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ON AN INTEGRAL OPERATOR AND ITS SPECTRUM

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1. Introduction

The action of the differential operator d/dx on the ultraspherical polynomials (spherical harmonics) $C_n^\nu(x)$ is given by

$$\frac{d}{dx} C_n^\nu(x) = 2\nu C_{n-1}^{\nu+1}(x). \tag{1.1}$$

This was used in [6] to provide a right inverse to d/dx. In this note we study the corresponding question for the Pollaczek polynomials $\{P_n^\nu(x; a, b)\}$ [3]. Recall [3] that the Pollaczek polynomials have the generating function

$$\sum_{n=0}^{\infty} P_n^\nu(x; a, b) t^n = (1 - te^{i\theta})^{-\nu+ih(x)}(1 - te^{-i\theta})^{-\nu-ih(x)}, \tag{1.2}$$

with

$$h(x) := \frac{ax + b}{\sqrt{1 - x^2}}, \qquad x = \cos\theta. \tag{1.3}$$

The branch of the square root is the branch that makes $\sqrt{x^2 - 1} \approx x$ as $x \to \infty$. Here

$$e^{i\theta} = x + \sqrt{x^2 - 1}. \tag{1.4}$$

The orthogonality relation of the Pollaczek polynomials is

$$\int_{-1}^{1} P_m^\nu(x; a, b) P_n^\nu(x; a, b) \rho(x; \nu)\, dx = \frac{2\pi\Gamma(n + 2\nu)\delta_{m,n}}{2^{2\nu}(n + a + \nu)n!}, \tag{1.5}$$

and the weight function $\rho(x; \nu)$ is

$$\rho(x; \nu) = (1 - x^2)^{\nu - 1/2} e^{(2\theta - \pi)h(x)} \Gamma(\nu + ih(x)) \Gamma(\nu - ih(x)). \tag{1.6}$$

The parameters a, b, ν are assumed to satisfy

$$a > |b| \quad \text{and} \quad \nu > 0. \tag{1.7}$$

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Define a linear operator T on polynomials through

$$TP_n^\nu(x; a, b) = 2\nu P_{n-1}^{\nu+1}(x; a, b). \tag{1.8}$$

The purpose of this note is to define a formal right inverse to T. The construction of the inverse operator depends on solving the connection coefficient problem expressing $P_n^\nu(x; a, b)$ in terms of $\{P_j^{\nu+1}(x; a, b)\}_{j=0}^n$. The solution of this connection coefficient problem is in Section 2. Section 3 contains an integral representation of a formal right inverse to T. The inverse operator T^{-1} is a linear integral operator with a non-symmetric kernel. It turns out that T^{-1} is compact, hence is trace class but not normal. In Section 2, we also find the eigenvalues and eigenfunctions of T^{-1} explicitly. The eigenvalues are related to zeros of Bessel functions $J_{\nu+a/x}(x)$. Section 4 contains q-analogues of the results of Sections 2 and 3. In Section 4 we first solve a connection coefficient problem for the q-Pollaczek polynomial. We then define a linear operator T_q by its action on the q-Pollaczek polynomials in a manner similar to (1.8). The definition is in (4.8). We also introduce a right inverse to T_q and identify its eigenvalues and eigenfunctions.

2. A connection coefficient problem

A theorem of Christoffel [7] asserts that if $\{p_n(x)\}$ are orthogonal with respect to $w(x)$, then the polynomials orthogonal with respect to $\pi(x)w(x)$, where $\pi(x)$ is a polynomial, are given by an explicit determinant expression.

The functional equation of the gamma function gives

$$\rho(x; \nu + 1) = (1 - x^2)[\nu^2 + h^2(x)]\rho(x; \nu).$$

Hence

$$\rho(x; \nu + 1) = [\nu^2(1 - x^2) + (ax + b)^2]\rho(x; \nu). \tag{2.1}$$

In this case the Christoffel formula becomes

$$\begin{aligned}
&[\nu^2(1 - x^2) + (ax + b)^2]P_n^{\nu+1}(x; a, b) \\
&\quad = \text{Constant} \begin{vmatrix} P_n^\nu(x; a, b) & P_{n+1}^\nu(x; a, b) & P_{n+2}^\nu(x; a, b) \\ P_n^\nu(x_1; a, b) & P_{n+1}^\nu(x_1; a, b) & P_{n+2}^\nu(x_1; a, b) \\ P_n^\nu(x_2; a, b) & P_{n+1}^\nu(x_2; a, b) & P_{n+2}^\nu(x_2; a, b) \end{vmatrix},
\end{aligned} \tag{2.2}$$

where x_1 and x_2 are the zeros of $\rho(x; \nu + 1)/\rho(x; \nu)$.

