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HANKEL OPERATORS ON COMPLEX ELLIPSOIDS

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1. Introduction

For (b_k) in $\ell^2 = \ell^2(\mathbb{C})$, the Hankel matrix $H = (h_{k,l})$ is the infinite matrix of which k, l entry is b_{k+l} which may be seen as an operator on ℓ^2. As it is well known [21], such an operator can be realized as an operator on $H^2(D)$ where D is the unit disc of $\mathbb{C}$: $H^2(D)$ identifies with ℓ^2 if $(b_k) \in \ell^2$ is identified with $\sum_k b_k z^k$. So, let $b(z) = \sum_k b_k z^k$. Given f in $H^2(D)$, the Hankel operator h is defined by

$$hf = S(b\overline{f}), \tag{1.1}$$

where S is the Szegö projection. Since the family (z^k) is an orthonormal basis of $H^2(D)$, the matrix H and the operator h (see [28]) satisfy

$$(h(z^k)/z^l) = \frac{1}{2i\pi}\int_T b(z)\overline{z}^{k+l}\frac{dz}{z} = b_{k+l} = h_{k,l}.$$

Hankel operators have been studied by many authors. They showed how the properties of the operator or its matrix depend on the symbol b. In 1957, Z. Nehari [19] showed that h is bounded if and only if b belongs to BMO and, in 1958, P. Hartman [11] proved that h is a compact operator if and only if b belongs to VMO. In 1979, V. V. Peller [20] proved that h is of the Schatten class $\mathcal{S}_p$, $1 \leq p < +\infty$ if and only if b is in the Besov space $B_p^{p,1/p}(D)$. An independent proof was given in 1980 by R. Coifman and R. Rochberg [5] for $p = 1$ and R. Rochberg extended it for $p \geq 1$ [22]. We follow their method.

Let $n \geq 2$ and let $\rho_k = \rho_{k_1,k_2,\ldots,k_n}$ be a sequence of positive real numbers. For b_k in the weighted space $\ell^2(\mathbb{C}^n, (\rho_k))$, the generalized Hankel matrix $H = (h_{k,l})$, $(k,l) = ((k_1,\ldots,k_n),(l_1,\ldots,l_n))$, is the matrix with entries

$$h_{k,l} = b_{k+l}\,\rho_{k+l}.$$

Let $\rho_k = \rho_{k_1,k_2,\ldots,k_n} = 1$. We denote by P^n the polydisc in $\mathbb{C}^n$ and by ∂P^n its boundary. The family $e_k(z) = z_1^{k_1}\cdots z_n^{k_n}$ is an orthonornal family of $H^2(P^n)$. Let $b(z) = \sum_k b_k e_k(z)$. The function b is in the Hardy space $H^2(P^n)$ and, again, we can define the Hankel operator h on $H^2(P^n)$ by the relation (1.1). Then we have

$$(h(e_k)/e_l) = \frac{1}{2\pi^n}\int_{\partial P^n} b(\zeta)e_k(\zeta)\overline{e}_l(\zeta)\,d\zeta_1\cdots d\zeta_n = b_{k+l},$$

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where $\partial P^n = \{z, |z_1| = |z_2| = \cdots = |z_n| = 1\}$. The projection S is the Szegö projection from $L^2(\partial P^n)$ onto $H^2(P^n)$. In this case, the results are partial for Hankel operators; see for instance M. Cotlar and C. Sadosky [8] and T. Nakazi [17]. The difficulty of the problem is that such operators are related to products of Hilbert transforms. Our aim, here, is to consider a family of weight for which the symbol b associated with (b_k) belongs to the Hardy space of a complex ellipsoïd. As ellipsoids are convex and pseudoconvex domains of finite type in $\mathbb{C}^n$, one may hope that the characterization for D extends in this case. More precisely, let $m = (m_1, \ldots, m_n)$ be an n-tuple of integers, and let

$$\rho_k = \frac{\pi^n \Gamma((k_1+1)/m_1) \cdots \Gamma((k_n+1)/m_n)}{m_1 \cdots m_n \Gamma((k_1+1)/m_1 + \cdots + (k_n+1)/m_n)}.$$

First, assume $m = (1, \ldots, 1)$. We consider the Hankel operator h defined on $H^2(B^n)$, where B^n is the unit ball of $\mathbb{C}^n$. The family $e_k(z)$ is an orthogonal basis of $H^2(B^n)$ and $\|e_k\|^2_{H^2(B^n)} = \frac{\pi^n k_1! \cdots k_n!}{(k_1+\cdots+k_n+n-1)!}$. Given (b_k) in the weighted space $\ell^2(\mathbb{C}^n, (\rho_k))$, the function $b(z) = \sum_k b_k e_k(z)$ is in $H^2(B^n)$ and we define the Hankel operator by the relation (1.1). In this case, $(h(e_k)/e_l) = b_{k+l} \|e_{k+l}\|^2_{H^2(B^n)}$ and the results on the disc have been extended by R. Coifman, R. Rochberg and G. Weiss [6], M. Feldman and R. Rochberg [9] and G. Zhang [29]. For the strictly pseudoconvex domains in $\mathbb{C}^n$ and finite type domains in $\mathbb{C}^2$, F. Beatrous and S-Y. Li proved that a Hankel operator H defined on Bergman space is bounded if and only b is in BMO and compact if and only if b is in VMO [3]. They give a sufficient condition on b so that H belongs to the Schatten class $\mathcal{S}_p$ [4]. For domains such that the Bergman kernel is non vanishing, they proved that this condition is also necessary. A characterization of Hankel operators on peudoconvex domains of finite type in $\mathbb{C}^2$ was given by S. Krantz, S-Y. Li and R. Rochberg [13] and [14].

