ILLINOIS JOURNAL OF MATHEMATICS
Volume 34, Number 3, Fall 1990

# ON THE MODULUS OF ABSOLUTE CONTINUITY OF HOLOMORPHIC FUNCTIONS IN THE BALL

BY

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## Introduction

The starting point of this work is the result proved in our previous paper [1] according to which if $f$ is a holomorphic function in the unit ball $B^n$ of $\mathbf{C}^n$ such that $R^n f \in H^1$, i.e., if

$$(1) \qquad \sup_r \int_S |R^n f(r\zeta)|\, d\sigma(\zeta) < \infty$$

(where $R = \sum_j z_j \partial/\partial z_j$ denotes the radial derivative), then $f$ is continuous up to the boundary and it is absolutely continuous along any smooth simple curve on the unit sphere $S$ (a particular case had been previously proved by F. Beatrous in [2]). Two natural questions arise. The first is to obtain a relation between the modulus of continuity of $R^n f$ as a function in $L^1(S)$ and the modulus of absolute continuity of $f$ in $\overline{B}$. The second is to find out which form does it take in this context the general principle first pointed out by E.M. Stein in [7] stating that holomorphic functions with some kind of boundary regularity are automatically twice as regular in the complex tangential directions.

In this paper we deal with these two questions (in section 1 and 2 respectively). Rather than relying on the results of [1] we carry over alternative proofs of the global continuity of $f$ and its absolute continuity on curves that also give the desired extra information. In some sense the methods used here are more direct and elementary than those of [1] but as a counterpart they just work under certain restrictions of $f$ and the type of curves being considered.

We will consider all curves parametrized by arc-length, i.e., $\langle \varphi'(t), \varphi(t)\rangle = 1$; we will also consider associated to $\varphi(t)$ the function $T(t)$, which we call its

Received April 8, 1988.
[1]Partially supported by a grant from the National Science Foundation.
[2]Partially supported by a grant from the CICYT.

*index of transversality*, defined by the relation

$$\langle \varphi'(t), \varphi(t) \rangle = iT(t).$$

The *complex-tangential* curves are those for which $T(t) \equiv 0$.

We will sometimes use the notation $B^n$ for the unit ball of $\mathbf{C}^n$. The (normalized) Lebesgue measure of $S$ is denoted $d\sigma$ and that of $B^n$ by $d\nu_n$.

## 1. On the modulus of continuity

Let $f$ be a holomorphic function in $B$ satisfying (1) and let $\omega(\delta)$ denote the modulus of absolute continuity of the $L^1$ function $R^n f$ in $S$:

$$\omega(\delta) = \sup\left\{ \int_E |R^n f(\zeta)|\, d\sigma(\zeta), \quad \sigma(E) \leq \delta \right\}.$$

In dimension $n = 1$, it is obvious by the fundamental theorem of calculus that $\omega$ dominates the modulus of continuity of $f$:

$$|f(z) - f(w)| \leq \omega(|z - w|).$$

It is therefore quite natural to try to relate the modulus of continuity of $f$ in $\overline{B^n}$, or which is the same, in $S$, with $\omega$, also when $n > 1$. In this section we will obtain such a relation, under the assumption that $\omega$ satisfies certain "regularity conditions". Concretely, set

$$\eta(\delta) = \omega(\delta^n).$$

We will assume throughout this section that $\eta$ is regular in the sense that

$$\eta(\delta)/\delta \nearrow \infty \quad \text{as } \delta \to 0$$

and

$$\int_0^\delta \frac{\eta(t)}{t}\, dt \leq C\eta(\delta), \quad \int_\delta^1 \frac{\eta(t)}{t^{3/2}}\, dt \leq C\frac{\eta(\delta)}{\delta^{1/2}}.$$

Note that trivially we will have, for $\alpha \geq 3/2$,

$$\int_\delta^1 \frac{\eta(t)}{t^\alpha}\, dt \leq C\frac{\eta(\delta)}{\delta^{\alpha-1}}.$$

LEMMA 1.1. *Under the above assumptions, the following estimate holds*:

$$(2) \qquad |Rf(z)| = O\left(\frac{\eta(1-r)}{1-r}\right), \quad r = |z|.$$

*Proof.* Recall ([5, p. 103]) that if $f_\zeta$, $\zeta \in S$, denotes the slice functions $f_\zeta(\lambda) = f(\lambda\zeta)$, then $Rf(\zeta) = \lambda f_\zeta'(\lambda)$. In estimating $Rf(z)$, by the unitary invariance we may assume that $z = (z_1, 0, \dots, 0)$ and estimate $\partial f/\partial z_1(z_1, 0, \dots, 0)$. First we will estimate $\partial^n f/\partial z_1^n$.