LEMMA 2.1. *Assume that*

$$\nu^2 + b^2 > a^2. \tag{2.3}$$

Then

$$P_n^\nu(x_1; a, b) = \frac{(2\nu)_n}{n!}\left(\frac{b+\sqrt{\Delta}}{\nu - a}\right)^n, \tag{2.4}$$

$$P_n^\nu(x_2; a, b) = \frac{(2\nu)_n}{n!}\left(\frac{b-\sqrt{\Delta}}{\nu - a}\right)^n, \tag{2.5}$$

where

$$x_1 := \frac{ab + \nu\sqrt{\Delta}}{\nu^2 - a^2}, \qquad x_2 := \frac{ab - \nu\sqrt{\Delta}}{\nu^2 - a^2} \tag{2.6}$$

and

$$\Delta := \nu^2 - a^2 + b^2. \tag{2.7}$$

In (2.4) and (2.5) we used the shifted factorial notation

$$(\sigma)_0 := 1, \quad (\sigma)_n := \prod_{j=1}^{n}(\sigma + j - 1). \tag{2.8}$$

THEOREM 2.2. *We have*

$$\begin{aligned} \frac{\rho(x; \nu+1)}{\rho(x; \nu)} P_n^{\nu+1}(x; a, b) &= \frac{(2\nu+n)(2\nu+n+1)(\nu+a)}{4(\nu+a+n+1)} P_n^\nu(x; a, b) \\ &\quad + \frac{(n+1)(2\nu+n+1)b}{2(\nu+a+n+1)} P_{n+1}^\nu(x; a, b) \\ &\quad - \frac{(n+1)(n+2)(\nu-a)}{4(\nu+a+n+1)} P_{n+2}^\nu(x; a, b). \end{aligned} \tag{2.9}$$

We now prove Lemma 2.1 and Theorem 2.2.

Proof of Lemma 2.1. Recall that x_1 and x_2 are the zeros of $[\nu + ih(x)][\nu - ih(x)]$. In fact $\nu + ih(x_1) = \nu - ih(x_2) = 0$. Thus (1.2) gives

$$\sum_{n=0}^{\infty} P_n^\nu(x_1; a, b)t^n = (1 - te^{i\theta_1})^{-\nu + ih(x_1)},$$

$x_1 = \cos\theta_1$, and the binomial theorem yields

$$P_n^\nu(x_1; a, b) = \frac{(\nu - ih(x_1))_n}{n!} e^{in\theta_1} = \frac{(2\nu)_n}{n!} e^{in\theta_1}.$$

A calculation using (1.4) and (2.6) establishes (2.4). Similarly we prove (2.5).

Proof of Theorem 2.2. Lemma 2.1, (2.1) and (2.2) show that the left-hand side of (2.9) is a constant multiple of

$$\frac{(2\nu+n)_2}{(n+1)_2}\frac{\nu+a}{\nu-a}P_n^\nu(x;a,b) + \frac{2b(2\nu+n+1)}{(\nu-a)(n+2)}P_{n+1}^\nu(x;a,b) - P_{n+2}^\nu(x;a,b). \tag{2.10}$$

The three-term recurrence relation [3]

$$(n+1)P_{n+1}^\nu(x;a,b) = 2[(n+\nu)x+b]P_n^\nu(x;a,b) - (n+2\nu-1)P_{n-1}^\nu(x;a,b) \tag{2.11}$$

and the initial conditions

$$P_0^\nu(x;a,b)=1, \quad P_1^\nu(x;a,b) = 2[(\nu+a)x+b] \tag{2.12}$$

show that

$$P_n^\nu(x;a,b) = \frac{2^n(\nu+a)_n}{n!}x^n + \text{ lower order terms}, \tag{2.13}$$

and the constant multiple of (2.10) can be found by equating coefficients of the highest power of x on both sides of (2.2). The constant is

$$\frac{(\nu-a)(n+1)(n+2)}{4(\nu+a+n+1)}.$$

A calculation now establishes (2.9).