The purpose of this paper is to study the Hankel operators when m is an n-tuple different from $(1, \ldots, 1)$. Let (b_k) in $\ell^2(\mathbb{C}^n, \rho_k)$ and $b(z) = \sum_k b_k e_k(z)$ in $H^2(\Omega)$, where Ω is the ellipsoïd related to m. We characterize the symbol b for which h, defined by (1.1), is bounded, compact or an element of the Schatten von-Neumann class $\mathcal{S}_p$, $1 \le p < +\infty$.

Let $m = (m_1, \ldots, m_n)$ be an n-tuple. We define

$$\Omega = \left\{ z \in \mathbb{C}^n,\ r(z) = \sum_{j=1}^n |z_j|^{2m_j} - 1 < 0 \right\}$$

and $\partial\Omega = \{z \in \mathbb{C}^n,\ r(z) = 0\}$. The complex ellipsoid Ω is a bounded convex, pseudoconvex domain of finite type in $\mathbb{C}^n$.

Before stating our results, let us recall the definition of $\mathcal{S}_p$. If Θ is a compact operator in a Hilbert space H we can consider (s_i) the sequence of eigenvalues of $(\Theta^*\Theta)^{1/2}$. The s_i are called singular values of Θ. The operator Θ is said to belong to $\mathcal{S}_p$ if and only if (s_i) is in ℓ^p. The space $\mathcal{S}_p$ endowed with the norm

$\|\Theta\|_{\mathcal{S}_p} = \left(\sum_{i=0}^{\infty} s_i^p\right)^{1/p}$ is a Banach space when $1 \le p < +\infty$. The space $\mathcal{S}_1$ is called the Trace Class of H and $\mathcal{S}_2$ is the Hilbert Schmidt class [10].

Let $q > -1$ and $dV_q = (-r(z))^q dV$, where dV is the Lebesgue measure of Ω. We denote by B_q the weighted Bergman projection: it is the orthogonal projection from $L^2(dV_q)$ onto the Bergman space $A^2(dV_q) = L^2(dV_q) \cap \mathcal{H}(\Omega)$, where $\mathcal{H}(\Omega)$ is the space of holomorphic functions in Ω. Let $f \in L^2(dV_q)$,

$$B_q f(z) = \int_\Omega B_q(z,\zeta) f(\zeta)\, dV_q(\zeta),$$

where $B_q(z,\zeta)$ is the weighted Bergman kernel. Let $B_0(z,\zeta) = B(z,\zeta)$ and $B_0 = B$. Then the following result holds.

THEOREM A. *Let $1 \le p < +\infty$ and $l \in \mathbb{N}$ such that $lp > n$. Let b be a holomorphic function and define h by $hf = S(b\overline{f})$. Then:*

(i) *If $b \in BMO(\partial\Omega)$ then h is bounded.*
(ii) *If $b \in VMO(\partial\Omega)$ then h is compact.*
(iii) *If $(-r(\zeta))^l \nabla^l b \in L^p(\Omega,\ B(\zeta,\zeta) dV(\zeta))$ then $h \in \mathcal{S}_p$.*

The condition $lp > n$ comes from the fact that the weight $(-r(z))^{pl} B(z,z)$ is an integrable function if and only if $lp > n$. It follows from the mean-value property that if $lp > n$ and $l' \in \mathbb{N}$, $(-r(\zeta))^l \nabla^l b$ is in $L^p(\Omega,\ B(\zeta,\zeta)dV(\zeta))$ if and only if $(-r(\zeta))^{l+l'} \nabla^{l+l'} b$ is in $L^p(\Omega,\ B(\zeta,\zeta)dV(\zeta))$ [12].

The conditions are the same as in the case of the ball [9]. To know whether the conditions are necessary is still open and, probably, difficult. We give some kind of necessary condition.

We shall use the homogeneity properties of the ellipsoid: Let us define σ as the measure on $\partial\Omega$ such that, for all continuous function f with compact support,

$$\int_{\mathbb{C}^n} f(z)\, dV(z) = \int_0^{+\infty} \left\{ \int_{\partial\Omega} f(\alpha^{1/m_1} z_1, \dots, \alpha^{1/m_n} z_n)\, d\sigma(z) \right\} \alpha^{2\widetilde{m}-1}\, d\alpha, \tag{1.2}$$

where $\widetilde{m} = \sum_{j=1}^n \frac{1}{m_j}$. J. D'Angelo gave an explicit formula for the Bergman kernel and an asymptotic formula for $B(z,z)$ [1]. The Szegö projection with respect to σ has been studied by A. Bonami and N. Lohoué [2]. They obtained an explicit formula for the Szegö kernel function and they defined an anisotropic pseudometric d to characterize its singularities. We use a family of polydiscs to give an equivalent definition of d. Let N be the holomorphic transverse vector field

$$N_z = \frac{z_1}{m_1}\frac{\partial}{\partial z_1} + \cdots + \frac{z_n}{m_n}\frac{\partial}{\partial z_n}.$$

Notice that $N_z r = 1$ on $\partial\Omega$ and, if $N_z = T + iL$, the real field L is tangent to $\partial\Omega$.

We consider the n complex tangent directions

$$L_j = \frac{\partial}{\partial z_j} - \frac{\partial r}{\partial z_j} N_z, \ 1 \leq j \leq n.$$

Since $\sum_{j=1}^n \frac{z_j}{m_j} L_j = 0$, the family $\{L_j, \ j \neq j_0\}$ spans the complex tangential space in the open set $\mathcal{V}_{j_0} = \{\zeta \in \mathbb{C}^n, \ |z_{j_0}| > 1/2\sqrt{n} > 0\}$. Let z in $\mathcal{V}_{j_0}$, $\delta > 0$ and

$$Q(z, \delta) = \{\zeta \in \mathbb{C}^n, \ \zeta = z + \alpha N_z + \sum_{j \neq j_0} \beta_j L_j, \ |\alpha| < \delta \text{ and } |\beta_j| < \tau_j(z, \delta)\},$$

where $\tau_j(z, \delta) = \inf\left\{\delta^{1/2m_j}, \frac{\delta^{1/2}}{|z_j|^{m_j-1}}\right\}$. The pseudometric d is given as follows.

