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QUATERNIONIC KAEHLER MANIFOLDS AND A CURVATURE CHARACTERIZATION OF TWO-POINT HOMOGENEOUS SPACES

BY

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0. Introduction

This note is a continuation of the study in [9] on a conjecture of Bob Osserman:

Conjecture. A nonflat Riemannian manifold is locally symmetric of rank one if the curvature operator $K_v = R(\cdot, v)v$ for any unit vector v has constant eigenvalues, counting multiplicities.

The affirmation of this conjecture would give us a very geometric understanding of two-point homogeneous spaces, for which the curvature condition in the conjecture is automatically satisfied.

The author showed in [9] that the conjecture is true if the dimension of the manifold is 4, an odd number, or 2 times an odd number, or if it is a Kaehler manifold of nonnegative or nonpositive curvature. The Kaehler case is a direct consequence of a result of Bishop and Goldberg stating that the maximal (minimal resp.) sectional curvature at each point of a Kaehler manifold is holomorphic, provided the manifold is nonnegatively (nonpositively resp.) holomorphically pinched [6]. Upon noticing that the curvature condition in the above conjecture implies the manifold is locally irreducible (Lemma 2), it is natural to study the quaternionic case after the Kaehler one in view of the short list of Berger on the possible holonomy groups for irreducible spaces [2], [18].

A quaternionic Kaehler manifold of dimension $4n$, $n \geq 2$, is a Riemannian manifold whose holonomy group lies in $Sp(n) \cdot Sp(1) \subset SO(4n)$[1] [5], [10], [11], [12], [15]. Such spaces bear some resemblence to, though differ much from Kaehler manifolds due to the fact that the $Sp(1)$ part of the holonomy

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[1]Since $Sp(1) \cdot Sp(1) = SO(4)$, a quaternionic Kaehler manifold of dimension 4 is just a general Riemannian manifold. However the conjecture of Osserman is true in this case as mentioned earlier.

group gives rise to a three-dimensional vector bundle $\mathscr{V}$ consisting at each point of certain complex structures compatible with the metric (see § 1); for instance, all quaternionic Kaehler manifolds are Einstein for $n \geq 2$. However, after some preliminaries we prove in Section two that a similar result to that of Bishop and Goldberg does hold for quaternionic Kaehler manifolds as well (Theorem 1), namely, the maximal (minimal resp.) sectional curvature of a quaternionic Kaehler manifold at each nonflat point is *quaternionic holomorphic*, i.e., is of the form $\langle R(X, JX)JX, X\rangle$ for some orthogonal J in $\mathscr{V}$, if the Einstein constant is nonnegative (nonpositive resp.). We then apply this result in Section three to the pullback bundle of $\mathscr{V}$ over the unit sphere bundle of the manifold and use some characteristic class arguments and basic harmonic theory to verify the conjecture for compact simply connected quaternionic Kaehler manifolds whose second Betti number vanishes, in a broader context, namely we show that the constancy of the maximal (minimal) eigenvalue of K_V alone, which may be dependent on the base point of v, will suffice for the conclusion (Theorem 2).

In Section 4 a modification of Theorem 2 implies that in the category of compact spaces whose integral cohomology rings are those of compact symmetric spaces of rank one, the eigenvalues (maximal eigenvalue if the scalar curvature is nonnegative) of K_v depending only on the base point of v characterizes the quaternionic projective spaces among simply connected quaternionic Kaehler manifolds (Theorem 3). In particular, this and the affirmation of the conjecture of Osserman in the Kaehler case mentioned in the beginning assert that the classification of two-point homogeneous spaces from our geometric-topological approach would be complete if one could prove that the space is a sphere when the holonomy group is special orthogonal (Corollaries 3 & 4).

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1. Preliminaries

We review some basic facts about quaternionic Kaehler manifolds in this section, leaving the details to [5], [10], [11], [12], [15].

