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CONTINUOUS SINGULAR MEASURES WITH ABSOLUTELY CONTINUOUS CONVOLUTION SQUARES ON LOCALLY COMPACT GROUPS

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1. Introduction

In this paper we show that on any locally compact non-discrete group G, there exist continuous singular measures, with respect to the left Haar measure, which have absolutely continuous convolution squares. The special case of this result regarding the existence of a singular measure on $[0, 2\pi]$ with absolutely continuous convolution squares, goes back to Wiener and Wintner [W-W]. The existence of such a measure on abelian groups is due to Hewitt and Zuckerman [H-Z]. Their construction is given by a Riesz product and the main difficulty in the general case is that the character group may admit no natural order. We also note two remarkable but difficult works of Saeki ([S_1] and [S_2]) on this subject.

Our work here differs from the above in that we use "Riesz products"

$$\prod_{k=1}^{\infty} (1 + a_k r_k(x))$$

based on a Rademacher system of functions $(r_n(x))_{n=0}^{\infty}$, which can be constructed on any non-discrete metrizable group.

In §3 we construct a singular measure $\prod_{k=1}^{\infty}(1 + a_k r_k(x))$ with absolutely continuous convolution square. Thus we obtain results without any use of characters and our method works even in the non-abelian case.

In §4 we examine this construction for the case of non-metrizable groups, and we discuss the general problem: if m is a continuous measure with convolution square $m * m \ll m$, find a measure μ which is singular with respect to m but $\mu * \mu \ll m$.

In the next section we give some preliminary notions and basic results on our Walsh system.

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2. Preliminaries

Let G be a locally compact non-discrete metrizable group. We denote by $M(G)$ the convolution measure algebra of G, which may be identified—as a Banach space—with the dual of $C_0(G)$, the space of all continuous functions on G that vanish at infinity. Let E be a Borel subset of G with left Haar measure $\lambda(E) = 1$. We shall denote by λ_E the restriction of λ on E. A system of *Rademacher* functions associated with E is a sequence of functions (r_n) which are zero off E, take the values 1 and -1 on subsets of E of equal measure, and are independent random variables, with respect to the probability measure λ_E. Since λ is non-atomic, we can certainly construct a system of Rademacher functions on E; in fact we divide E into two subsets of equal measure then divide similarly each of these two subsets to define the second Rademacher function, etc. The n-partition of E in 2^n sets of equal measure will be indexed $E_{n,1}, E_{n,2}, \ldots, E_{n,2^n}$. The *Walsh system* $(w_n)_{n=0}^{\infty}$ associated with (r_n) will be $w_0 = r_0$, and the Walsh function w_n is the finite product of Rademacher functions r_j in a way such that

$$n = 2^{j_1-1} + 2^{j_2-1} + \cdots + 2^{j_p-1}, \quad j_1 < j_2 < \cdots < j_p$$

and $w_n = r_{j_1} r_{j_2} \cdots r_{j_p}$.

A "Riesz product" associated with a sequence (r_{j_n}) and a sequence (a_n) of real numbers satisfying $|a_n| \leq 1$ is a product $\prod_{n=1}^{\infty}(\chi_E(x) + a_n r_{j_n}(x))\,dx$ where the limit is obtained in the weak* sense of $M(G)$. For simplicity we shall let (r_{j_n}) be denoted by (r'_n). We examine the convergence of this product in the weak* topology of $M(G)$ and the case where this limit μ is in the space $M_a(G)$ of all absolutely continuous measures of G.

LEMMA (2.1). *On any non-discrete metrizable group G we can find a set E and a partition of E as above such that:*

(i) $\max_{1 \leq k \leq 2^n} \operatorname{diam} E_{n,k} \to 0, \quad n \to \infty$;

(ii) *Any Walsh system associated with this partition of E is complete in* $L_2(E)$.

(iii) *If* $f_n(x) = \prod_{j=1}^{n}(X_E(x) + a_j r'_j(x))$ *where* $|a_j| \leq 1$ *then* $f_n(x)$ *converges weak* in* $M(G)$.

Proof. (i) This construction on groups homeomorphic with $\mathbf{R}^n$ is obvious. In general one can choose E to be a compact totally disconnected perfect set of positive measure which is homeomorphic with the Cantor group D. Thus a partition on D provides a partition on E satisfying (i) and such that the measure on each $E_{n,k}$ is 2^{-n}.

(ii) and (iii) These follow easily from (i). For more details we refer to [K].