DEFINITION. *Let z and ζ in $\mathbb{C}^n$. Then,*

$$d(z, \zeta) = \inf\{\delta > 0, \ \zeta \in Q(z, \delta)\}.$$

Let z on $\partial\Omega$ and $\delta > 0$. We denote the anisotropic ball of $\partial\Omega Q(z, \delta) \cap \partial\Omega$ by $B(z, \delta)$. Let f in $L^1_{loc}(\partial\Omega)$. For z on $\partial\Omega$ and $\delta > 0$, let

$$m(f, z, \delta) = \frac{1}{\sigma(B(z, \delta))} \int_{B(z,\delta)} f(\zeta)\, d\sigma(\zeta),$$

$$\operatorname{osc}(f, z, \delta) = \frac{1}{\sigma(B(z, \delta))} \int_{B(z,\delta)} |f(\zeta) - m(f, z, \delta)|\, d\sigma(\zeta).$$

A function f in $L^1_{loc}(\partial\Omega)$ is in the anisotropic space $BMO(\partial\Omega)$ if

$$\|f\|_{BMO} = \sup_{z, \ \delta > 0} \operatorname{osc}(f, z, \delta) < +\infty.$$

Let $f \in BMO(\partial\Omega)$ and $0 < r < 1$. Let $M_r(f) = \sup \operatorname{osc}(f, z, \delta)$ where the supremum is considered for z on $\partial\Omega$ and $0 < \delta \leq r$. The function f is in $VMO(\partial\Omega)$ if $\lim_{r \to 0} M_r(f) = 0$.

The proof of (i) is classical. The Szegö projection is a singular integral operator with respect to the pseudometric d [26]. We can consider $\mathcal{C}_b$ the commutator associated to b. Let f in $L^2(\partial\Omega)$, $\mathcal{C}_b f = S(bf) - bSf$. Since $\mathcal{C}_b \overline{Sf} = hf$ we only have to study the commutator. The proof of S. Janson [15] extends to this context to show that $\mathcal{C}_b$ is bounded.

Part (ii) of the theorem follows from the first one by routine arguments. We approximate h by finite rank operators. Choose b in $BMO(\partial\Omega)$. Let $\delta > 0$. We consider $b_\delta(z) = m(b, z, \delta)$. By part (i) of the theorem,

$$\|\mathcal{C}_b - \mathcal{C}_{b_\delta}\| \leq C \|b - b_\delta\|_{BMO(\partial\Omega)}.$$

We need to prove that $\mathcal{C}_b$ is compact only when b is continuous. By the theorem of Stone-Weierstrass, b is uniformly approximated by polynomials P_n. For each P_n, $\mathcal{C}_{P_n}$ is a finite rank operator and therefore $\mathcal{C}_{P_n}$ is compact. We take the limit in the sense of operators to conclude that $\mathcal{C}_b$ is compact. It remains to show that $\lim_{\delta\to 0}\|b-b_\delta\|_{BMO(\partial\Omega)}=0$. In the case of the ball, R. Coifman, R. Rochberg and G. Weiss [6] proved that there exists $C>0$ such that

$$\operatorname{osc}(b-b_\delta, r, z)\leq C\left(M_\delta(b)+M_{C\delta}(b)\right).$$

The result is still valid in the case of complex ellipsoids. By definition of $VMO(\partial\Omega)$, $\lim_{\delta\to 0}M_\delta(b)=0$. □

Let us prove (iii). If (e_i) and (f_i) are two orthonormal basis, a compact operator Θ in a Hilbert space H has the Schmidt decomposition

$$\Theta=\Theta(\lambda)=\sum_{i=0}^{\infty}\lambda_i(./e_i)f_i, \tag{1.3}$$

where (/) is the inner product in H. If Θ is given by (1.3), then $\lambda_i=s_i$. The family (e_k) is an orthogonal family in $H^2(\Omega)$ but the relation (1.3) with (e_k) does not allow us to prove that (s_i) is in ℓ^p. We begin to give a generalization of the Schmidt decomposition: we prove that a compact operator Θ defined as in (1.3) where e_i and f_i are only nearly weakly orthonormal (Definition 3.1) and (λ_i) in ℓ^p, $1\leq p<+\infty$, satisfies $\sum_i s_i^p\leq C\sum_i\lambda_i^p$. Then we prove that a Hankel operator is a finite sum of operators of type (1.3). This sum follows from the theorem of atomic decomposition of Bergman spaces [5], [27]. Let $1\leq p<+\infty$. There exists a sequence $K_j(z)$ in A^p such that F in A^p may be written as $F(z)=\sum_j\lambda_jK_j(z)$ and $\|F\|_{A^p}\simeq(\sum_i|\lambda_i|^p)^{1/p}$. The functions $K_i(z)$ are built with the weighted Bergman Kernel $B_q(z,w_i)$. We use the relation between the Szegö and the Bergman kernel given in the next part to obtain the nearly weakly orthonormal sequences.

2. The Szegö kernel

The aim of this section is to give the fundamental properties of the Szegö kernel for the measure σ. We give pointwise estimates for $N_z^kS(z,\zeta)$, $k\in\mathbb{N}$. When $n=2$ such estimates follow from [18]. When Ω is an ellipsoid of $\mathbb{C}^n$, $n\geq 3$, we use a direct method.