A quaternionic Kaehler manifold M of dimension $4n$, $n \geq 2$, is an oriented Riemannian manifold whose holonomy group lies in

$$Sp(n) \cdot Sp(1) \subset SO(4n).$$

For such a manifold, fix an orientation and choose a local positively oriented orthonormal frame $X_1, X_2, \ldots, X_{4n}$ adapted to $Sp(n) \cdot Sp(1)$. Then quaternionic multiplications on the right on the quaternionic vector space $\mathbf{H}^n = \mathbf{R}^{4n}$

by $-i$, $-j$, $-k$, where $1, i, j, k$ stand for the standard basis for the quaternion algebra, induce three local orthogonal complex structures I, J, K with respect to the chosen frame such that $IJ = K$. The complex structures $\bar{I}, \bar{J}, \bar{K}$ for another adapted frame are related to I, J, K by an orthogonal transformation in $SO(3)$, therefore gluing $\mathscr{V} = \operatorname{Span}\langle I, J, K\rangle$ into a vector bundle of dimension 3 over M. Let $\mathscr{S} = \{aI + bJ + cK;\ a^2 + b^2 + c^2 = 1\}$ be the unit sphere bundle of $\mathscr{V}$. For notational convenience we will refer to I, J, K as I_1, I_2, I_3 from now on. Let θ^u, ω_v^u and Ω_v^u be the dual forms, connection forms, and curvature forms for the adapted frame. The connection ∇ on the manifold induces a connection on $\mathscr{V}$ also denoted by ∇. Then

$$\nabla I_i = \sum_j S_i^j I_j, \tag{1.1}$$

where

$$\begin{aligned} S_2^3 &= \omega_{2+4k}^{1+4k} - \omega_{4+4k}^{3+4k}, \\ S_3^1 &= \omega_{3+4k}^{1+4k} - \omega_{2+2k}^{4+4k}, \\ S_1^2 &= \omega_{4+4k}^{1+4k} - \omega_{3+4k}^{2+4k} \end{aligned} \tag{1.2}$$

for $0 \le k \le n-1$. Taking exterior derivatives with respect to (1.2), one finds the curvature forms Λ_v^u for the connection on $\mathscr{V}$ to be

$$\begin{aligned} \Lambda_2^3 &= \Omega_{2+4k}^{1+4k} - \Omega_{4+4k}^{3+4k}, \\ \Lambda_3^1 &= \Omega_{3+4k}^{1+4k} - \Omega_{2+4k}^{4+4k}, \\ \Lambda_1^2 &= \Omega_{4+4k}^{1+4k} - \Omega_{3+4k}^{2+4k} \end{aligned} \tag{1.3}$$

for $0 \le k \le n-1$.

In fact by examining the representation of $Sp(n) \cdot Sp(1)$ over $\mathbf{R}^{4n}$, it can be shown via (1.3) that the manifold is Einstein with Einstein constant $= (n+2)\lambda$, where

$$3\lambda = \sum_{i=1}^{3} \langle R(X, I_i X) I_i X, X\rangle, \tag{1.4}$$

and

$$\lambda = \sum_{i=0}^{3} \langle R(Y, I_i X) I_i X, Y\rangle, \tag{1.5}$$

for all unit X and unit Y perpendicular to all $I_i X$, where I_0 denotes the identity transformation. Also

$$\Lambda_2^3 = -\lambda I, \quad \Lambda_3^1 = -\lambda J, \quad \Lambda_1^2 = -\lambda K, \tag{1.6}$$

where whenever without confusion I, J, K denote the skew-symmetric forms associated with I, J, K. Hence the first Pontrjagin class of $\mathscr{V}$ is represented by the harmonic 4-form

$$p = \lambda^2/4\pi^2(I \wedge I + J \wedge J + K \wedge K). \tag{1.7}$$

Furthermore $H^{4s}(M, \mathbf{R}) \neq 0$, $s \leq n$, since $I \wedge I + J \wedge J + K \wedge K \in H^4(M, \mathbf{R})$, where $H^*(M, \mathbf{R})$ as usual denotes the real cohomology ring of M. Finally using (1.1) one derives