Formula (2.9) has a dual expressing $P_n^\nu(x;a,b)$ in terms of $\{P_j^{\nu+1}(x;a,b)\}_{j=0}^n$. It is easier to derive this dual directly instead of using (2.9).

THEOREM 2.3. *If $\nu+a$ is not a negative integer then*

$$(\nu+a+n)P_n^\nu(x;a,b) = (\nu+a)P_n^{\nu+1}(x;a,b) + 2bP_{n-1}^{\nu+1}(x;a,b) + (a-\nu)P_{n-2}^{\nu+1}(x;a,b). \tag{2.14}$$

Proof. From (1.2) we find

$$\sum_{n=0}^\infty P_n^\nu(x;a,b)t^n = (1-te^{i\theta})(1-te^{-i\theta})\sum_{n=0}^\infty P_n^{\nu+1}(x;a,b)t^n = (1-2xt+t^2)\sum_{n=0}^\infty P_n^{\nu+1}(x;a,b)t^n.$$

Hence

$$P_n^\nu(x;a,b) = P_n^{\nu+1}(x;a,b) - 2xP_{n-1}^{\nu+1}(x;a,b) + P_{n-2}^{\nu+1}(x;a,b) \tag{2.15}$$

and we eliminate $xP_{n-1}^{\nu+1}(x;a,b)$ between (2.15) and (2.11) (with ν replaced by $\nu+1$). The result is (2.14).

It is worth noting that (2.9) and (2.14) are equivalent and can be derived from each other [1], [5].

3. An integral operator

Let $g \in L_2(-1,1,\rho(x,\nu+1))$ and let its orthogonal series be

$$g(x) \sim \sum_{n=0}^{\infty} g_n P_n^{\nu+1}(x;a,b). \tag{3.1}$$

Then

$$g_n = \frac{2^{2\nu+2}(n+1+\nu+a)n!}{2\pi\Gamma(n+2\nu+2)} \int_{-1}^{1} P_n^{\nu+1}(y;a,b)\rho(y,\nu+1)g(y)\,dy. \tag{3.2}$$

Since $TP_n^\nu(x;a,b) = 2\nu P_{n-1}^{\nu+1}(x;a,b)$ one can define T^{-1} through its action on the polynomials $P_n^\nu(x;a,b)$ via

$$T^{-1}P_n^{\nu+1}(x;a,b) = P_{n+1}^\nu(x;a,b)/(2\nu).$$

One can then extend the definition of T^{-1} to all of $L_2(-1,1,\rho(x,\nu+1))$ in the following manner:

$$T^{-1}\left(\sum_{n=0}^{\infty} g_n P_n^{\nu+1}(x;a,b)\right) = \sum_{n=0}^{\infty} \frac{g_n}{2\nu} P_{n+1}^\nu(x;a,b).$$

More precisely, if $g(x)$ has the orthogonal series (3.1) define

$$(T^{-1}g)(x) = \int_{-1}^{1} g(y)K_\nu(x,y)\rho(y,\nu+1)dy, \tag{3.3}$$

where

$$K_\nu(x,y) := \sum_{n=0}^{\infty} \frac{2^{2\nu}(n+1+\nu+a)n!}{\pi\nu\Gamma(n+2\nu+2)} P_{n+1}^\nu(x;a,b)P_n^{\nu+1}(y;a,b). \tag{3.4}$$

Our next objective is to find the discrete spectrum of T^{-1}. Observe that T^{-1} maps $L_2(-1, 1, \rho(x, \nu + 1))$ into $L_2(-1, 1, \rho(x, \nu))$. So if $T^{-1}g = Eg$ then

$$g \in L_2(-1, 1, \rho(x, \nu)) \cap L_2(-1, 1, \rho(x, \nu + 1)). \tag{3.5}$$

Now assume (3.5) holds and

$$T^{-1}g = Eg. \tag{3.6}$$

Therefore (3.3) implies

$$g(x) \sim \sum_{n=0}^{\infty} a_n(E) P_n^{\nu}(x; a, b), \quad a_0 = 0. \tag{3.7}$$

Therefore $Ea_n(E)$ is the coefficient of $P_n^{\nu}(x; a, b)$ in $T^{-1}g$; that is

$$Ea_n(E) = \frac{2^{2\nu}(n + \nu + a)(n-1)!}{\pi \nu \Gamma(n + 2\nu + 1)} \int_{-1}^{1} g(y)\rho(y, \nu + 1) P_{n-1}^{\nu+1}(y; a, b) dy.$$