Let $E$ be any set in the $z_1$-plane and let $E_0$ (resp. $E_1$) be the set of points in $S$ (resp. $B^{n-1}$) whose first component is in $E$. Then

$$\int_{E_0} |R^n f(\zeta)|\, d\sigma(\zeta) = \int_{E_1} d\nu_{n-1}(\zeta') \int_0^{2\pi} \left| R^n f\left(\zeta', (1 - |\zeta'|^2)^{1/2} e^{i\theta}\right)\right| \frac{d\theta}{2\pi}$$

with $\zeta' = (\zeta_1, \dots, \zeta_{n-1})$, by formula (2) in [5, p. 15]. By subharmonicity the last integral is greater than

$$\begin{aligned} &\int_E |R^n f(\zeta', 0)|\, d\nu(\zeta') \\ &= \int_e d\nu_1(\zeta_1) \int_{|\zeta_2|^2 + \cdots + |\zeta_{n-1}|^2 < 1 - |\zeta_1|^2} \\ &\quad \times |R^n f(\zeta_1, \zeta_2, \dots, \zeta_{n-1}, 0)|\, d\nu_{n-2}(\zeta_2, \dots, \zeta_{n-1}) \\ &\geq \int_E (1 - |\lambda|^2)^{n-2} \left| \frac{\partial^n f}{\partial \lambda^n}(\lambda, 0, \dots, 0)\right| dA(\lambda). \end{aligned}$$

If $E$ is the disc $E = \{\lambda: |\lambda - z_1| \leq \frac{1}{2}(1 - |z_1|)\}$, then $\sigma(E_0)$ is comparable to $(1 - |z_1|)^n$ and thus we obtain the estimate

$$\int_E \left| \frac{\partial^n f}{\partial z_1^n}(\lambda, 0, \dots, 0)\right| dA(\lambda) = O\left( \frac{\eta(1 - r)}{(1 - r)^{n-2}} \right).$$

By the mean value property we conclude that

$$\left| \frac{\partial^n f}{\partial z_1^n}(z_1, 0, \dots, 0)\right| = O\left( \frac{\eta(1 - r)}{(1 - r)^n} \right), \quad r = |z_1|.$$

Then (2) follows by integrating $(n - 1)$ times in $r$ and using the regularity of $\eta$. ■

In order to establish that $\eta$ is the modulus of continuity of $f$ we will no longer need the original hypothesis $R^n f \in H^1$, the result will apply to any holomorphic function satisfying (2). In what follows we will closely follow the arguments in [5, Section 6.4].

THEOREM 1.2. *Let $f$ be a holomorphic function in $B^n$ satisfying*

$$(2) \qquad |Rf(z)| = O\left(\frac{\eta(1-r)}{1-r}\right), \quad r = |z|$$

*with $\eta$ regular. Then $f$ is in the ball algebra, i.e., $f \in C(\overline{B^n})$. Furthermore:*

(a) *if $v$ is a unit direction, and $D_v$ denotes differentiation in the direction of $v$, then*

$$|D_v f(z)| = O\left(\frac{\eta(1-r)}{(1-r)^{1/2}} + |\langle v,\zeta\rangle|\frac{\eta(1-r)}{(1-r)}\right), \quad z = r\zeta$$

*and in particular*

$$|D_v f(z)| = O\left(\frac{\eta(1-r)}{1-r}\right).$$

(b) *The modulus of continuity of $f$ in $S$ is dominated by $\eta$, i.e.,*

$$|f(z) - f(w)| = O(\eta(|z-w|), \quad z,w \in S.$$

(c) *If $\varphi: I \to S$ is a $C^1$-curve on $S$ with $|\varphi'(t)| = 1$ and index of transversality $T(t)$, then*

$$|f(\varphi(t_1)) - f(\varphi(t_2))| = O\left(\eta\left(|t_1 - t_2|^2 + \int_{t_1}^{t_2} T(t)\,dt\right)\right).$$

*In particular $f(\varphi(t))$ has modulus of continuity $\eta(\delta^2)$ if $\varphi$ is complex tangential.*

(d) *$\eta$ is also the non-isotropic modulus of continuity of $f$ in $S$, i.e.,*

$$|f(z) - f(w)| = O(\eta(|1 - \langle z,w\rangle|)), \quad z,w \in S.$$

*Proof.* Since $\eta(\delta)/\delta$ is convergent near 0 it is clear that $f$ has a continuous boundary value $f(\zeta)$, and also, by regularity of $\eta$

$$(3) \qquad |f(\zeta) - f(r\zeta)| = O(\eta(1-r)).$$

To prove (e) we use that there is $c$ such that if

$$|\lambda| \le c\frac{1-r}{(1-r)^{1/2} + |\langle v,\zeta\rangle|}$$

then $1 - |r\zeta + \lambda v|^2 \ge c(1-r)$. Let $g(\lambda)$ denote the function of one complex-variable

$$g(\lambda) = Rf(r\zeta + \lambda v)$$

in the above disc. Then $g$ is bounded by $O[\eta(1-r)/1-r]$ and hence, by the Cauchy inequality,

$$|D_v Rf(r\zeta)| = O\left[\frac{\eta(1-r)}{1-r}\,\frac{(1-r)^{1/2} + |\langle v, \zeta\rangle|}{1-r}\right]. \tag{4}$$

Then part (a) follows by integrating this from 0 to $r$ (use formula 6.4.5 (2) in [5]).

Of course (b) is less precise than (d), so we proceed to prove (c). If $u(r,t) = f(r\varphi(t))$, part (a) gives

$$\left|\frac{\partial u}{\partial t}(r,t)\right| = O\left[\frac{\eta(1-r)}{(1-r)^{1/2}} + |T(t)|\frac{\eta(1-r)}{1-r}\right]. \tag{5}$$

Now write

$$\begin{aligned}|f(\varphi(t_1)) - f(\varphi(t_2))| \le |f(\varphi(t_1)) - u(r,t_1)| + |u(r,t_1) - u(r,t_2))|\\ + |u(r,t_2) - f(\varphi(t_2))|.\end{aligned}$$

The first and third terms are bounded by $\eta(1-r)$, by (3), and the second, using (5), by

$$|t_1 - t_2|\frac{\eta(1-r)}{(1-r)^{1/2}} + \frac{\eta(1-r)}{1-r}\int_{t_1}^{t_2}|T(t)|\,dt.$$

Now it suffices to choose $1 - r = |t_1 - t_2|^2 + \int_{t_1}^{t_2}|T(t)|\,dt$ to finish the proof of (c).