PROPOSITION (2.2). *Let $d\mu = \prod_{n=1}^{\infty}(\chi_E(x) + a_n r_n'(x))\,dx$ be the "Riesz product" associated with (a_n) and (r_n'). Then μ is in $M_a(G)$ if and only if $\sum_{n=1}^{\infty} a_n^2 < \infty$.*

Proof. Let f_n be as in Lemma (2.1). Then since r_n are independent random variables,

$$\int f_n^2\,dx = \prod_{k=1}^{n}(1 + a_k^2)$$

and so if $\sum_{n=1}^{\infty} a_n^2 < \infty$, f_n is Cauchy in $L_2(G)$. For the rest of the proof see [G-M, 7.2.2] or [Z, V §7, §8].

3. The main construction

We shall prove the following theorem:

THEOREM (3.1). *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $|a_n| \le 1$, $\sum_{n=1}^{\infty} a_n^2 = \infty$ and $\sum_{n=1}^{\infty} a_n^4 < \infty$. Then there exists a subsequence (r_n') of Rademacher functions such that, if*

$$d\mu = \prod_{n=1}^{\infty}(\chi_E(x) + a_n r_n'(x))\,dx$$

then μ is a singular measure with absolutely continuous convolution square.

For this theorem we need some elementary lemmas.

LEMMA (3.2). *Let $f \in L_1(G)$. Then for any $p \in [1,\infty)$, $\|f * w_n\|_p \to 0$ and $\|w_n * f\|_p \to 0$ as $n \to \infty$.*

Proof. The "Riemann Lebesgue" property, $\int f(x) w_n(x)\,dx \to 0$ $(n \to \infty)$, is elementary. It can be used for any translate f_x of f, so that, for every x, $f * w_n(x)$ tends to zero. With f compactly supported this implies immediately the convergence in L_p for every $p < +\infty$.

LEMMA (3.3). *Let μ, ν, μ_n, ν_n, $n = 1, 2, \ldots$, be measures in the unit ball of $M(G)$ having the same compact support such that $\mu_n \to \mu$ and $\nu_n \to \nu$, $n \to \infty$, in the weak* topology. Then $\mu_n * \nu_n \to \mu * \nu$, $n \to \infty$, weak* in $M(G)$.*

Proof. This property is obvious for tensor products and so for convolution products.

Proof of (3.1). First we shall show by induction that there exists a subsequence (r'_n) of Rademacher functions such that if $f_0(x) = r_0(x)$,

$$f_k(x) = \prod_{n=1}^{k} (1 + a_n r'_n(x)), \quad k = 1, 2, \dots,$$

then for any k we have

$$\|f_k * f_k\|_2^2 \le \prod_{n=1}^{k} (1 + 2a_n^4). \tag{1}$$

Supposing that $r'_1, \dots, r'_k$ have been chosen so that (1) holds, we have

$$\begin{aligned} \|f_{k+1} * f_{k+1}\|_2^2 &= \|f_k(1 + a_{k+1}r'_{k+1}) * f_k(1 + a_{k+1}r'_{k+1})\|_2^2 \\ &= \|f_k * f_k + a_{k+1}(f_k r'_{k+1}) * f_k + a_{k+1} f_k * (f_k r'_{k+1}) \\ &\quad + a_{k+1}^2 (f_k r'_{k+1}) * (f_k r'_{k+1})\|_2^2 \\ &\le \|f_k * f_k\|_2^2 + a_{k+1}^4 \|(f_k r'_{k+1}) * (f_k r'_{k+1})\|_2^2 \\ &\quad + 14 \text{ terms of the form } |I_k(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)|, \end{aligned} \tag{2}$$

$$I_k(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \int \left(f_k r'^{\varepsilon_1}_{k+1} * f_k r'^{\varepsilon_2}_{k+1}\right)\left(f_k r'^{\varepsilon_3}_{k+1} * f_k r'^{\varepsilon_4}_{k+1}\right) dx$$

where $\varepsilon_i = 0, 1$, $i = 1, 2, 3, 4$ and at least one $\varepsilon_i \ne 0$ and one $\varepsilon_i = 0$.

One can find a Rademacher r'_{k+1} such that

$$\begin{aligned} |I_k(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)| &\le \frac{a_{k+1}^4}{14} \|f_k * f_k\|_2^2 \\ &\le \frac{a_{k+1}^4}{14} \prod_{n=1}^{k} (1 + 2a_n^4). \end{aligned} \tag{3}$$

In fact, we observe that for any $\varepsilon, \varepsilon' = 0, 1$ and any x we have

$$\left|\left(f_k r'^{\varepsilon}_{k+1}\right) * \left(f_k r'^{\varepsilon'}_{k+1}\right)(x)\right| \le f_k * f_k(x). \tag{4}$$

When one of the terms inside the integral $I_k(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ is of the kind $f_k r'_{k+1} * f_k$, using Schwarz's inequality and (4) above, one obtains

$$|I_k(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)| \le \|f_k * f_k\|_2 \|f_k r'_{k+1} * f_k\|_2. \tag{5}$$

Furthermore, by elementary calculation we have

$$|I_k(0,0,1,1)| = |I_k(1,1,0,0)| \le C \cdot \|f_k r'_{k+1} * g\|_2 \tag{6}$$

where the constant C and the function g depend only on k. Then we apply Lemma (3.2) on (5) and (6) to find r'_{k+1} satisfying (3).