$$R(X,Y)I_i Z = I_i R(X,Y)Z + \sum_k \Lambda_i^k(X,Y)I_k Z. \tag{1.8}$$

2. Sectional curvature in terms of quaternionic holomorphic sectional curvatures

Motivated by [6] for the Kaehler case we first prove an algebraic identity for any sectional curvature in terms of certain quaternionic holomorphic sectional curvatures for quaternionic Kaehler manifolds. Similar to [6] for $1 \leq i \leq 3$ we define

$$Q_i(X) = \langle R(X, I_i X)I_i X, X\rangle, \; H_i(X) = Q_i(X)/\|X\|^4$$

and

$$K(X,Y) = \langle R(X,Y)Y, X\rangle.$$

LEMMA 1. *Let X, Y be perpendicular unit vectors. Let $\cos(\beta_i) = \langle I_i X, Y\rangle$. Then*

$$\begin{aligned} 24K(X,Y) = {} & -12\lambda\left[1 - \sum_i (\cos(\beta_i))^2\right] \\ & + \left[\sum_i 3(1 + \cos(\beta_i))^2 H_i(X + I_i Y)\right. \\ & \left. + 3(1 - \cos(\beta_i))^2 H_i(X - I_i Y)\right]. \end{aligned}$$

Proof. By algebraic computation one finds that

$$\begin{aligned}
&3Q_i(X+I_iY)+3Q_i(X-I_iY)\\
&\quad=6[Q_i(X)+Q_i(Y)+K(X,Y)+K(I_iX,I_iY)\\
&\qquad+2\langle R(I_iX,X)Y,I_iY\rangle+2\langle R(I_iX,I_iY)Y,X\rangle]\\
&\quad=6\Big[Q_i(X)+Q_i(Y)+4K(X,Y)+2\langle R(I_iX,X)Y,I_iY\rangle\\
&\qquad+3\sum_k\Lambda_i^k(X,Y)\langle I_kY,I_iX\rangle+\sum_k\Lambda_i^k(I_iX,I_iY)\langle I_kY,I_iX\rangle\Big],
\end{aligned}$$

where the last equality is obtained by applying (1.8) on the fourth and the last terms of the previous equality. Similarly,

$$\begin{aligned}
&Q_i(X+Y)+Q_i(X-Y)\\
&\quad=2[Q_i(X)+Q_i(Y)+2\langle R(I_iX,Y)Y,I_iY\rangle\\
&\qquad+2\langle R(I_iX,Y)X,I_iY\rangle+\langle R(I_iX,Y)Y,I_iX\rangle\\
&\qquad+\langle R(I_iY,X)X,I_iY\rangle]\\
&\quad=2\Big[Q_i(X)+Q_i(Y)+2\langle R(I_iX,X)Y,I_iY\rangle\\
&\qquad+4\langle R(I_iX,Y)X,I_iY\rangle-\sum_k\Lambda_i^k(I_iX,Y)\langle I_kX,Y\rangle\\
&\qquad-\sum_k\Lambda_i^k(X,I_iY)\langle Y,I_kX\rangle\Big].
\end{aligned}$$

One therefore yields

$$\begin{aligned}
&3Q_i(X+I_iY)+3Q_i(X-I_iY)-Q_i(X+Y)\\
&\quad-Q_i(X-Y)-4Q_i(X)-4Q_i(Y)\\
&\quad=24K(X,Y)+8\langle R(I_iX,X)Y,I_iY\rangle-8\langle R(I_iX,Y)X,I_iY\rangle\\
&\qquad+18\sum_k\Lambda_i^k(X,Y)\langle I_kY,I_iX\rangle+6\sum_k\Lambda_i^k(I_iX,I_iY)\langle I_kY,I_iX\rangle\\
&\qquad-2\sum_k\Lambda_i^k(I_iX,Y)\langle I_kX,Y\rangle-2\sum_k\Lambda_i^k(X,I_iY)\langle Y,I_kX\rangle\\
&\quad=24K(X,Y)+8\langle R(X,Y)I_iY,I_iX\rangle\\
&\qquad+\text{four remaining terms involving }\Lambda_j^i\text{ in the previous equality}\\
&\quad=32K(X,Y)+26\sum_k\Lambda_i^k(X,Y)\langle I_kY,I_iX\rangle\\
&\qquad+\text{three remaining terms involving }\Lambda_j^i\text{ in the previous equality,}
\end{aligned}$$