Finally, for part (d) we use the fact that given $z, w$ on $S$, there is a curve

$$\varphi\colon [0, L] \to S, |\varphi'(t)| = 1,$$

which is complex tangential, $L \simeq |1 - \langle z, w\rangle|^{1/2}$, with $\varphi(0) = z$, $\varphi(L) = w$ and apply part (c). ■

COROLLARY 1.3. *Suppose $R^n f \in H^1$ and let $\eta(\delta)$ be defined as above. If $\eta$ is regular, then $\eta$ dominates the modulus of continuity of $f$ in $\overline{B}$.*

To illustrate the corollary, assume that $R^n f \in H^p$ for $p > 1$. If $q$ is the conjugate exponent to $p$, then

$$\omega(\delta) = O(\delta^{1/q}), \quad \eta(\delta) = O(\delta^{n/q}).$$

Regularity of $\eta$ means $p < n/(n-1)$, and the conclusion of Corollary 6.3 is that $f$ satisfies a Hölder condition with exponent $n[1 - 1/p]$ (with respect to both the Euclidean and non-isotropic distance). This was proved by Graham [3] and Krantz [4].

If $\eta(\delta)/\delta \to 0$ as $\delta \to 0$ as for instance in the above situation when $p > n/(n-1)$, then $f$ is nicer and one can estimate the modulus of continuity of certain derivatives of $f$.

*Remark.* Note that, conversely, if $f$ is a holomorphic function in the ball algebra with modulus of continuity $\eta$, then (2) holds. This follows by applying Cauchy's inequality in a suitable disc placed in the normal direction. Hence Theorem 1.2 seems to say that a holomorphic function behaves twice as well in the complex tangential directions. We wish to point out here that this must be interpreted very carefully. For $\eta(\delta) = O(\delta^\alpha)$, $\alpha < 1/2$, $n = 2$, the hypothesis of Theorem 1.2 is equivalent to the condition that the slice functions $f_\zeta$ have a uniformly bounded Lip $\alpha$ norm. It may happen that the restriction of $f$ to every transverse curve is actually smoother than its restriction to certain complex tangential curves. For example, if $n = 2$ and

$$f(z,w) = \frac{w}{(1-z)^\alpha}, \quad 0 < \tfrac{1}{2}$$

then

$$|Rf(z,w)| = O(1 - |z| - |w|)^{-1/2-\alpha}$$

so that $f$ is in Lip$[\frac{1}{2} - \alpha]$. Now suppose $\varphi(t)$ is a transverse curve. We claim that $f \circ \varphi$ is of class Lip$(1-\alpha)$. We may assume that $\varphi(0) = (1, 0)$, transversality implying then $\varphi_2'(0) \neq 0$. So if $\varphi = (\varphi_1, \varphi_2)$,

$$1 - \varphi_1(t) = ct + O(t^2), \quad \varphi_2(t) = O(t)$$

and hence for $t$ near 0,

$$f(\varphi(t)) = O(t^{1-\alpha}).$$

In an analogous way

$$\frac{d}{dt} f(\varphi(t)) = O(t^{-\alpha})$$

and the assertion follows. On the other hand

$$f(\cos t, \sin t) = \frac{\sin t}{(1 - \cos t)^\alpha}$$

is in Lip$(1-2\alpha)$ (as indicated by Theorem 1.2) but vanishes exactly to order $1-2\alpha$ as $t \to 0$ and hence it cannot be in Lip$(1-\alpha)$. Since $1-\alpha > 1-2\alpha$ this shows that $f$ is less smooth on the complex tangential curve $(\cos t, \sin t)$ than it is on any transverse curve. The reason, of course, is that the Lip$(1-\alpha)$ norm of all the slice functions is not uniformly bounded, whereas the Lip$(\frac{1}{2}\alpha)$ norm is uniformly bounded.

Another example of the same type is the following. Consider in $B^2$ the function $f(z,w) = w/(1-z)$. A similar analysis shows that the restriction of $f$ to any transverse curve is $\mathbf{C}^\infty$ but the function is not even bounded on the complex-tangential curve $(\cos t, \sin t)$.

## 2. On the modulus of absolute continuity

Suppose $R^n f \in H^1$ and $\omega(\delta)$, $\eta(\delta)$ are as in Section 1. We know that the modulus of continuity of $f$ is dominated by $\eta$. In this section we will prove that $f$ is absolutely continuous on any real-analytic curve and that the "modulus of absolute continuity" can be estimated in terms on $\eta$ and the index of transversality. This result gives the estimate $\eta(\delta)$ for a general curve and the estimate $\eta(\delta^2)$ for a complex-tangential curve, thus giving a precise meaning to the phrase "twice as absolutely continuous". We emphasize that the method of [1] proves absolute continuity on any $C^2$ curve but without any estimate of its modulus of absolute continuity.

To simplify the development we will just consider the case $n = 2$.