We apply (3) and (4) to (2):

$$\|f_{k+1} * f_{k+1}\|_2^2 \leq \|f_k * f_k\|_2^2 + 2a_{k+1}^4 \|f_k * f_k\|_2^2.$$

In case where $k = 0$, $\|f_0 * f_0\|_2^2 \leq 1$ and so the last inequality completes our inductive proof.

Now,

$$\begin{aligned}\|f_k * f_k - f_{k+1} * f_{k+1}\|_2^2 &= \|a_{k+1}(f_k r'_{k+1}) * f_k + a_{k+1} f_k * (f_k r'_{k+1}) \\ &\qquad + a_{k+1}^2 (f_k r'_{k+1}) * (f_k r'_{k+1})\|_2^2 \\ &\leq a_{k+1}^4 \|(f_k r'_{k+1}) * (f_k r'_{k+1})\|_2^2 \\ &\qquad + 8 \text{ terms } |I_k(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)|\end{aligned}$$

and so as in (2) we have

$$\|f_k * f_k - f_{k+1} * f_{k+1}\|_2^2 < 2a_{k+1}^4 \prod_{n=1}^{\infty} (1 + 2a_n^4).$$

Since $\sum_{k=1}^{\infty} a_k^4 < \infty$, it is clear that $f_n * f_n$ converges in $L_2(G)$ and so in $L_1(G)$. Thus, since $d\mu = \lim f_n(x)\,dx$, by Lemma (3.3) and Proposition (2.2) μ is as we claimed. □

4. Comments

1. We see that the main result can be extended to locally compact non-discrete groups.

It is well known and easy to see that if G is a σ-compact non-discrete group, then there exists a compact normal subgroup H of G, such that G/H is a metrizable locally compact non-discrete group. Now if T: $C_0(G) \to C_0(G/H)$ is the usual canonical map, then the adjoint map T' is an isometric isomorphism of $M(G/H)$ onto $M(G)$. One can easily show the following, modifying similar proofs in [W].

The measure μ on G/H is absolutely continuous if and only if $T'(\mu)$ is too. Furthermore $T'(\mu * \mu) = T'(\mu) * T'(\mu)$; hence if μ is as in Theorem (3.1), $T'(\mu)$ is singular with absolutely continuous convolution square on the σ-compact group G. Note that any non-discrete group G contains σ-compact non-discrete subgroups.

2. It seems that the general problem "given a measure m, is there a measure μ singular with respect to m, such that $\mu * \mu \ll m$" remains open.

The case where m is an absolutely continuous measure such that $m * m \ll m$, follows with minor modifications from the proof of (3.1), provided of course that the Rademacher system is determined with respect to the measure m.

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$$\sum_{n=1}^{\infty} a_n^4 \prod_{j=1}^{n-1} \left(1 + a_j^2\right) < \infty$$

which we had in our initial proof.

REFERENCES

[G-M] C.C. GRAHAM and O.C. MCGEHEE, *Essays in commutative harmonic analysis*, Springer-Verlag, vol. 238, New York, 1983.

[H-Z] E. HEWITT and H. ZUCKERMAN, *Singular measures with absolutely continuous convolution squares* Proc. Cambridge Philos. Soc., vol. 62 (1966), pp. 399–420.

[K] C. KARANIKAS, *Examples of Riesz product-type measures on metrizable groups*, Boll. Un. Mat. Ital. A, (7) (4-A) (1990), pp. 331–341.

[S_1] S. SAEKI, *Singular measures having absolutely continuous convolution powers*, Illinois J. Math., vol. 21 (1977), pp. 395–412.

[S_2] ______, *On convolution squares of singular measures* Illinois J. Math., vol. 24 (1980), pp. 225–232.

[W-W] N. WIENER and A. WINTNER, *Fourier-Stieltjes transforms and singular infinite convolutions*, Amer. J. Math., vol. 60 (1938), pp. 513–522.

[W] J.H. WILLIAMSON, *A theorem on algebras of measures on topological groups*, Proc. Edinburgh Math. Soc., vol. 11 (1958/59), pp. 195–206.

[Z] A. ZYGMUND, *Trigonometric series*, two volumes, Cambridge University Press, Cambridge, Mass., 1959.

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