where the last equality is obtained by applying (1.8) to the second term of the previous equality. Summing up over i, using (1.6), and recalling the definition of $Q_i(X)$ and $H_i(X)$, one gets

$$\begin{aligned}96K(X,Y) = {} & 4\left[3\sum_i (1+\cos(\beta_i))^2 H_i(X+I_iY)\right.\\ & \left. +3\sum_i (1-\cos(\beta_i))^2 H_i(X-I_iY)\right]\\ & -4\sum_i [H_i(X+Y)+H_i(X-Y)+H_i(X)+H_i(Y)]\\ & +48\lambda\sum_i (\cos(\beta_i))^2,\end{aligned}$$

which proves the lemma when we notice that the second bracketed summation is 12λ by (1.4). Q.E.D.

THEOREM 1. *The maximal (resp. minimal) sectional curvature at each nonflat point of a quaternionic Kaehler manifold is quaternionic holomorphic if the scalar curvature is nonnegative (resp. nonpositive).*

Proof. We will prove the case when the scalar curvature is nonnegative; the other case is similar.

Let X and Y be perpendicular unit vectors spanning the section of the maximal curvature which is a nonnegative constant δ by the assumption on the scalar curvature. By Lemma 1

$$\begin{aligned}\delta &= K(X,Y)\\ &\le -\lambda/2\left[1-\sum_i (\cos(\beta_i))^2\right]\\ &\quad +\delta/8\sum_i\left[(1+\cos(\beta_i))^2+(1-\cos(\beta_i))^2\right]\\ &= \delta-\left[1-\sum_i(\cos(\beta_i))^2\right][\delta/4+\lambda/2].\end{aligned}$$

If either δ or λ is greater than zero then $\Sigma_i(\cos(\beta_i))^2 = 1$, as desired. Otherwise $0=\lambda=\delta$, which implies all the sectional curvatures at the point are zero. Q.E.D.

COROLLARY 1. *Given a quaternionic Kaehler manifold such that the maximal (minimal resp. if the scalar curvature is negative) eigenvalue of* $R(\cdot, v)v$, $\|v\| = 1$, *has multiplicity* 3 *and depends only on the base point of* v. *Then the* *space must be covered by the quaternionic projective space or its noncompact dual.*

Proof. First of all no points can be flat, otherwise at the flat points the multiplicity of $R(\cdot, v)v$ for any unit v would not be 3. Theorem 1 then says that the maximal (minimal resp. if the scalar is negative) eigenvalue of $R(\cdot, v)v$ must be quaternionic holomorphic, and hence by (1.4) must be equal to $\lambda/3$. This implies all the quaternionic holomorphic sectional curvatures are equal to $\lambda/3$. Hence the result follows from an argument in [11]. Q.E.D.

3. Affirmation of the conjecture when $H^2(M, \mathbf{R}) = 0$

By Corollary 1 the conjecture of Osserman is true once the multiplicity of the maximal (minimal resp. if the scalar curvature is negative) eigenvalue for each K_v is 3. We will prove in this section that this is the case when the second Betti number is zero.

THEOREM 2. *A compact simply connected quaternionic Kaehler manifold M with $H^2(M, \mathbf{R}) = 0$ must be the quaternionic projective space, provided the maximal* (*resp. minimal if the scalar curvature is negative*) *eigenvalue of the curvature operator $K_v = R(\cdot, v)v$ for each unit v depends only on the base point of v with a fixed constant multiplicity for all v.*