We introduce the "disc" $\Pi = \{z = r\varphi(t),\ 0 \le r \le 1,\ t \in I\}$ and the measure $d\mu$ supported on $\Pi$ defined by

$$\int F\,d\mu = \int_I \int_0^1 F(r\varphi(t))\left((1-r^2)^{1/2} + |T(t)|\right) dr\,dt.$$

For $F \in H^1(B^2)$ we let

$$\omega_F(\delta) = \sup\left\{\int_E |F(\zeta)|\,d\sigma(s) \colon E \subset S,\ \sigma(E) \le \delta\right\}.$$

Lemma 2.1. *The measure $d\mu$ is a Carleson measure, i.e.,*

$$\int |F|\,d\mu \le C\|F\|_1,$$

*and if*

$$\tau_F(\delta) = \sup\left\{\int_E |F|\,d\mu \colon E \subset \Pi \colon \mu(E) \le \delta\right\}$$

*then $\tau_F(\delta) \le C\omega_F(\delta)$.*

*Proof.* If $\varphi(t) = (\varphi_1(t), \varphi_2(t))$ we introduce

$$\psi(t) = \left(-\overline{\varphi_2(t)}, \overline{\varphi_1(t)}\right).$$

Then $\psi(t)$ is a unit vector and $\langle \varphi(t), \psi(t) \rangle = 0$; i.e., it spans the complex tangent space at each point $r\varphi(t)$, $0 < r \leq 1$.

The idea to relate $\tau_F(\delta)$ with $\omega_F(\delta)$ is the same used in the proof of Lemma 1.1, where there $\Pi$ was the slice $z_2 = 0$. That is, we would use subharmonicity along complex tangent directions to write, with $x^2 = 1 - r^2$,

$$(6) \qquad |F(r\varphi(t))| \leq \frac{1}{2\pi}\int_0^{2\pi} |F(r\varphi(t)) + xe^{i\theta}\psi(t))|\, d\theta.$$

Then we will integrate in $dr$, $dt$. To compare $d\sigma$ on $S$ with $dr\,dt\,d\sigma$ we first study the properties of the map

$$\Phi(r, t, \theta) = r\varphi(t) + (1 - r^2)^{1/2} e^{i\theta}\psi(t)$$

from $(0, 1) \times I \times [0, 2\pi]$ to $S$.

LEMMA 2.2. *Suppose $\varphi$: $I = [a, b] \to R^n$ is a real analytic curve. Let $V$ be the smallest linear subspace of $R^n$ that contains $\{\varphi(t)$: $t \in I\}$, and let $W$ be the orthogonal complement of $V$ with respect to the usual inner product, $\langle\ ,\ \rangle$, on $R^n$. Then there is an integer $m \geq 0$ such that if $x \notin W$ then the equation $\langle x, \varphi(t)\rangle = 0$ has at most $m$ solutions in $I$.*

*Proof.* If $x \notin W$ then $x = x_1 + x_2$ where $x_1 \in W$, $x_2 \in V$ and $x_2 \neq 0$. The equation $\langle x, \varphi(t)\rangle = 0$ is equivalent to the equation $\langle (x_2)/(|x_2|), \varphi(t)\rangle = 0$. So we will show that there is an $m$ such that if $x \in V$, $|x| = 1$, then there are at most $m$ solutions to the equation $\langle x, \varphi(t)\rangle = 0$. First we do this locally. Fix $x \in V$, $|x| = 1$ and $t_0 \in I$. Since $\langle x, \varphi(t)\rangle \not\equiv 0$, it follows from the fact that $\varphi$ is real analytic that there is a least $m = m(t_0, x)$, such that $\langle x, \varphi^{(m)}(t_0)\rangle \neq 0$. And from this it follows from Rouche's theorem that there is a neighborhood $O_x$ of $x$ and an $\varepsilon_x > 0$ so that the equation $\langle y, \varphi(t)\rangle = 0$ has at most $m$ solutions in $t$, $|t - t_0| < \varepsilon_x$, for each $y \in O_x$. By compactness of the unit sphere in $V$ it follows that there is an $\varepsilon > 0$ and an integer $m$ so that for each $x \in V$, $|x| = 1$, the equation $\langle x, \varphi(t)\rangle = 0$ has at most $m$ solutions $t$, $|t - t_0| < \varphi$. Now by compactness of $I$, there are a finite number of intervals $I_1 \cup \cdots \cup I_N \supseteq I$ and integers $m_j$, $j = 1, \ldots, N$, so that the equation $\langle x, \varphi(t)\rangle = 0$ has at most $m_j$ solutions in $I_j$, for each $x \in V$, $|x| = 1$. Now just let $m = m_1 + \cdots + m_N$ and the lemma is proved.

Note that if $\varphi(t) \not\equiv 0$ then the real dimension of $W$ is less than $n$. This will be used in the next lemma.

LEMMA 2.3. *Suppose that $\varphi$ is a real analytic curve.*
(a) *There is an integer $m$ such that if $U = (0,1) \times I \times [0, 2\pi)$ then $d\sigma$ almost all points in $\Phi(U)$ have at most $m$ preimages.*
(b) *Let $A(t)$, $B(t)$ be defined by*

$$\varphi_1'(t)\varphi_2(t) - \varphi_1(t)\varphi_2'(t) = A(t) + iB(t).$$

*Then, up to multiplicities,*

$$d\sigma(\zeta) = |-T(t)r + x(A(t)\sin\theta - B(t)\cos\theta)|\, dr\, dt\, d\sigma$$

(*here and in the following, $x$ stands for $(1 - r^2)^{1/2}$*). *In particular for any measurable set $V$ in $U$ and any function $h$,*

$$\begin{aligned}\int_{\Phi(V)} h(\zeta)\, d\sigma(\zeta) \leq \int_V & h(\Phi(r,t,\theta)) \\ & \times |-T(t)r + x(A(t)\sin\theta - B(t)\cos\theta)|\, dr\, dt\, d\sigma \\ \leq m & \int_{\Phi(V)} h(\zeta)\, d\sigma(\zeta).\end{aligned}$$

*Remark.* Part (b) applies whenever part (a) holds. This follows from the Jacobian formula for surface integrals, see, for example, Theorem 1.2 on page 388 of [6].