Recall that $S(z,\zeta)=\sum_k\|e_k\|^{-2}_{L^2(\partial\Omega)}e_k(z)\overline{e}_k(\zeta)$; hence

$$S(z,\zeta)=\frac{m_1\cdots m_n}{\pi^n}\sum_k\frac{\Gamma((k_1+1)/m_1+\cdots+(k_n+1)/m_n)}{\Gamma((k_1+1)/m_1)\cdots\Gamma((k_n+1)/m_n)}z^k\overline{\zeta}^k, \tag{2.4}$$

where $z^k\overline{\zeta}^k=(z_1\overline{\zeta}_1)^{k_1}\cdots(z_n\overline{\zeta}_n)^{k_n}$ [2]. Let z in $\Omega\setminus\{(0,\ldots,0)\}$. There exists (z',λ) on $\partial\Omega\times\mathbb{R}_+^*$ such that $z=(\lambda^{1/m_1}z_1',\ldots,\lambda^{1/m_n}z_n')$. We define the projection on $\partial\Omega$

by $\pi(z) = z'$ and $\lambda(z) = \lambda$. In a neighborhood of $\partial\Omega$, $1 - \lambda(z) \simeq -r(z) \simeq \delta(z) = \operatorname{dist}(z, \partial\Omega)$. Let z in $\overline{\Omega}$, ζ on $\partial\Omega$ and $D(z,\zeta) = \delta(z) + d(\pi(z), \zeta)$. We shall rely on the following proposition:

PROPOSITION 2.1. *Let $k \in \mathbb{N}$. There exists $C(k) > 0$ such that*

$$|N_z^k S(z,\zeta)| \leq \frac{C(k)}{D(z,\zeta)^k \sigma(B(\pi(z), D(z,\zeta)))}. \tag{2.5}$$

Proof. Such a proposition may be deduced from the result of [2] or from the more general results of J. Mac-Neal for decoupled domains [16]. Let us remark that the derivatives of S and B_q are linked by the following relations:

LEMMA 2.2. *Let z in Ω and ζ in $\overline{\Omega}$. Then:*

(i) $N_z S(z,\zeta) = \frac{1}{2} B(z,\zeta) - \widetilde{m} S(z,\zeta)$.
(ii) $B_{q+1}(z,\zeta) = \frac{1}{q+1}\big(N_z B_q(z,\zeta) + (\widetilde{m} + 1 + q) B_q(z,\zeta)\big)$.

Proof of the lemma. Since $e_k(z) = z_1^{k_1} \cdots z_n^{k_n}$ is an orthogonal basis of $A^2(dV)$, the Bergman kernel satisfies

$$B(z,\zeta) = \sum_k a \|e_k\|_{L^2(dV)}^{-2} e_k(z) \overline{e}_k(\zeta).$$

We use the definition of σ to compute $\|e_k\|_{L^2(dV)}^{-2}$:

$$\begin{aligned} \|e_k\|_{L^2(dV)}^2 &= \int_\Omega |\zeta_1|^{2k_1} \cdots |\zeta_n|^{2k_n} \, dV(\zeta) \\ &= \frac{2}{k_1/m_1 + \cdots k_n/m_n + \widetilde{m}} \|e_k\|_{L^2(\partial\Omega)}^2 \\ &= \frac{\pi^n}{m_1 \cdots m_n} \frac{\Gamma((k_1+1)/m_1) \cdots \Gamma((k_n+1)/m_n)}{\Gamma((k_1+1)/m_1 + \cdots + (k_n+1)/m_n + 1)}. \end{aligned}$$

The relation $N_z z^k \overline{\zeta}^k = (k_1/m_1 + \cdots + k_n/m_n) z^k \overline{\zeta}^k$ and the fact that $\Gamma(z+1) = z\Gamma(z)$ give (i).

The second relation follows from similar arguments. □

The following remark is an immediate consequence of the lemma.

REMARK 2.3. *There exist real numbers $a_0, a_1, \ldots, a_{q+1}$ such that*

$$B_q(z,w) = \sum_{k=0}^{q+1} a_k N_z^k S(z,w) = \sum_{k=0}^{q+1} a_k \overline{N}_\zeta^k S(z,w).$$

3. Schatten class

The Schmidt decomposition is not available to obtain the singular values s_i in this particular case. We recall the following characterization which does not require the spectral theory:

$$s_i = \inf\left\{\|\Theta - \Xi\|_{\mathcal{L}(H)}\ ;\ \operatorname{rank}(\Xi) \le i\right\}. \tag{3.6}$$

We use (3.6) to prove that h is in the Schatten class. We follow the method developed by R. Rochberg and S. Semmes [23], [24]. Let $\Theta(\lambda) = \sum_{i=0}^{\infty} \lambda_i(./e_i)f_i$, where (e_i) and (f_i) are two nearly weakly orthogonal (N.W.O.) families (Definition 3.1) and (λ_i) is in ℓ^p. We use geometrical arguments to prove that $\Theta(\lambda)$ is in $\mathcal{S}_p$, $1 \le p < +\infty$.

Let us define a Whitney covering of Ω by polydiscs $Q(w, \eta\delta(w))$, $0 < \eta < 1$. Let w_i be the center of the polydisc and let $Q_i = Q(w_i, \eta\delta(w_i))$. We fix $C_0 > 0$ such that $Q(w_i, \eta\delta(w_i)/C_0) \cap Q(w_{i'}, \eta\delta(w_{i'})/C_0) = \emptyset$ if $i \ne i'$. Let $\widetilde{Q}_i = Q(w_i, \eta\delta(w_i)/C_0)$, $\overline{Q}_i = Q(w_i, C_0\eta\delta(w_i))$ and $B_i = \pi(Q_i)$.

DEFINITION 3.1. *The family (e_i) in $L^2(\partial\Omega)$ is a N.W.O. family if and only if*

(i) *there exists $C > 0$ independent of i such that $\|e_i\|_{L^2(\partial\Omega)} \le C$ and*
(ii) *the maximal operator T^* defined on $L^2(\partial\Omega)$ by*

$$T^* f(z) = \sup_{z\in B_i} \frac{1}{\sigma(B_i)^{1/2}} \left| \int_{\partial\Omega} f(\zeta) e_i(\zeta)\, d\sigma(\zeta) \right|$$

is bounded in $L^2(\partial\Omega)$.

