, t)$) is the Fourier transform of $\phi(\ , p)$ ($\phi(\ , t)$), i.e.,

$$f(u, p) = \int_{[1, p] \times [1,2] \times [3,4]} e^{i\langle u, x(t, s, r)\rangle}\, dr\, dt\, ds,$$

so

$$H(s, t) = \int_{R^d} \left(\int_{[0, p] \times [0, t] \times ([1,2] \times [3,4])^2} e^{i\langle u, X(p', s, r) - X(t', s', r')\rangle} \times dr'\, dt\, ds'\, dr\, dp\, ds \right) du.$$

From this we see that for a positive function $g(\ ,\ )$

$$\begin{aligned} E\left[\int_{[1,2]^2} g(p, t) H(dp, dt) \right] &\le C \int_{[1,2]^2} g(p, t)\, dp\, dt \\ &\quad \times \int_{([1,2] \times [3,4])^2} \frac{dr\, ds\, dr'\, ds'}{(|p - t| + |r - r'| + |s - s'|)^{d/2}} \\ &\le K \int_{[1,2]^2} g(p, t) |p - t|^{2 - d/2}\, dp\, dt. \end{aligned}$$

Now consider $g(p, t) = |p - t|^{-\alpha}$.

Case 1. $d = 5$. For λ_5 a.e. x,

$$\int_{[1,2]^2} \frac{\phi(x, dp) \phi(x, dt)}{|p - t|^{\alpha}} < \infty$$

if $\alpha < \frac{1}{2}$. Frostmans Theorem tells us that therefore a.e. x which has $\phi(x, 2)$

> 0 also has the support of $\phi(x, dt)$ on a set of dimension greater than or equal to $\frac{1}{2}$. But the support of this measure is a.e. contained in the set

$$\{t\colon X(t, s, r) = x \text{ for some } (s, r) \in [1,2] \times [3,4]\}.$$

Therefore the set

$$\left\{x \in R^5\colon \dim\left\{t\colon \{t\} \times [1,2] \times [3,4] \cap W^{-1}(\{x\}) \text{ is nonempty}\right\} \geq \tfrac{1}{2}\right\}$$

has positive λ_5 measure a.s. So by the argument after Lemma 1 in Section 3 we conclude that with positive probability $\dim\{t\colon \{t\} \times [1,2] \times [3,4] \cap W^{-1}(\{\mathbf{0}\})$ is non-empty$\}$ has dimension greater than or equal to $\frac{1}{2}$. This is equivalent to

$$\dim\{t \in [1,2]\colon W(t, s) = W(t, r) \text{ for some } (s, r) \in [1,2] \times [3,4]\} \geq \tfrac{1}{2}$$

This completes the proof.

Case 2. $d = 4$. The argument is essentially the same and we conclude that

$$\dim\{t \in [1,2]\colon W(t, s) = W(t, r) \text{ for some } (s, r) \cap [1,2] \times [3,4]\} > 1.$$

Higher Multiplicities

One way to think of the problem of whether or not the Ornstein-Uhlenbeck process hits k-multiple paths is to treat it as a question of whether a $k + 1$ time parameter Gaussian process hits points in $(R^d)^{k-1}$ space with positive probability. Given this perspective Lyons results for higher dimensions seem natural. The condition for k-multiple points can be written as:

$$k\text{-multiple points have positive capacity if } 2(k + 1) > (k - 1)d.$$

In a private communication Rosen has shown how using the methods of Rosen (1984) the results of the previous section can be rederived. He also computes the Hausdorff dimension of times of multiple point paths for multiplicities higher than two.

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