*Proof.* Note that if $\Phi(r, t, \theta) = \zeta$ then $\langle \zeta, \varphi(t)\rangle = r$ and hence $\operatorname{Im}\langle \zeta, \varphi(t)\rangle = \operatorname{Re}\langle i\zeta, \varphi(t)\rangle = 0$. Since the real part of the complex inner product in $\mathbf{C}^2$ is the real inner product in $\mathbf{R}^4$, the last lemma applies. We conclude that there is a proper real linear subspace $W$ in $\mathbf{C}^2$ and an integer $m$ so that if $\zeta \notin iW$ the equation $\operatorname{Im}\langle \zeta, \varphi(t)\rangle = 0$ has at most $m$ solutions in $I$. Note that once $t$ and $\zeta$ are given, the equation $r\varphi(t) + xe^{i\theta}\psi(t) = \zeta$ determines $r$ and $\theta$ uniquely, $0 < r < 1$, $0 \leq \theta < 2\pi$. This shows that if $\zeta \notin iW$ then the equation $\Phi(r, t, \theta) = \zeta$ has at most $m$ solutions in $(0, 1) \times I \times [0, 2\pi)$. Since the intersection of $iW$ and the unit sphere in $\mathbf{C}^2$ has $\sigma$ measure zero, the proof of part a) is complete.

Now we compute the image of $d\sigma$ under $\varphi$. Let

$$\zeta_1 = r\varphi_1 - xe^{i\theta}\overline{\varphi}_2,$$

$$\zeta_2 = r\varphi_2 + xe^{i\theta}\overline{\varphi}_1, \quad \frac{dx}{dr} = -\frac{r}{x}.$$

Then

$$
\begin{aligned}
\frac{\partial}{\partial r} &= \left(\varphi_1 + \frac{r}{x} e^{i\theta} \overline{\varphi}_2\right) \frac{\partial}{\partial \zeta_1} + \left(\varphi_2 - \frac{r}{x} e^{i\theta} \overline{\varphi}_1\right) \frac{\partial}{\partial \zeta_2} \\
\frac{\partial}{\partial t} &= \left(r\varphi_1' - x e^{i\theta} \overline{\varphi}_2'\right) \frac{\partial}{\partial \zeta_1} + \left(r\varphi_2' + x e^{i\theta} \overline{\varphi}_1'\right) \frac{\partial}{\partial \zeta_2} \\
\frac{\partial}{\partial \theta} &= -ixe^{i\theta} \overline{\varphi}_2 \frac{\partial}{\partial \zeta_1} + ixe^{i\theta} \overline{\varphi}_1 \frac{\partial}{\partial \zeta_2}.
\end{aligned}
$$

From this it follows immediately (recall that $|\varphi'(t)| = 1$) that

$$
\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = \frac{1}{x^2}, \quad \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle = 1, \quad \left\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right\rangle = x^2.
$$

For the mixed terms,

$$
\begin{aligned}
\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial t} \right\rangle = {} & r\varphi_1 \overline{\varphi}_1' - xe^{i\theta} \varphi_1 \varphi_2' + \frac{r^2}{x} e^{i\theta} \overline{\varphi}_2 \overline{\varphi}_1' - r\overline{\varphi}_2 \varphi_2' \\
& + r\varphi_2 \overline{\varphi}_2' + xe^{-i\theta} \varphi_2 \varphi_1' - \frac{r^2}{x} e^{i\theta} \overline{\varphi}_1 \overline{\varphi}_2' - r\overline{\varphi}_1 \varphi_1'.
\end{aligned}
$$

Let us write $\varphi_1' \varphi_2 - \varphi_1 \varphi_2' = A(t) + iB(t)$. Note that this is the component of $\varphi'(t)$ in the complex tangent direction, hence $A^2 + B^2 = 1 - T^2$. Then,

$$
\left\langle \frac{\partial}{\partial e}, \frac{\partial}{\partial t} \right\rangle = 2irT + xe^{-i\theta}(A + iB) + \frac{r^2}{x} e^{i\theta}(A - iB),
$$

and taking real parts, and noting that $T$ is real, we have

$$
\begin{aligned}
\frac{\partial}{\partial r} \cdot \frac{\partial}{\partial t} &= x(A\cos\theta + B\sin\theta) + \frac{r^2}{x}(A\cos\theta + B\sin\theta) \\
&= \frac{A\cos\theta + B\sin\theta}{x}.
\end{aligned}
$$