Let (λ_i) be in ℓ^p, $1 \le p < +\infty$, and let (e_i) and (f_i) be two N.W.O. families. We follow the method of [24] to prove that $\Theta(\lambda)$ is in $\mathcal{S}_p$. We approximate $\Theta(\lambda)$ by the finite rank operators $\Theta_k(\lambda) = \sum_{j=0}^{k-1} \lambda_i(\cdot/e_i)f_i$. We define the sequence $(M(\lambda)_i)$ by

$$M(\lambda)_i = \frac{1}{\sigma(B_i)} \sum_{w_k\in T_i} |\lambda_k| \sigma(B_k), \tag{3.7}$$

where $T_i = \{\zeta \in \Omega,\ \pi(\zeta) \in B_i \text{ and } r(w_i) < r(\zeta) < 0\}$ is the tent over the ball B_i. We use (3.6) and the following propositions to estimate the singular values of $\Theta(\lambda)$ [9], [24]. We follow the method given for $\mathbb{R}^n$. We have to do it carefully to control the constants.

PROPOSITION 3.2. *Let (e_i) and (f_i) be two N.W.O. families and let $\Theta = \sum_i \lambda_i(\cdot/e_i)f_i$. There exists $C > 0$ such that, for $k \in \mathbb{N}$ and $f, g \in L^2(\partial\Omega)$,*

$$|((\Theta - \Theta_k)f/g)| \le C\|f\|_{L^2(\partial\Omega)}\|g\|_{L^2(\partial\Omega)} M(\lambda)^*_k,$$

where $M(\lambda)^*$ is the nonincreasing rearrangement of $M(\lambda)$.

Proof. Let (λ_i) a bounded sequence. We consider the discret measure

$$\Lambda(\lambda) = \sum_i |\lambda_i| \sigma(B_i) \delta_{w_i},$$

where δ_{w_i} is the Dirac measure at w_i. Let $\|\Lambda(\lambda)\|_{\mathrm{Carl}}$ the Carleson norm of the measure $\Lambda(\lambda)$. Then $\|\Lambda(\lambda)\|_{\mathrm{Carl}} = \sup_i |M(\lambda)_i|$. Let (ν_i) be a bounded sequence and let $\nu(z) = \sum_i \nu_i \varphi_i(z)$, where $\varphi_i(z)$ is a continuous function such that $|\varphi(z)| \leq 1$ and

$$\begin{aligned} \varphi_i(z) &= 1 \text{ if } z \in Q(w_i, \eta\delta(w_i)/C_0^2), \\ \varphi_i(z) &= 0 \text{ if } z \notin \widetilde{Q}_i. \end{aligned}$$

Then

$$\sum_i \lambda_i \sigma(B_i) \nu_i = \int_\Omega \nu(\zeta)\, d\Lambda(\zeta) \leq C \|\Lambda(\lambda)\|_{\mathrm{Carl}} \int_{\partial\Omega} \nu^*(z)\, d\sigma(z),$$

where $\nu^*(z) = \sup_{z \in B_i} |\nu_i|$ [25]. Let f and g in $L^2(\partial\Omega)$. The choice $\nu_i = \frac{|(f/e_i)|}{\sigma(B_i)^{1/2}} \frac{|(g/f_i)|}{\sigma(B_i)^{1/2}}$ gives

$$\begin{aligned} |((\Theta - \Theta_k) f/g)| &\leq C \|\Lambda(\lambda^k)\|_{\mathrm{Carl}} \int_{\partial\Omega} T^* f(z) T^* g(z)\, d\sigma(z) \\ &\leq C \|\Lambda(\lambda^k)\|_{\mathrm{Carl}} \|f\|_{L^2(\partial\Omega)} \|g\|_{L^2(\partial\Omega)}, \end{aligned}$$

where (λ_i^k) is the sequence deduced from (λ_i) and defined by $\lambda_i^k = 0$ for $i = 0, \ldots, k-1$ and $\lambda_i^k = \lambda_i$ for $i \geq k$. We suppose that $M(\lambda)$ is a nonincreasing sequence. It remains to prove that there exists $C > 0$ such that, for $0 \leq i \leq k-1$,

$$M(\lambda^k)_i \leq C M(\lambda)_k.$$

We estimate $M(\lambda^k)_i$ with terms $M(\lambda^k)_l = M(\lambda)_l$, $l \geq k$. Consider the order relation on $A_{i,k} = \{w_l,\ w_l \in T_i \text{ and } l \geq k\}$ given by

$$w_l \prec w_{l'} \text{ iff } w_l \in T_{l'} \text{ and } r(w_{l'}) < r(w_l).$$

Let w_l denote the maximal elements for this relation. There exist such maximal elements as there is at most a finite number of $w_{l'}$ for which $w_l \prec w_{l'}$. Moreover, any w_l in $A_{i,k}$ is contained in some $T_{l'}$, with $w_{l'}$ maximal. The sequence (w_l) satisfies the following technical lemma:

LEMMA 3.3. *Let* $0 \le i \le k-1$. *Then*:

(i) $B_i \subset \overline{\bigcup_l B_l}$.

(ii) $\pi\left(Q(w_l, \eta\delta(w_l)/C_0^3)\right) \bigcap \pi\left(Q(w_{l'}, \eta\delta(w_{l'})/C_0^3)\right) = \emptyset$ *if* $l \neq l'$.