Proof. Recall that $\mathscr{V}$ is the 3-dimensional vector bundle and $\mathscr{Q}$ its unit sphere bundle associated with the manifold. It suffices to show the multiplicity of the maximal (minimal if the scalar curvature is negative) eigenvalue for K_v is 3 by Corollary 1. Suppose the contrary. For each v denote by $\mathscr{E}_v$ the eigenspace of the maximal (minimal resp.) eigenvalue for K_v. By Theorem 1, $\mathscr{E}_v$ is contained in $\mathscr{W}_v = \operatorname{Span}\langle Iv, Jv, Kv\rangle$, and either $\mathscr{E}_v$ or $\mathscr{E}_v^{\perp}$, the orthogonal complement of $\mathscr{E}_v$ in $\mathscr{W}_v$, is 1-dimensional. We may assume $\dim \mathscr{E}_v = 1$ without loss of generality. Let SM be the unit sphere bundle of M with the projection π: $SM \to M$. The vertical tangent space of SM at v, denoted $SM_v^{\perp}$, can be identified with the space $v^{\perp}$ of tangent vectors perpendicular to v; say g_v: $SM_v^{\perp} \to v^{\perp}$ is the identification. Via g_v^{-1}, $\mathscr{W}_v$ and $\mathscr{E}_v$ define two continuous vector bundles $\mathscr{W}$ and $\mathscr{E}$ respectively over SM with $\mathscr{W} \supset \mathscr{E}$. Also $\mathscr{W}$ is easily seen to be bundle isomorphic to $\pi^{-1}\mathscr{V}$, the pullback bundle of $\mathscr{V}$, via the map

$$f: (v, J) \in \pi^{-1}\mathscr{V} \subset SM \times \mathscr{V} \to Jv \in \mathscr{W}.$$

Since $\mathscr{E}$ is 1-dimensional and since SM is simply connected, $\mathscr{E}$ is a trivial bundle. Let s be a section of $\mathscr{E}$ of unit length, i.e., $s_v = aIv + bJv + cKv$, where $a^2 + b^2 + c^2 = 1$. Then

$$\bar{s} = \bar{\pi} f^{-1} s \colon SM \to \mathcal{G}$$

is a continuous bundle map over M, where $\bar{\pi}$ denotes the projection of $SM \times \mathcal{V}$ onto $\mathcal{V}$; locally this simply says

$$\bar{s}_v = aI + bJ + cK$$

when $s_v = aIv + bJv + cKv$. Moreover $g_v^{-1} \bar{s}_v g_v \colon \mathcal{E}_v^{\perp} \to \mathcal{E}_v^{\perp}$ defines a complex structure on the 2-dimensional bundle $\mathcal{E}^{\perp}$, making $\mathcal{E}^{\perp}$ into a 1-dimensional complex line bundle; for, after making a change of coordinates we may assume $\bar{s}_v = I$ so that $\mathcal{E}_v = \langle Jv, Kv \rangle$, from which it follows that I leaves $\mathcal{E}_v$ invariant. Since $\mathcal{W} = \mathcal{E} \oplus \mathcal{E}^{\perp}$ and since $\mathcal{E}$ is trivial, one has

$$p_1(\mathcal{W}) = p_1(\mathcal{E}^{\perp}) = \left[c_1(\mathcal{E}^{\perp})\right]^2 = 0, \tag{3.1}$$

where c_1 and p_1 denote the first Chern and Pontrjagin classes, and the last equality is gotten by the assumption that $H^2(M, \mathbf{R}) = 0$ and hence $H^2(SM, \mathbf{R}) = 0$ by the Gysin sequence with $m = 2$, $k = \dim M - 1$:

$$\begin{aligned} \cdots &\to H^{m-k}(M, \mathbf{R}) \to H^{m+1}(M, \mathbf{R}) \xrightarrow{\pi^*} H^{m+1}(SM, \mathbf{R}) \\ &\to H^{m-k+1}(M, \mathbf{R}) \to \cdots \end{aligned}$$

where π^* is the induced map by π. However $0 = p_1(\mathcal{W}) = \pi^* p_1(\mathcal{V})$ since $\mathcal{W} \simeq \pi^{-1}\mathcal{V}$. Therefore $p_1(\mathcal{V}) = 0$ because π^* is an isomorphism in the Gysin sequence with $m = 3$. It follows that p, the representative of $p_1(\mathcal{V})$ given in (1.6), is exact besides being harmonic; thus $p = 0$ by the Hodge theory. This forces $\lambda = 0$, implying that the manifold is hyperkaehlerian, i.e., $\mathcal{V}$ is trivial. Let I in $\mathcal{G}$ be a global complex structure such that it is parallel with respect to the connection ∇. Then (M, I) is a Kaehler manifold; therefore $H^2(M, \mathbf{R}) \neq 0$. This contradiction completes the proof. Q.E.D.