Also

$$
\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right\rangle = ixe^{-i\theta} \varphi_1 \varphi_2 + ir|\varphi_2|^2 - ixe^{-i\theta} \varphi_2 \varphi_1 + ir|\varphi_1|^2 = ir,
$$

hence

$$
\frac{\partial}{\partial r} \cdot \frac{\partial}{\partial \theta} = 0,
$$

and finally

$$\begin{aligned}\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial \theta} \right\rangle &= ixr\varphi_1' e^{-i\theta}\varphi_2 - ix^2\varphi_1'\varphi_2 - ixre^{-i\theta}\varphi_2'\varphi_1 - ix^2\overline{\varphi}_1'\varphi_1 \\ &= Tx^2 + ixre^{-i\theta}(A + iB), \\ \frac{\partial}{\partial t} \cdot \frac{\partial}{\partial \theta} &= Tx^2 + xr(A\sin\theta - B\cos\theta).\end{aligned}$$

Now we have to compute

$$\Lambda = \det\begin{pmatrix} \dfrac{1}{x^2} & \dfrac{A\cos\theta + B\sin\theta}{x} & 0 \\ \dfrac{A\cos\theta + B\sin\theta}{x} & 1 & Tx^2 + xr(A\sin\theta - B\cos\theta) \\ 0 & Tx^2 + xr(A\sin\theta - B\cos\theta) & x^2 \end{pmatrix}.$$

Easy manipulations and use of the identity

$$1 - (A\cos\theta + B\sin\theta)^2 = T^2 + (A\sin\theta - B\cos\theta)^2$$

yield

$$\Lambda = [-Tr + x(A\sin\theta - B\cos\theta)]^2.$$

Hence we conclude that $d\sigma$ is mapped onto

$$|-T(t)r + x(A(t)\sin\theta - B(t)\cos\theta)|\, dr\, dt\, d\theta$$

and part (b) follows. ∎

If $E \subset \Pi$, $E = \{r\varphi(t), (r, t) \in E_0\}$, from (6) it follows that for all $F \in H^1$,

$$\text{(7)} \quad \int_E |F|\, d\mu \le \frac{1}{2r}\int\int_{E_0}\int_0^{2\pi} |F(r\varphi(t) + xe^{i\theta}\psi(t))|[x + |T(t)|]\, d\theta\, dt\, dr.$$

But according to Lemma 7.2, the right hand side is comparable to

$$\int_{\Phi(E_0\times[0,2r])} |F(\zeta)|\, d\sigma(\zeta)$$

only when $A = B = 0$, i.e., when $T = 1$ (slices). Thus in the general case we need to use subharmonicity more accurately. This is the content of the next lemma.

LEMMA 2.4. *If $h \in H^1(B^1)$ and $u$ is real, let*

$$I_u = \frac{1}{2\pi}\int_0^{2\pi}|h(e^{i\theta})||1 + u\sin t|\,d\theta.$$

*Then*
(a) $I_u \geq \max(\frac{1}{2}|u|), 1 - |u|)|h(0)| \geq \max[(|u|)/2, \frac{1}{3}]|h(0)|$ *and*
(b) $I_u \geq \frac{1}{8}|u|\,|h'(0)|$.

*Proof.* If $z = e^{i\theta}$, then $|1 + u\sin\theta| = |2iz + u(z^2 - 1)|/2$. Letting $g(z) = h(z)|2iz + u(z^2 - 1)/2$ we see that

$$I_u = \frac{1}{2\pi}\int_{-\pi}^{\pi}|g(e^{i\theta})|\,d\theta \geq |g(0)| = \frac{|u|}{2}|h(0)|.$$

On the other hand, if $0 < |u| < 1$ it is obvious that

$$I_u \geq (1 - |u|)\frac{1}{2\pi}\int_{-\pi}^{\pi}|h(e^{i\theta})|\,d\theta \geq (1 - u)|h(0)|.$$

This proves (a) since it is clear that $\max[\frac{1}{2}|u|, 1 - |u|] \geq \frac{1}{3}$.
For (b) note that

$$\frac{1}{2\pi}\int h(e^{i\theta})(1 + u\sin\theta)\,d\theta = h(0) + \frac{u}{2i}h'(0)$$

and so

$$\begin{aligned} I_u &\geq \left|h(0) + \frac{u}{2i}h'(0)\right| \geq \frac{|u|}{2}|h'(0)| - |h(0)| \\ &\geq \frac{|u|}{2}|h'(0)| - 3|I_u|, \qquad \text{by part (a)}. \end{aligned}$$

Part (b) follows. ■

*End of proof of Lemma* 2.1. Let $h(\lambda) = F(r\varphi(t) + x\lambda\Psi(t))$. We will bound from below

$$(8)\quad I: = \frac{1}{2\pi}\int_0^{2\pi}|h(e^{i\theta})||-T(t)r + x(A(T)\sin\theta - B(t)\cos\theta)|\,d\theta.$$

Recall that $x^2 = 1 - r^2$, and $A(t)^2 + B(t)^2 = 1 - T(t)^2$. Let $A - iB = \alpha = |\alpha|e^{i\eta}$. Then

$$A\sin\theta - B\cos\theta = \operatorname{Im}\alpha\cdot e^{i\theta} = |\alpha|\sin(\theta + \eta).$$

Changing from $\theta$ to $\theta + \eta$, we thus may assume that $B = 0$ and $A = \sqrt{1 - T^2}$.