Apply (2.10) to obtain

$$\begin{aligned} 2\nu E a_n(E) &= \frac{\nu + a}{(n - 1 + \nu + a)} a_{n-1}(E) + \frac{2b}{n + \nu + a} a_n(E) \\ &+ \frac{a - \nu}{n + \nu + a + 1} a_{n+1}(E). \end{aligned} \tag{3.8}$$

In view of (1.5) the function g of (3.7) is in $L_2(-1, 1, \rho(x, \nu))$ if and only if

$$\sum_{n=1}^{\infty} |a_n(E)|^2 \frac{\Gamma(n + 2\nu)}{(n + 1)!} < \infty. \tag{3.9}$$

In order to determine the large n asymptotics of $a_n(E)$ we set

$$a_n(E) = \left(i \sqrt{\frac{\nu + a}{\nu - a}} \right)^{n-1} \frac{\nu + a + n}{\nu + a + 1} a_1(E) b_{n-1}(E), \ n > 0. \tag{3.10}$$

Since $a_0(E) = 0$ and $a_1(E)$ is an arbitrary constant, we see that the b_n's are generated via

$$b_{-1}(E) = 0, \quad b_0(E) = 1, \tag{3.11}$$

$$b_{n+1}(E) + b_{n-1}(E) = \frac{2i}{\sqrt{\nu^2 - a^2}} [\nu(\nu + a + 1 + n)E - b] b_n(E). \tag{3.12}$$

At this stage we note that it is more convenient to renormalize E and b_n through

$$u := i\nu E / \sqrt{\nu^2 - a^2}, \quad B := -ib / \sqrt{\nu^2 - a^2}, \quad c_n(u) = b_n(E). \tag{3.13}$$

Therefore

$$c_{n+1}(u) = 2[(n+\nu+a+1)u + B]c_n(u) - c_{n-1}(u). \tag{3.14}$$

Now formula (3.14) is (1.11) in [4] and in the notation of [4] we have

$$a_n(E) = \left(i\sqrt{\frac{\nu+a}{\nu-a}}\right)^{n-1} \frac{\nu+a+n}{\nu+a+1} a_1(E)\tau_{n-1}(u, 1+a+\nu, B). \tag{3.15}$$

In view of

$$\Gamma(a+z)/\Gamma(b+z) \approx z^{a-b}, \quad z \to \infty, \quad |\arg z| < \pi, \tag{3.16}$$

(3.13) in [4] and (3.15) establish

(3.17)

$$a_n(E)\sqrt{\frac{\Gamma(n+2\nu)}{\Gamma(n+2)}} \approx \frac{n^\nu}{\nu+a+1}\left(i\sqrt{\frac{\nu+a}{\nu-a}}\right)^{n-1} a_1(E) J_{\nu+a+B/u}(1/u)$$
$$\times (2u)^{n+\nu+a-1+B/u}\Gamma(n+\nu+a+B/u) \ as\ n \to \infty.$$

This shows that the series (3.9) diverges unless $1/u$ is a nontrivial zero of $J_{\nu+a+Bz}(z)$ or $u = 0$. If $u = 0$ then (3.14) implies

$$c_n(0) = U_n(B),$$

$\{U_n(x)\}$ a Chebyshev polynomial of the second kind. Therefore

$$\sqrt{1-B^2}\, c_n(0) \approx \frac{1}{2}\left(\frac{b+\sqrt{\Delta}}{\sqrt{\nu^2-a^2}}\right)^{\pm(n+1)} \quad \text{as } n \to \infty$$

according as $b > 0$ or $b < 0$, respectively. When $b = 0$ then

$$|c_n(0)| = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

Thus we conclude that the series (3.9) diverges at $E = 0$.