Proof of the lemma. Let $z \in B_i$. Then, for $\varepsilon > 0$ small enough, $z \in \pi(Q_p) \subset B(z, \varepsilon)$, with w_p close to the boundary. Then w_p belongs to some T_l, with w_l which is maximal and so $\pi(w_p)$ belongs to $\cup_l B_l$. As ε is arbitrarily small, z is in its closure. Let z satisfy

$$\begin{aligned} d(z, \pi(w_l)) &< \eta\delta(w_l)/C_0^3 \\ d(z, \pi(w_{l'})) &< \eta\delta(w_{l'})/C_0^3. \end{aligned}$$

Assume $\delta(w_{l'}) \le \delta(w_l)$. Since d is a C_0 pseudometric, $d(\pi(w_{l'})), \pi(w_l)) \le \eta\delta(w_l)/C_0$. Thus $w_{l'}$ belongs to T_l and hence $w_{l'}$ is not a maximal point. □

The sequence (w_l) may be infinite but it is an immediate consequence of Lemma 3.3 that $\sigma\,(B_i)\,/C \le \sum_l \sigma\,(B_l) \le C\sigma\,(B_i)$. Then

$$M(\lambda^k)_i \le \frac{1}{\sigma(B_i)} \sum_l \sigma(B_l) M(\lambda^k)_l \le \frac{M(\lambda^k)_k}{\sigma(B_i)} \sum_l \sigma(B_l) \le CM(\lambda)_k. \qquad \square$$

It remains to use the folowing proposition to prove that $\Theta(\lambda)$ defined as in (1.3) with e_i and f_i N.W.O. is in $\mathcal{S}_p$:

PROPOSITION 3.4. *M is bounded in* ℓ^p, $1 \le p < +\infty$.

Proof of the proposition. For $p > 1$, we use the Schur lemma with the sequence $(\sigma) = (\sigma(B_i))$ [30]. First, let us remark that, for $p \ge 1$,

$$M(\sigma^p)_i = \frac{1}{\sigma(B_i)} \sum_{w_k \in T_i} \sigma(B_k)^{1+p} \le \sigma(B_i)^p \sum_{w_k \in T_i} \left(\frac{\sigma(B_k)}{\sigma(B_i)}\right)^{1+p}.$$

Since there exists a finite number of points w_k such that $r(w_k) < -1/2$, we denote by k' the index such that, for $k \ge k'$, $-1/2 < r(w_k) < 0$. Let $i \ge k'$. We denote by j_0 the index such that $w_i \in \mathcal{V}_{j_0}$. For w in $\mathcal{V}_{j_0}$,

$$\sigma(B(\pi(w), \delta)) \simeq \delta \prod_{\substack{j=1 \\ j \neq j_0}}^{n} \tau_j(w, \delta)^2 \simeq \sqrt{\mathrm{Vol}(Q(w, \delta))} \prod_{\substack{j=1 \\ j \neq j_0}}^{n} \tau_j(w, \delta).$$

We obtain

$$M(\sigma^p)_i \le C\sigma(B_i)^p \sum_{w_k \in T_i} \left(\frac{\mathrm{Vol}(Q_k)}{\mathrm{Vol}(Q_i)}\right)^{\frac{1+p}{2}} \prod_{\substack{j=1 \\ j \neq j_0}}^{n} \left(\frac{\tau_j(w_k, \delta(w_k))}{\tau_j(w_i, \delta(w_i))}\right)^{1+p}.$$

Since w_k is in T_i, $\delta(w_k) \le \delta(w_i)$ and $\tau_j(w_k, \delta) \simeq \tau_j(w_i, \delta)$, $\delta > 0$. Then

$$M(\sigma^p)_i \le C\sigma(B_i)^p \left(\sum_{w_k \in T_i} \frac{\mathrm{Vol}(Q_k)}{\mathrm{Vol}(Q_i)} \right)^{\frac{1+p}{2}} .$$

The polydiscs Q_k are almost disjoint and $\mathrm{Vol}(Q_i) \simeq \mathrm{Vol}(T_i)$, thus $M(\sigma^p)_i \le C\sigma_i^p$. Let p and p' such that $1/p + 1/p' = 1$,

$$\begin{aligned} M(\sigma^p)_i &\le C\sigma(B_i)^p, \\ M(\sigma^{p'})_i &\le C\sigma(B_i)^{p'}. \end{aligned}$$

The Schur lemma implies that M is bounded in ℓ^p, $1 < p < +\infty$.

Assume $p = 1$. Let us remark that $\sum_i M(\lambda)_i = \sum_k |\lambda_k| \sigma(B_k) \sum_{w_l \in A_k} \frac{1}{\sigma(B_l)}$, where $A_k = \{w_l,\ w_k \in T_l\}$. We have only to show that there exists $C > 0$ such that

$$\sigma(B_k) \sum_{w_l \in A_k} \frac{1}{\sigma(B_l)} < C. \tag{3.8}$$

We consider A_k^s the partition of A_k given as follows:

DEFINITION 3.5. *Let w_k in Ω. Then*:

(i) $A_k^1 = \{w_l \in A_k,\ Q_l \cap Q_k \neq \emptyset\}$.
(ii) $A_k^{s+1} = \{w_l \in A_k \setminus \cup_{i=1}^s A_k^i,$ *such that* $Q_i \cap Q_k \neq \emptyset$ *for some* $w_i \in A_k^s\}$.

The estimation (3.8) follows from the following technical lemma.

Lemma 3.6. *There exist $N = N(\Omega) \in \mathbb{N}$ and $R = R(\Omega, \eta) > 1$ such that*:

(i) *There are at most N points w_l in A_k^s.*
(ii) *Let w_l in A_k^s, and $s \ge 2$. Then $\delta(w_l) \ge R^s \delta(w_k)$.*

Proof of the lemma. Let us prove (i). Since $\delta(w_l) \ge \delta(w_k)$ when $w_l \in A_k$, the number N is less than the number N' of domains $Q(z_i, \eta\delta(w_k))$ such that

$$Q(z_i, \eta\delta(w_k)) \cap Q(w_k, \eta\delta(w_k)) \neq \emptyset \tag{3.9}$$

$$Q(z_i, \eta\delta(w_k)/C_0) \cap Q(z_{i'}, \eta\delta(w_k)/C_0) = \emptyset \text{ if } i \neq i'. \tag{3.10}$$

From (3.9) that there exists $C_1 > 0$ such that $Q(z_i, \eta\delta(w_k)) \subseteq Q(w_k, C_1\eta\delta(w_k))$ and therefore $\tau_j(w_k, \delta) \simeq \tau_j(z_i, \delta)$, $\delta > 0$ [7]. Moreover, from (3.10),

$$N'\delta(w_k) \left(\prod_{j=1}^n \tau_j(w_k, \delta(w_k)) \right)^2 \le C \sum_i \mathrm{Vol}\left(Q(z_i, \eta\delta(w_k)/C_0)\right)$$

$$\leq C \operatorname{Vol}(Q(w_k, C_1\eta\delta(w_k)))$$

$$\leq C\delta(w_k)\left(\prod_{j=1}^{n}\tau_j(w_k,\delta(w_k))\right)^2,$$

where C is independent of the Whitney covering.