Remark 1. All the quaternionic Kaehler symmetric spaces of compact type classified in [19], except the complex Grassmann $G(n + 2, 2)$, have trivial second cohomology group with real coefficients [20]. Our result shows that one can not prescribe constant values as the eigenvalues of $R(\cdot, v)v$ on these spaces except $\mathbf{H}P^n$.

Remark 2. The result in [15] stating, in particular, that a quaternionic Kaehler manifold of positive scalar curvature with $H^2(M, \mathbf{Z}_2) = 0$ is isometric to $\mathbf{H}P^n$ does not imply Theorem 2, since on the one hand a manifold with $H^2(M, \mathbf{R}) = 0$ may have $H^2(M, \mathbf{Z}_2) \neq 0$, as can be seen by the examples in

Remark 1. On the other hand Theorem 2 is true for any scalar curvature, not just the positive ones.

COROLLARY 2. *A compact simply connected quaternionic Kaehler manifold whose second Betti number is zero must be the quaternionic projective space, provided* $K_v = R(\cdot, v)v$ *has constant eigenvalues, counting multiplicities. In particular, the statement holds if the real cohomology ring of the space is that of a quaternionic projective space.*

4. Implications of the previous results

DEFINITION. We say a compact manifold M is *homologically modelled on symmetric spaces of rank one* if its integral cohomology ring $H^*(M, \mathbf{Z})$ is that of a compact symmetric space of rank one.

LEMMA 2. *Suppose the maximal eigenvalue of* K_v *is positive and depends only on the base point of* v. *Then the manifold is locally irreducible.*

Proof. Suppose locally the manifold splits as the Riemannian product $M \times N$. Select unit vectors x and y with the same base point such that x is tangent to M and y is tangent to N. Consider $w = (x + y)/\sqrt{2}$. It is easy to see that K_x annihilates vectors tangent to N and so does K_y to those tangent to M and $K_w = (K_x + K_y)/2$, since $R(\cdot, x)y = 0$ by the Bianchi identity. Let r be the common maximal eigenvalue and let $z = a + b$, a tangent to M and b tangent to N, be an arbitrary unit vector. Then

$$r = \sup_z \langle K_w(z), z\rangle = \operatorname{Sup}\big(\langle K_x(a), a\rangle + \langle K_y(b), b\rangle\big)/2 \le r/2,$$

which is impossible. Q.E.D.

THEOREM 3. *In the category of compact manifolds homologically modelled on symmetric spaces of rank one, the eigenvalues (maximal eigenvalue if the scalar curvature is nonnegative) of the operator* K_v *depending only on the base point of* v *with constant multiplicities for all* v *characterizes the quaternionic projective spaces among simply connected quaternionic Kaehler manifolds.*