Now $I = |T(t)| r I_u$, with $u = -xA/Tr$. Applying Lemma 2.4 it is immediate that

$$I \geq \max\left[\frac{\sqrt{1 - T^2}}{2}, \frac{|T(t)|r}{3}\right] |h(0)|.$$

If $r \geq 1/2$, this is $\geq c(x + |T(t)|)|h(0)|$. Hence we conclude

$$\begin{aligned}
&(x + |T(t)|)|F(r\varphi(t))| \\
&\quad \leq c \int_0^{2\pi} |F(r\varphi(t) + xe^{i\theta}\psi(t))| \, |-T(t)r + x(A\sin\theta - B\cos\theta)| \, d\theta.
\end{aligned}$$

If $E = \{r\varphi(t), (r, t) \in E_0\}$, integrating in $r, t$ and using Lemma 2.3 we obtain

$$\int_E |f| \, d\mu \leq c \int_{\Phi(E_0 \times [0, 2\pi])} |F(\zeta)| \, d\sigma(\zeta) \leq c\omega_F(\sigma(\Phi(E_0 \times [0, 2\pi]))).$$

Also, by Lemma 2.3, and since $|A\sin\theta - B\cos\theta| \leq 1$,

$$\begin{aligned}
&\sigma(\Phi(E_0 \times [0, 2\pi])) \\
&\quad = \iint_{E_0} \int_0^{2\pi} |-T(t)r + x(A(t)\sin\theta - B(t)\cos\theta)| \, dr \, dt \, d\theta \\
&\quad \leq \iint_{E_0} (|T(t)| + x) \, dr \, dt = \mu(E).
\end{aligned}$$

This finishes the proof of Lemma 2.1. ■

With Lemma 2.1 we can now measure the absolute continuity of $f$ along curves, when $F = R^2 f$ is in $H^1$:

THEOREM 2.5. *Assume that $f$ is holomorphic in $B^2$ and that $R^2 f \in H^1$. Define*

$$\begin{aligned}
\omega(\delta) &= \sup\left\{\int_E |R^2 f(\zeta)| \, d\sigma(\zeta), \, E \subset S, \, \sigma(E) \leq \delta\right\}, \\
\eta(\delta) &= \omega(\delta^2).
\end{aligned}$$

*Assume also that $\eta$ is regular in the sense of Section 1. Let $\varphi\colon I \to S$ be a real analytic curve such that $|\varphi'(t)| \equiv 1$, with index of transversality $T(t)$. Then, if*

*$I_k = [a_k, b_k]$ $k = 1, \ldots, m$ are disjoint subintervals of $I$,*

$$\sum_{k=1}^{m} |f(\varphi(b_k)) - f(\varphi(a_k))| \leq C\eta\left(\int_{UI_k} |T(t)|\,dt + \left(\sum_{k=1}^{m} (b_k - a_k)\right)^2\right).$$

Note that the estimate is entirely analogous to the one in Theorem 1.2 (c) and gives the precise meaning of the term "twice as absolutely continuous". If the curve is transverse, then

$$\int_{UI_k} |T(t)|\,dt \approx \sum (b_k - a_k)$$

and the modulus of absolute continuity is $\eta(\sum_{k=1}^{m}(b_k - a_k))$. If the curve is complex tangential, then $T = 0$ and the modulus of absolute continuity is then

$$\eta\left(\left(\sum_{k=1}^{m} (b_k - a_k)\right)^2\right).$$

*Proof.* Write $u(r, t) = f(r\varphi(t))$. For an interval $[a, b]$,

$$|u(x, b) - u(x, a)| < \int_a^b |u_t(x, t)|\,dt,$$

$$|u_t(x, t) - u_t(s, t)| \leq \int_s^x |u_{rt}(r, t)|\,dr$$

and hence

$$|u(x, b) - u(x, a)| \leq \int_a^b |u_t(s, t)|\,dt + \int_a^b \int_s^x |u_{rt}(r, t)|\,dr\,dt.$$

If $I_k = [a_k, b_k]$ are as in the statement, making $x \to 1$ and adding on $k$, we have

$$(9) \qquad \begin{aligned} &\sum_{k=1}^{m} |f(\varphi(b_k)) - f(\varphi(a_k))| \\ &\leq \int_{UI_k} |u_t(s, t)|\,dt + \int_{UI_k} \int_s^1 |u_{tr}(r, t)|\,dr\,dt \\ &:= I_1 + I_2. \end{aligned}$$

where $s$ will be chosen later.

Let $\nabla_T$ denote the tangential gradient, i.e.,

$$\nabla_T h(z) = -\bar{z}_2 \frac{\partial h}{\partial z_1}(z) + \bar{z}_1 \frac{\partial h}{\partial z_2}(z).$$

The $t$ derivative is the derivative in the direction of $\varphi'(t)$. Since

$$\varphi' = \langle \varphi', \varphi \rangle \varphi + \langle \varphi', \psi \rangle \psi$$

we can write

$$u_{tr}(r,t) = \langle \varphi'(t), \varphi(t) \rangle R^2 f(r\varphi(t)) + \langle \varphi'(t), \psi(t) \rangle \nabla_T Rf(r\varphi(t)).$$

Therefore

$$|u_{tr}(r,t)| \le |T(t)|\,|R^2 f(r\varphi(t))| + \left(1 - T(t)^2\right)^{1/2} |\nabla_T Rf(r\varphi(t))|$$

and

$$\begin{aligned} I_2 &\le \int_{UI_k} \int_s^1 |T(t)|\,|R^2 f(r\varphi(t))|\,dr\,dt \\ &\quad + \int_{UI_k} \int_s^1 \left(1 - T(t)^2\right)^{1/2} |\nabla_T Rf(r\varphi(t))\,dr\,dt \\ &:= I_3 + I_4. \end{aligned}$$

In the $dr$ integral of the last term we use the following lemma (see [1] for a proof).