The remaining candidates for eigenvalues are when u, as per (3.13), is a nontrivial zero of $J_{\nu+a+B/u}(1/u)$. Ismail [4] established the recursion relation

$$\begin{aligned} J_{\nu+a+B/u+n}(1/u) &= \tau_n(u, \nu+a, B) J_{\nu+a+B/u}(1/u) \\ &\quad - \tau_{n-1}(u, \nu+a+1, B) J_{\nu+a-1+B/u}(1/u). \end{aligned} \tag{3.18}$$

When $J_{\nu+a+B/u}(1/u)$ vanishes then (3.18) yields

$$\begin{aligned} J_{\nu+a-1+B/u}(1/u)\tau_{n-1}(u, \nu+a+1, B) &= -J_{\nu+a+n+B/u}(1/u) \\ &\approx -\frac{(2u)^{-\nu-a-n-B/u}}{\Gamma(\nu+a+n+1+B/u)} \end{aligned} \tag{3.19}$$

as $n \to \infty$. It also follows from [4] that $J_{\nu+a-1+B/u}(1/u)$ does not vanish at the nontrivial zeros of $J_{\nu+a+B/u}(1/u)$. Finally (3.15) and (3.19) imply the convergence of the series (3.9). This establishes the following theorem.

THEOREM 3.4. *The eigenvalues* $\{\lambda_n\}$ *of the integral operator* T^{-1} *of* (3.3) *are precisely the reciprocals of the non-trivial zeros of* $J_{\nu+a-bz/\nu}(-i\sqrt{\nu^2-a^2}z/\nu)$. *The corresponding eigenfunctions are*

$$(3.20)\quad g(x,\lambda_n) = \sum_{k=1}^{\infty}\left(i\sqrt{\frac{\nu+a}{\nu-a}}\right)^{k-1}\frac{\nu+a+k}{\nu+a+1} \times \tau_{k-1}(1/\eta_n, 1+\nu+a, -ib/\sqrt{\nu^2-a^2})P_k^{\nu}(x;a,b),$$

where $\{\eta_j\}$ *are the zeros of* $J_{\nu+a+Bz}(z)$, *and*

$$(3.21)\qquad \lambda_n = -\frac{i\sqrt{\nu^2-a^2}}{\nu\eta_n}.$$

4. The q-Pollaczek polynomials

In this section we study the same problem for q-Pollaczek polynomials. By [2], q-Pollaczek polynomials are defined by the following three-term recurrence formula:

$$(4.1)\qquad \begin{aligned} F_0(x;U,V,\Delta;q) &= 1,\\ F_1(x;U,V,\Delta;q) &= 2[(1-\Delta U)x+V]/(1-q),\\ \left(1-q^{n+1}\right)F_{n+1}(x;U,V,\Delta;q) &= 2\left[\left(1-U\Delta q^n\right)x+Vq^n\right]F_n(x;U,V,\Delta;q)\\ &\quad -\left(1-\Delta^2q^{n-1}\right)F_{n-1}(x;U,V,\Delta;q),\ (n\geq 2). \end{aligned}$$

For convenience, we use the simpler notations

$$F_n(x) = F_n(x;U,V,\Delta;q);\qquad G_n(x) := F_n(x;U,qV,q\Delta;q).$$

From [4] we know that the q-Pollaczek polynomials have the generating function

$$(4.2)\qquad F(x,t) = \sum_{n=0}^{\infty}F_n(x)t^n = \frac{(t/\xi;q)_\infty(t/\zeta;q)_\infty}{(t/\alpha;q)_\infty(t/\beta;q)_\infty}$$

where α and β are roots of $t^2-2xt+1=0$, so that $\alpha=e^{i\theta}$, $\beta=e^{-i\theta}$; and ξ and ζ are given by

$$\Delta^2t^2-2(U\Delta x-V)t+1 = (1-t/\xi)(1-t/\zeta).$$

In (4.2) we use the notation

$$(a;q)_0 := 1, \qquad (a;q)_n = \prod_{j=1}^{n}(1-aq^{j-1}),\ j = 1,2,\ldots \text{or } \infty.$$

Now we want to express $F_n(x)$ as a linear combination of $G_j(x)$, $(0 \le j \le n)$. This will be a q-analogue of Theorem 2.3. It is clear that (4.2) implies

$$\text{(4.3)} \quad \sum_{n=0}^{\infty} G_n(x)t^n = \frac{(tq/\xi;q)_\infty (tq/\zeta;q)_\infty}{(t/\alpha;q)_\infty (t/\beta;q)_\infty} = \frac{F(x,t)}{(1-t/\xi)(1-t/\zeta)} = \frac{F(x,t)}{\Delta^2 t^2 - 2(U\Delta x - V)t + 1};$$

hence

$$\text{(4.4)} \qquad F_n(x) = G_n(x) - 2(U\Delta x - V)G_{n-1}(x) + \Delta^2 G_{n-2}(x).$$