Let $s \geq 2$ and w_l in A_k^{s+1}. Let us remark that there exists $C_2 \geq 1$ such that for w in $Q(z, c\delta(z))$ and $c > 0$ small enough,

$$\frac{1}{C_2}(1-c)\delta(z) \leq \delta(w) \leq C_2(1+c)\delta(z). \tag{3.11}$$

We denote by w_i the point of A_k^s such that $Q_l \cap Q_i \neq \emptyset$. Since $\widetilde{Q}_l \cap \widetilde{Q}_i = \emptyset$ and $Q_l \cap Q_i \neq \emptyset$, it follows from the relation (3.11) with $c = \frac{\eta}{C_0}$ that $\delta(w_l) \geq R\delta(w_i)$, where $R = \frac{C_2^2(C_0+\eta)}{C_0-\eta} > 1$. $\square$

Let w_k in Ω. It follows from Lemma 3.6 that

$$\sum_{w_l \in A_k}\frac{\sigma(B_k)}{\sigma(B_l)} \leq \sum_{w_l\in A_k}\frac{\delta(w_k)}{\delta(w_l)}\left(\prod_{j=1}^{n}\frac{\tau_j(w_k,\delta(w_k))}{\tau_j(w_l,\delta(w_l))}\right)^2.$$

Since $\tau_j(w_k,\delta) \simeq \tau_j(w_i,\delta)$, $\delta > 0$ there exists $n(w_k) > 0$ such that

$$\sum_{w_l\in A_k}\frac{\sigma(B_k)}{\sigma(B_l)} \leq \sum_{w_l\in A_k}\left(\frac{\delta(w_k)}{\delta(w_l)}\right)^{n(k)} \leq 2N + N\sum_{s=2}^{s_0}R^{-s\,n(k)} \leq C,$$

where C depends on η and Ω. $\square$

The following proposition provides the N.W.O. families that we will use to study the Hankel operators.

PROPOSITION 3.7. *Let $\alpha \geq 0$ and $k \in \mathbb{N}$. The family (e_i) defined by*

$$e_i(z) = \sigma(B_i)^{1/2}\delta(w_i)^{k+\alpha}N_z^k S(z,w_i)$$

is a N.W.O. family.

Proof. Let $B_l = B(\pi(w_i), 2^l\delta(w_i))$ and $C_l = B_{l+1}\setminus B_l$, the corona of $\partial\Omega$. Then

$$\|e_i\|^2_{L^2(\partial\Omega)} = \sigma(B_i)\delta(w_i)^{2k+2\alpha}\int_{B(\pi(w_i),\delta(w_i))}|N_\zeta^k S(\zeta,w_i)|^2\,d\sigma(\zeta)$$
$$+\sum_{l\geq 1}\sigma(B_i)\delta(w_i)^{2k+2\alpha}\int_{C_l}|N_\zeta^k S(\zeta,w_i)|^2\,d\sigma(\zeta).$$

On B_1, we use the fact that $|N_\zeta^k S(\zeta, w_i)| \leq C\delta(w_i))^{-k}\sigma(B(\pi(w_i), 2^l\delta(w_i)))^{-1}$. On C_l, by Proposition 2.1, $|N_\zeta^k S(\zeta, w_i)| \leq C(2^l\delta(w_i))^{-k}\sigma(B_l)^{-1}$. Then

$$\|e_i\|^2_{L^2(\partial\Omega)} \leq C\delta(w_i)^{2\alpha+2k}\sum_l (2^l\delta(w_i))^{-2k}\frac{\mathrm{Vol}(B_{l+1})}{\mathrm{Vol}(B_l)} \leq C.$$

Let $f \in L^2(\partial\Omega)$ and z on $\partial\Omega$. By definition,

$$T^*f(z) \leq \sup_{z\in B_i}\delta(w_i)^{k+\alpha}|N_{w_i}^k Sf(w_i)|.$$

The function $N_z^k Sf$ is holomorphic, so

$$|N_{w_i}^k Sf(w_i)| \leq \frac{C}{\mathrm{Vol}(Q_i)}\int_{Q_i}|N_\zeta^k Sf(\zeta)|\delta(\zeta)^{k+\alpha}\,dV(\zeta),$$

hence

$$T^*f(z) \leq CM(\delta(\cdot)^{k+\alpha}N_z^k Sf),$$

where M is the Hardy-Littlewood maximal function with respect to the pseudometric d, defined by

$$MF(z) = \sup_{w\in\Omega,\delta>0}\frac{1}{\mathrm{Vol}(Q(w,\delta))}\int_{Q(w,\delta)}|F(\zeta)|\,dV(\zeta).$$

The operator M is bounded in $L^2(dV)$. Then

$$\|T^*f\|_{L^2(\partial\Omega)} \leq \|\delta(\cdot)^{k+\alpha}N_z^k Sf\|_{L^2(dV)}.$$

It follows from the mean-value property that $\|\delta(\cdot)^k\nabla^k Sf\|_{L^2(dV)} \leq C\|Sf\|_{L^2(dV)}$ [12]. Then

$$\|T^*f\|_{L^2(\partial\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}.$$ □

