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A SIMPLE PROOF OF BOGOMOLOV'S THEOREM ON CLASS ∇II_0 SURFACES WITH $b_2 = 0$[1]

BY[1]

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In memory of Kuo-Tsai Chen

1. Introduction

In his famous paper [B_1] and [B_2], Bogomolov proved that all the class ∇II_0 surfaces with $b_2 = 0$ are those in the Kodaira-Inoue's list. His argument is by studying the representation

$$\rho\colon \pi_1(S) \to GL(2, \mathbf{C})$$

induced by the holomorphic affine structure (see §3), and case by case eliminate all the possibilities besides the Kodaira-Inoue list. The proof is extremely complicated and in many cases, difficult to follow. In this article, we use the existence result of Li-Yau [L-Y] to give a short proof of Bogomolov's theorem.

2. Preliminary on ∇II_0 surfaces and the statement of Bogomolov's theorem

In the classification of compact complex surfaces, ∇II_0 is the only class which has not been completed:

$$\nabla II_0 = \{\text{minimal surfaces with } b_1 = 1\}.$$

If the algebraic dimension $a(S)$ is equal to one, then S is an elliptic surface. On the other hand, if $a(S)$ is equal to zero, $b_2(S) = 0$ and S contains curves, then S is a Hopf surface (i.e., surfaces with universal covering $C^2 \setminus \{(0,0)\}$), as was observed by Kodaira many years ago. Hence one need only consider

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the following two subsets of ∇II_0, in the notation of M. Inoue [I_1]:

$$(A) = \{ S \in \nabla II_0 | b_2(S) = 0, \; S \text{ does not contain any curve} \}$$
$$(B) = \{ S \in \nabla II_0 | b_2(S) > 0 \}.$$

Set (B) is still not completely classified yet, though many examples have been found; see Nakamura [N] for details.

For set (A), Inoue constructed three types of examples: S_M, $S_t^{(+)}$, $S^{(-)}$, parametrized by some matrices and integers and $t \in \mathbf{C}$. They all have the form $\mathbf{H} \times \mathbf{C}/G$, where $\mathbf{H}$ stands for the upper half plane, and G is some discrete subgroup of the affine group acting freely and properly discontinuously on $\mathbf{H} \times \mathbf{C}$. He also proved the following [I_2]:

THEOREM 1 (INOUE). *Let $S \in (A)$. Assume that there is a line bundle L on S such that*

$$H^0(T_S \otimes L) \neq 0,$$

where T_S is the tangent bundle. Then S must be one of the examples S_m, $S_t^{(+)}$, $S^{(-)}$.

Hence the remaining surfaces in set (A) must belong to

$$(A_0) = \{ S \in (A) | H^0(T_S \otimes L) \equiv 0 \text{ for any } L \in Pic(S) \}.$$

Now we can state the Bogomolov's theorem.

THEOREM 2 (BOGOMOLOV). $(A_0) = \varnothing$.

3. Affine structures on a surface in (A_0)

Let $S \in (A_0)$ be a possible surface, and T_S, Ω_S denote its tangent, cotangent bundle. Recall that a type (1,0) connection on T_S is called holomorphically affine if, with respect to a holomorphic frame, the connection forms are holomorphic one forms. The obstruction to the existence of such a connection comes from a class in $H^1(S, \Omega_S \otimes \Omega_S \otimes T_S)$ and the difference of two such connections is an element in $H^0(S, \Omega_S \otimes \Omega_S \otimes T_S)$.

The next result was proved in [5].

THEOREM 3 [I-K-O]. *If $S \in (A_0)$, then $H^i(S, \Omega_S \otimes \Omega_S \otimes T_S) = 0$ $(i = 0, 1, 2)$.*

Hence there exists a unique holomorphic affine connection ∇_0 on the tangent bundle T_S of S.

In fact, the curvature and torsion of ∇_0 vanish, and so ∇_0 is integrable and gives a complex affine structure on S. This complex affine structure gives a representation

$$\rho: \pi_1(S) \to GL(n, C).$$

Bogolomov's argument is based on a delicate analysis of all the topological possibilities induced from this representation, and eliminate them case by case. In the following, we are going to produce a flat hermitian connection ∇ on T_S, which will violate the uniqueness of holomorphic affine connections on S (Theorem 3), and hence prove Theorem 2.