In (4.1), replace Δ by $q\Delta$, V by qV, and eliminate $xG_{n-1}(x)$ between (4.1) and (4.4). Then

$$\text{(4.5)} \qquad (1-U\Delta q^n)F_n(x) = (1-U\Delta)G_n(x) + 2VG_{n-1}(x) + \Delta(\Delta - U)G_{n-2}(x)$$

From [2] the polynomials $\{F_n(x)\}$ are orthogonal with respect to the weight function

$$\text{(4.6)} \qquad w(x) = w(x;U,V,\Delta;q) := \frac{(1-U\Delta)\left(\Delta^2, q, e^{2i\theta}, e^{-2i\theta}; q\right)_\infty}{2\pi \sin\theta \left(e^{i\theta}/\xi, e^{-i\theta}/\xi, e^{i\theta}/\zeta, e^{-i\theta}/\zeta; q\right)_\infty}.$$

Formula (4.5) has a dual expressing $G_n(x)W(x)$ in terms of $\left\{F_j(x)w(x)\right\}_{j=0}^{\infty}$, where the weight function W is the weight function for $\{G_n\}$. In other words

$$W(x) = w(x;U,qV,q\Delta;q).$$

The dual formula to (4.5) is

$$\text{(4.7)} \qquad G_n(x)W(x) = \sum_{j=n}^{n+2} C_{n,j}F_j(x)w(x)$$

where

$$I_n := \int_{-1}^{1} F_n^2(x)w(x)\,dx = \frac{(1-U\Delta)\left(\Delta^2;q\right)_n}{(1-U\Delta q^n)\,(q;q)_n}$$

$$I_n^* := \int_{-1}^{1} G_n^2(x)W(x)\,dx = \frac{(1-U\Delta q)\left(\Delta^2 q^2; q\right)}{\left(1-U\Delta q^{n+1}\right)(q;q)_n}$$

$$C_{n,j} = \frac{1}{I_j}\int_{-1}^{1} G_n(x)F_j(x)W(x)\,dx$$

$$C_{n,n} = \frac{I_n^*(1-U\Delta)}{I_n\,(1-U\Delta q^n)} = \frac{(1-U\Delta q)\left(1-\Delta^2 q^n\right)\left(1-\Delta^2 q^{n+1}\right)}{\left(1-U\Delta q^{n+1}\right)\left(1-\Delta^2\right)\left(1-\Delta^2 q\right)}$$

$$C_{n,n+1} = \frac{I_n^* 2V}{I_{n+1}\left(1-U\Delta q^{n+2}\right)} = \frac{2V(1-U\Delta q)\left(1-q^{n+1}\right)\left(1-\Delta^2 q^{n+1}\right)}{\left(1-U\Delta q^{n+1}\right)\left(1-\Delta^2\right)(1-U\Delta)\left(1-\Delta^2 q\right)}$$

$$C_{n,n+2} = \frac{I_n^*\Delta(\Delta-U)}{I_{n+2}\left(1-U\Delta q^{n+2}\right)} = \frac{\Delta(\Delta-U)(1-U\Delta q)\left(1-q^{n+1}\right)\left(1-q^{n+2}\right)}{\left(1-U\Delta q^{n+1}\right)\left(1-\Delta^2\right)(1-U\Delta)\left(1-\Delta^2 q\right)}.$$

It is worth mentioning that in the following calculation, the first expression of the above $C_{n,j}$ is more convenient.

Consider a linear operator T_q defined on the span of the F_n's through

$$T_q F_n(x) = \frac{2(1-\Delta)q^{(1-n)/2}}{1-q}\, G_{n-1}(x). \tag{4.8}$$

This defines T_q on a dense subset of $L_2(-1, 1, w)$. We now seek a linear operator T_q^{-1} for which

$$T_q^{-1} G_n(x) = \frac{(1-q)q^{n/2}}{2(1-\Delta)}\, F_{n+1}(x).$$

For $g(x) \in L_2(-1, 1, W(x))$. Let $g(x) \sim \sum_{n=0}^{\infty} g_n G_n(x)$. We define T_q^{-1} as the integral operator

$$\left(T_q^{-1}\right)(x) = \int_{-1}^{1} g(y)K_q(x, y)W(y)\,dy, \tag{4.9}$$

where the kernel $K_q(x, y)$ is defined by

$$\begin{aligned} K_q(x, y) &:= \sum_{n=0}^{\infty} \frac{(1-q)\left(I_n^*\right)^{-1} q^{n/2}}{2(1-\Delta)}\, F_{n+1}(x)G_n(y) \\ &= \sum_{n=0}^{\infty} \frac{(1-q)(1-U\Delta q^{n+1})q^{n/2}(q;q)_n}{2(1-\Delta)(1-U\Delta q)(q^2\Delta^2;q)_n}\, F_{n+1}(x)G_n(y). \end{aligned} \tag{4.10}$$