LEMMA 2.6. *If $g(r)$ is differentiable in $(0,1)$ then*

$$\int_s^1 |g(r)| \le \int_s^1 (1-r)|g'(r)|\,dr + (1-s)|g(s)|, \quad 0 < s < 1.$$

We obtain

$$\begin{aligned} I_4 &\le \int_{UI_k} (1-s)|\nabla_T Rf(s\varphi(t))|\,dt \\ &\quad + \int_{UI_k} \int_s^1 (1-r)\left(1 - T(t)^2\right)^{1/2} |\nabla_T R^2 f(r\varphi(t))|\,dr\,dt \\ &:= I_5 + I_6. \end{aligned}$$

In conclusion, the left hand side of (9) is bounded by $I_1 + I_3 + I_5 + I_6$. By estimate (5),

$$I_1 \le c\left\{\frac{\eta(1-s)}{(1-s)^{1/2}} \sum_{k=1}^{m} |I_k| + \frac{\eta(1-s)}{1-s} \int_{UI_k} |T(t)|\, dt\right\}$$

and by estimate (4) with $D_v = \nabla_T$,

$$I_5 \le c\frac{\eta(1-s)}{(1-s)^{1/2}} \sum_{k=1}^{m} |I_k|.$$

If $E = \{r\varphi(t),\ 1 - s \le r \le 1,\ t \in UI_k\}$, by Lemma 2.1,

$$I_3 \le c\omega(\mu(E)).$$

Now let us bound $I_6$. For $|\lambda| \le 1$ let

$$h(\lambda) = R^2 f(r\varphi(t) + x\lambda\psi(t))$$

and let $I$ be the integral in (8). Recall that $I = |T(t)| r I_u$ with $u = -xA/Tr$. Applying Lemma 2.4 (b), we get

$$\sqrt{1 - T^2}\, x |h'(0)| \le 8I,$$

that is,

$$\begin{aligned}\sqrt{1 - T(t)^2}\,(1 - r^2)\left|\nabla_T R^2 f(r\varphi(t))\right| &\le \frac{3}{\pi}\int_0^{2\pi} \left|R^2 f\left(r\varphi(t) + xe^{i\theta}\psi(t)\right)\right| \\ &\quad \times |-T(t)r + x(A(t)\sin\theta - B(t)\cos\theta)|\, d\theta.\end{aligned}$$

With $F = R^2 f$ this is exactly the same bound obtained in the proof of Lemma 2.1. Integrating in $dr\,dt$ we thus obtain for $I_6$ the same bound as for $I_3$:

$$I_6 \le c\omega(\mu(E)).$$

On the other hand,

$$\begin{aligned}\mu(E) &= \int_s^1 \int_{UI_k} \left[(1 - r^2)^{1/2} + |T(t)|\right] dr\,dt \\ &= (1 - s)\int_{UI_k} |T(t)|\, dt + c(1 - s)^{3/2} \sum_{k=1}^{m} |I_k|.\end{aligned}$$

Hence, we finally obtain

$$\sum_{k=1}^{m} |f(\varphi(b_k)) - f(\varphi(a_k))|$$
$$\leq c\left\{\frac{\eta(1-s)}{(1-s)^{1/2}}\sum_{k=1}^{m}|I_k| + \frac{\eta(1-s)}{1-s}\int_{UI_k}|T(t)|\,dt\right\}$$
$$+\,\omega\left((1-s)\int_{UI_k}|T(t)|\,dt + (1-s)^{3/2}\sum_{k=1}^{m}|I_k|\right).$$

Now, it just remains to choose $1 - s = (\Sigma_{k=1}|I_k|)^2 + \int_{UI_k}|T(t)|\,dt$ to finish the proof of the theorem. ■

*Remark.* The constant $C$ in Theorem 2.5 depends on the curve $\varphi(t)$. In fact, from the proof we notice that it only depends on the integer $N$ in Lemma 2.3. In particular, the constant can be chosen the same for two curves $\varphi_1(t)$, $\varphi_2(t)$ related by a real orthonormal transformation; i.e., there exists a linear transformation $U \in O(2n)$ such that $U\varphi_1(t) = \varphi_2(t)$. For example, it is the same constant for all slices of $B^2$ and the curve $t| \to (\cos t, \sin t)$, so generally speaking, the function $f$ is twice as absolutely continuous along complex tangential directions.

*Remark.* The assumption that $\varphi$ is real analytic is probably not necessary. Indeed, we can prove the theorem for any *transverse* curve that is of class $C^2$. For this problem the complex tangential case seems more difficult.

To check the sharpness of the theorem, let us consider the example $f(z,w) = (1-z)^\alpha$, $\alpha > 0$. Then it is immediate that $R^2 f \in H^1(B^2)$. Also,

$$\omega(\delta) = \sup_{\sigma(E)\leq\delta}\int_E |R^2 f(\zeta)|\,d\sigma(\zeta) = \sup_{A(\zeta)\leq\delta}\int_E |f''(z)|\,dA(z).$$

Here $dA$ denotes the area element. Now

$$\int_E |f''(z)|\,dA = \int r^{\alpha-1}\chi E(re^{i\theta})\,dr\,d\theta$$

using polar coordinates centered at $(1,0)$. From this it is clear that the maximum will occur when $E$ is a disc centered at $z = 1$ and in this case $\omega(\delta) = O(\delta^{\alpha/2})$, and hence $\eta(\delta) = O(\delta^\alpha)$. This shows that the last theorem cannot be improved.

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