4. Hermitian-Yang-Mills connections on stable vector bundles over non-Kähler manifold

In [L-Y], Li and Yau generalized the work of Uhlenbeck-Yau to the non-Kähler case. More precisely, let M^n be a compact hermitian manifold with metric form ω. By a conformal change of the metric, one may always assume that $\partial\bar{\partial}(\omega^{n-1}) = 0$.

So for a torsion-free coherent sheaf $\mathscr{F}$ on M, the degree

$$\mu(\mathscr{F}) = \int_M c_1(\mathscr{F}) \wedge \omega^{n-1}/\text{rank}(\mathscr{F})$$

will be well defined, and one can extend the definition of μ-stability to this case, namely $\mathscr{F}$ is called μ-stable if $\mu(\mathscr{F}') < \mu(\mathscr{F})$ for any (torsion free) subsheaf $\mathscr{F}'$ of $\mathscr{F}$ with $0 < \text{rank}(\mathscr{F}') < \text{rank}(\mathscr{F})$. Just as in the Kähler case, one need only to examine the inequality for reflexive subsheaves $\mathscr{F}'$, (i.e., $\mathscr{F}' \cong \mathscr{F}'^{**}$. A rank one reflexive sheaf is necessarily locally free and is therefore a line bundle.)

In case $\mathscr{F}$ is locally free, i.e., a vector bundle, a hermitian connection ∇ on $\mathscr{F}$ is called hermitian-Yang-Mills (with respect to the base metric ω) if $Tr_\omega\Theta_\nabla = \mu I_r$. Here Θ_∇ is the curvature matrix of ∇, I_r stands for the unit matrix of rank $r = \text{rank}(\mathscr{F})$, and μ is a constant.

Parallel to the Kähler case, one has:

THEOREM 4 (LI-YAU [L-Y]). *Let M^n be a compact hermitian manifold with metric form ω satisfying $\partial\bar{\partial}(\omega^{n-1}) = 0$, and E be a rank r holomorphic vector bundle over M. Then E is μ-stable iff it admits a hermitian-Yang-Mills connection.*

5. The proof of Bogomolov's theorem: $(A_0) = \varnothing$

Let S be a possible surface in (A_0). Then the condition $H^0(T_S \otimes L) = 0$ for any $L \in \text{Pic}(S)$ says that there are no rank one reflexive subsheaves of T_S, hence T_S is μ-stable for any base metric ω with $\partial\bar{\partial}\omega = 0$. By Theorem 4, one has a hermitian connection ∇ on T_S, Namely $Tr_\omega(\Theta) = \mu I_2$, Θ being the curvature matrix of ∇. From the usual Chern form calculation, one proves curvature of the connection ∇ is equal to zero by the following argument.

Let $\{e_1, e_2\}$ be a holomorphic frame of T_S, and $\theta = (\theta_{ij})_{2\times 2}$ be the connection matrix of ∇. Then the curvature matrix of ∇ is $\Theta = \bar{\partial}\theta$, since ∇ is hermitian. Now the Yang-Mills equation reads

$$\Theta_{ij} \wedge \omega = \mu\delta_{ij}\omega \wedge \omega \tag{1}$$

and the Chern forms are given by

$$\psi_1 = \frac{\sqrt{-1}}{2\pi}\,\mathrm{tr}\,\Theta, \quad \psi_2 = \frac{-1}{4\pi^2}\det\Theta. \tag{2}$$

Since the two Chern number $c_1^2(S)$ and $c_2(S)$ are both zero, the usual algebraic calculation yields $\Theta = \bar{\partial}\theta = 0$. Hence θ is a matrix of holomorphic 1-forms and ∇ is holomorphic affine. But the torsion of a holomorphic affine connection is an element in

$$H^0(S, \Omega_S \otimes \Omega_S \otimes T_S).$$

By Theorem 3, one knows that the hermitian connection ∇ has vanishing torsion and S is Kähler. This is impossible as $b_1(S) = 1$. So we get a contradiction from the existence of an $S \in (A_0)$. Hence (A_0) must be empty, and the proof is completed.

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