Next, we find the discrete spectrum of T_q^{-1}. It is easy to check that T_q^{-1} maps $L_2(-1, 1, W(x))$ into $L_2(-1, 1, w(x))$. Hence if

$$T_q^{-1} g = E\, g, \tag{4.11}$$

then $g \in L_2(-1, 1, W(x)) \cap L_2(-1, 1, w(x))$. Now assume that (4.11) holds; then by (4.9),

$$g(x) \sim \sum_{n=0}^{\infty} A_n(E) F_n(x), \quad A_0(E) := 0. \tag{4.12}$$

Combining (4.9) with (4.11), we get

$$\begin{aligned} E A_n(E) &= \int_{-1}^{1} g(y) \frac{(1-q)q^{(n-1)/2}}{2(1-\Delta) I_{n-1}^*} G_{n-1}(y) W(y)\, dy \\ &= \frac{(1-q)(1-U\Delta)q^{(n-1)/2}}{2(1-\Delta)\left(1-U\Delta q^{n-1}\right)} A_{n-1}(E) \\ &\quad + \frac{2V(1-q)q^{(n-1)/2}}{2(1-\Delta)(1-U\Delta q^n)} A_n(E) \\ &\quad + \frac{\Delta(\Delta-U)(1-q)q^{(n-1)/2}}{2(1-\Delta)\left(1-U\Delta q^{n+1}\right)} A_{n+1}(E). \end{aligned} \tag{4.13}$$

We take

$$U = q^a, \quad \Delta = q^\nu, \quad a < \nu$$

and renormalize as

$$A_n(E) = D^{n-1} \frac{1-q^{n+a+\nu}}{1-q^{1+a+\nu}} q^{-(n-1)^2/4} A_1(E) b_{n-1}(LE) \tag{4.14}$$

where

$$\begin{aligned} D &= i \left(\frac{1-q^{a+\nu}}{q^{2\nu+a}(q^a - q^\nu)} \right)^{\frac{1}{2}}, \\ L &= \frac{i q^{\frac{1}{4}+\frac{a}{2}} (1-q^\nu)}{(1-q)\sqrt{(1-q^{a+\nu})(q^a-q^\nu)}}. \end{aligned} \tag{4.15}$$

Then (4.13) becomes

(4.16)

$$b_n(E) = 2\left[E\left(1-q^{n+a+\nu}\right) + M q^{(n+a+\nu)/2}\right] b_{n-1}(E) - q^{n+a+\nu-1} b_{n-2}(E)$$

where

$$M = \frac{-Vi}{q^{\frac{1}{4}+\frac{\nu}{2}}\sqrt{(1-q^{a+\nu})(q^a-q^\nu)}}.$$

We shall first consider the symmetric case $V = 0$. In this case, $M = 0$ and (4.16) becomes

$$b_n(E) = 2E\left(1-q^{n+a+\nu}\right)b_{n-1}(E) - q^{n+a+\nu-1}b_{n-2}(E) \tag{4.17}$$

which is (1.22) of [4]. Ismail [4] introduced a q-analogue of the Lommel polynomials. Comparing (4.17) with (3.6) in [4] we arrive at the identification

$$\begin{aligned} b_n(E) &= h_{n,a+\nu+1}(E;q) \\ &= \sum_{j=0}^{[n/2]} \frac{(2E)^{n-2j}(-1)^j\left(q^{a+\nu+1};q\right)_{n-j}(q;q)_{n-j}}{(q;q)_j\left(q^{a+\nu+1};q\right)_j(q;q)_{n-2j}} q^{j(j+a+\nu)}. \end{aligned} \tag{4.18}$$

By (3.9) of [4] we get

$$h_{n,a+\nu+1}(E;q) \approx \frac{(q;q)_\infty J^{(2)}_{a+\nu}(1/E;q)}{(2E)^{-n-a-\nu}} \text{ as } n \to \infty \tag{4.19}$$

where $J^{(2)}_{a+\nu}$ is a q-Bessel function [4].