Proof. Given a quaternionic Kaehler manifold M of dimension $4n$ in the category. $H^4(M, \mathbf{R}) \neq 0$ and thus $H^4(M, \mathbf{Z}) \neq 0$. It follows that $H^*(M, \mathbf{Z}) = H^*(\mathbf{C}P^{2n}, \mathbf{Z})$, or $H^*(\mathbf{H}P^n, \mathbf{Z})$, where as usual $\mathbf{C}P^n$ and $\mathbf{H}P^n$ denote the complex and quaternionic projective spaces. If $H^*(M, \mathbf{Z}) = H^*(\mathbf{H}P^n, \mathbf{Z})$, then $H^2(M, \mathbf{R}) = 0$ in particular. Therefore Theorem 2 implies the space is the quaternionic projective space. We will next show $H^*(M, \mathbf{Z}) \neq H^*(\mathbf{C}P^{2n}, \mathbf{Z})$. Suppose the contrary; then by [14] the second Stiefel-Whitney class of $M = (2n+1)\omega = \omega \neq 0$, where $\omega \in H^2(M, \mathbf{Z}_2) = \mathbf{Z}_2$ denotes the generator. Let $\mathscr{a}$ be the generator of $H^2(M, \mathbf{Z})$ such that $\mathscr{a}^{2n}$ is the preferred fundamental cycle induced by the isomorphism $H^*(M, \mathbf{Z}) = H^*(\mathbf{C}P^{2n}, \mathbf{Z})$. By a direct computation $\mathscr{p}^n = (-1)^n[\lambda^2/4\pi^2]^n\, \mathscr{vol}$, where $\mathscr{p}$ is as in (1.7) and $\mathscr{vol}$ denotes the Riemannian volume form of the space. Therefore by choosing $\mathscr{vol}$ to be compatible or reverse to $\mathscr{a}^{2n}$ properly, we may always require $[\mathscr{p}_1(\mathscr{V})]^n$ be a negative multiple of $\mathscr{a}^{2n}$ if $\lambda \neq 0$. However, notations as in Theorem 2, we already saw $\mathscr{p}_1(\mathscr{V}) = \mathscr{c}^2$ for some $\mathscr{c}$ in view of (3.1) if $\mathscr{W}$ splits; hence letting $\mathscr{c} = t\mathscr{a}$ for some integer t, $[\mathscr{p}_1(\mathscr{V})]^n = t^{2n}\mathscr{a}^{2n}$, which is always a positive multiple of $\mathscr{a}^{2n}$. This contradiction shows $\lambda = 0$ so that M is hyper-Kaehlerian and so $\mathscr{V}$ is trivial, which implies the second Stiefel-Whitney class of $\mathscr{V}$ is zero and hence so is that of M by [13], [15], contradicting the fact that the second Stiefel-Whitney class of M is never zero by the discussions above. This establishes that $\mathscr{W}$ never splits, the same arguments as in Theorem 2 then shows that the space is isometric to $\mathbf{H}P^n$, so that $H^*(M, \mathbf{Z}) \neq H^*(\mathbf{C}P^{2n}, \mathbf{Z})$, a contradiction. Q.E.D.

COROLLARY 3. *A compact simply connected manifold of nonnegative curvature homologically modelled on rank-one symmetric spaces whose holonomy group is not special orthogonal must be the appropriate projective space, provided the maximal eigenvalue of K_v depends only on the base point of v with a fixed multiplicity for all v.*

Proof. Notice that the scalar curvature must be everywhere positive since the existence of a point where the scalar curvature vanishes would imply the space is flat by the constant multiplicity of the maximal eigenvalue of K_v, and so the space would not be compact. This implies that the holonomy group of M lies in $U(2n)$, $Sp(n) \cdot Sp(1)$, or Spin(9) by [2], [18], and Lemma 2, i.e., M is Kaehler, quaternionic Kaehler, or isometric to the Cayley plane [8]. If M is Kaehler then M is isometric to $\mathbf{C}P^{2n}$ by [9]. If M is quaternionic Kaehler then Theorem 3 completes the proof. Q.E.D.

Since a Blaschke manifold is homologically modelled on symmetric spaces of rank one and must be simply connected unless it is the standard real projective space [4], Corollary 3 provides a curvature criterion for Blaschke manifolds of nonnegative curvature whose holonomy group is not special orthogonal to be rank-one symmetric.

In particular, a compact two-point homogeneous space must be a Blaschke manifold of nonnegative curvature satisfying the curvature condition in Corollary 3, hence

COROLLARY 4. *A compact two-point homogeneous space whose holonomy group is not special orthogonal must be the appropriate projective space.*

This semi-classification of two-point homogeneous spaces from the present geometric-topological approach leaves only the sphere case to tackle.

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