Equation (4.12) is true iff

$$\sum_{n=1}^{\infty} |A_n(E)|^2 \frac{(1-U\Delta)\left(\Delta^2;q\right)_n}{(1-U\Delta q^n)(q;q)_n} < \infty. \tag{4.20}$$

We consider the asymptotic behavior of $A_n(E)$ so as to determine the solution of (4.11). We have

$$\begin{aligned} |A_n(E)| &\approx D^{n-1}\frac{A_1}{1-q^{1+a+\nu}} q^{-(n-1)^2/4} b_{n-1}(LE) \\ &\approx D^{n-1}\frac{A_1(2LE)^{n-1+a+\nu}}{1-q^{1+a+\nu}} q^{-(n-1)^2/4}(q;q)_\infty J^{(2)}_{a+\nu}(1/LE;q) \text{ as } n \to \infty. \end{aligned}$$

If $E \neq 0$ or $J^{(2)}_{a+\nu}(1/LE;q) \neq 0$, then $|A_n(E)| \to \infty$ as $n \to \infty$; i.e., (4.11) has no solutions in this case. If $E = 0$ from (4.13) we get

$$A_{2n}(0) = 0, \quad A_{2n+1}(0) = \left(1-q^{2n+a+\nu+1}\right)\left(\frac{1-q^{a+\nu}}{q^\nu(q^a-q^\nu)}\right)^n$$

and (4.20) is not true. If $J^{(2)}_{a+\nu}(1/LE;q) = 0$, then by (1.19) and Theorem 4.3 of [4] we obtain

$$q^{n(a+\nu)+n(n-1))/2} J^{(2)}_{a+\nu+n}(1/LE;q) = -h_{n-1,a+\nu+1}(LE;q) J^{(2)}_{a+\nu-1}(1/LE;q).$$

On the other hand we know that

$$J^{(2)}_{a+\nu+n}(1/LE; q) \approx (2LE)^{-(a+\nu+n)} \frac{(q^{a+\nu+n+1}; q)_\infty}{(q; q)_\infty} \text{ as } n \to \infty$$

so (4.20) holds. Summarizing the above, we get Theorem 4.1.

THEOREM 4.1. *The eigenvalues $\{\lambda_n(q)\}$ of the T_q^{-1} of* (4.11) *are the reciprocals of zeros of $J^{(2)}_{a+\nu}\left(L^{-1}\xi; q\right)$, where L is given by* (4.15). *The corresponding eigenfunctions are in the form of* (4.12).

Finally, we come to the nonsymmetric case $V \neq 0$. From the Birkhoff-Tritjinski theory for difference equations we see that the second order difference equation (4.16) has two linearly independent solutions $b_{n,1}(E)$ and $b_{n,2}(E)$ such that

$$\begin{aligned} b_{n,1}(E) &= (2E)^n O(1) \quad \text{as } n \to \infty, \\ b_{n,2}(E) &= \left(q^{a+\nu+1/2}/2E\right)^n q^{n^2/2} O(1) \quad \text{as } n \to \infty. \end{aligned} \tag{4.21}$$

Thus there are functions $C(E)$ and $D(E)$ such that

$$b_n(E) = C(E) b_{n,1}(E) + D(E) b_{n,2}(E). \tag{4.22}$$

By (4.14) and (4.20) we can verify that the spectrum of the integral operator (4.11) consists of the zeros of $C(E)$ and possibly the origin. But when $E = 0$ the recurrence relation (4.16) degenerates to

$$b_n(0) = 2Mq^{(n+a+\nu)/2} b_{n-1}(0) - q^{n+a+\nu-1} b_{n-2}(0). \tag{4.23}$$

The change of variables

$$b_n(0) = q^{(n(n-1)/4} c_n$$

changes (4.23) into the second order difference equation with constant coefficients

$$c_n = 2Mq^{(a+\nu+1)/2} c_{n-1} - q^{a+\nu+1/2} c_{n-2}. \tag{4.24}$$

The two linear independent solutions of (4.24) are asymptotically like

$$|c_n| = \left| q^{(a+\nu)/2} (Mq^{1/2} \pm \sqrt{M^2 q - q^{1/2}}) \right|^n O(1).$$

From here it is clear that (4.20) does not hold.

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