It remains to show that a Hankel operator h is a finite sum of operators of type $\Theta(\lambda)$ and hence is in $\mathcal{S}_p$ by (3.6). The N.W.O. families and (λ) sequences are built via the atomic decomposition of Bergman spaces A^p [5], [27]. Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ in $\mathbb{R}^n$ and $\mu(\zeta)^{\boldsymbol{\beta}} = \prod_{j=1}^n (\tau_j(\zeta, \delta(\zeta)))^{\beta_j}$. Let α in $\mathbb{R}$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ in $\mathbb{R}^n$ and $dV_{\alpha,\boldsymbol{\beta}}(\zeta) = (-r(\zeta))^\alpha\mu(\zeta)^{\boldsymbol{\beta}}dV(\zeta)$. Let $1 \leq j \leq n$, since $\delta(z)^{1/2} \leq \mu(\zeta) \leq \delta(\zeta)^{1/2m_j}$, we consider the mapping g_j defined by $g_j(x) = 2$ if $x < 0$ and $g_j(x) = 2m_j$ if $x > 0$. We consider a Whitney covering of Ω by domains of type $Q(w, \eta\delta(w))$ with $\eta > 0$ small enough. Let w_i be the center of such domains and (K_i) the family of elements of $A^p(dV_{\alpha,\boldsymbol{\beta}}(\zeta))$ defined by

$$K_i(z) = \delta(w_i)^{t-\alpha/p}\mu(w_i)^{-\boldsymbol{\beta}/p}\,\mathrm{Vol}(Q_i)^{1-1/p}B_t(z, w_i),$$

where the parameter t is strictly greater than t_0 where $t_0 = \frac{\alpha}{p} + \frac{1}{p}\sum_{j=1}^{n} \frac{\beta_j}{g_j(-\beta_j)} + \frac{1}{p} - 1$. The following theorem is the theorem of atomic decomposition of the weighted Bergman space $A^p(dV_{\alpha,\boldsymbol{\beta}}(\zeta))$ (see [26] for details).

THEOREM 3.8. *Let $1 \le p < +\infty$, α in $\mathbb{R}$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ in $\mathbb{R}^n$ such that $1 + \alpha + \sum_{j=1}^{n} \frac{\beta_j}{g_j(\beta_j)} > 0$. Let $F \in A^p(dV_{\alpha,\boldsymbol{\beta}}(\zeta))$. There exists (λ_i') in ℓ^p such that*

(i) $F(z) = \sum_i \lambda_i' K_i(z)$,
(ii) $\|F\|_{\alpha,\boldsymbol{\beta},p} \simeq \left(\sum_i |\lambda_i'|^p\right)^{1/p}$.

In the theorem, the family (K_i) is not a basis of $A^p(dV_{\alpha,\boldsymbol{\beta}}(\zeta))$ because the decomposition is not unique.

Let $s > -1$ and $D_s = (1+s)^{-1}((N_z + (1+s+\widetilde{m})I)$. The field D_s is transverse and $D_s B_s(z,\zeta) = B_{s+1}(z,\zeta)$. Suppose that $\nabla^l b \in A^p(\delta(z)^{pl} B(z,z)dV(z))$, the function $D_{t-l+1}\cdots D_{t-1}b$ also belongs to $A^p(\delta(z)^{pl}B(z,z)dV(z))$. Recall that $B(z,z) \simeq \delta(z)^{-1}(\prod_{j=1}^{n} \tau_j(z,\delta(z)))^{-2}$. It follows from the theorem of atomic decomposition with $\alpha = -1 + lp$ and $\boldsymbol{\beta} = \mathbf{-2} = (-2, \dots, -2)$ that there exists (λ_i') in ℓ^p such that

$$D_{t-l+1}\cdots D_{t-1}b(z) = \sum_i \lambda_i' \delta(w_i)^{t-l+1/p} \mu(w_i)^{\mathbf{2}/p} \operatorname{Vol}(Q_i)^{1-1/p} B_t(w_i, z),$$

Let $s = t - l$ and $u_i = \left(\frac{\mu(w_i)^{\mathbf{-2}}}{\sigma(B_i)}\right)^{-1/p} \left(\frac{\operatorname{Vol}(Q_i)}{\delta(w_i)\sigma(B_i)}\right)^{1-1/p} \simeq 1$. Let $\nu_i = u_i\lambda_i'$. The sequence (ν_i) is in ℓ^p and

$$b(z) = \sum_i \nu_i \delta(w_i)^{1+s} \sigma(B_i) B_s(w_i, z).$$

According to Remark 2.3,

$$b(z) = \sum_i \nu_i \delta(w_i)^{1+s} \sigma(B_i) \sum_{k=0}^{s+1} a_k N_z^k S(z, w_i).$$

Choose F in $H^2(\Omega)$. Then

$$\begin{aligned} hF(z) &= \int_{\partial\Omega} S(z,\zeta) b(\zeta) \overline{F}(\zeta)\, d\sigma(\zeta) \\ &= \sum_{k=0}^{s+1} a_k \sum_i \nu_i \delta(w_i)^{1+s} \sigma(B_i) \int_{\partial\Omega} N_\zeta^k S(\zeta, w_i) b(\zeta) S(z,\zeta) \overline{F}(\zeta)\, d\sigma(\zeta) \end{aligned}$$

Since $N_\zeta^k S(\zeta, w_i) = \overline{N}_{w_i}^k S(\zeta, w_i)$ and the function $\zeta \to S(z,\zeta)\overline{F}(\zeta)$ is antiholomorphic,

$$\int_{\partial\Omega} N_\zeta^k S(\zeta, w_i) b(\zeta) S(z,\zeta) \overline{F}(\zeta)\, d\sigma(\zeta)$$

$$= \overline{N}^k_{w_i}\left(S(z, w_i)\overline{F}(w_i)\right)$$
$$= \sum_{q=0}^{k} C_k^q N_z^q S(z, w_i) \int_{\partial\Omega} N_z^{k-q} S(w_i, \zeta) b(\zeta) \overline{F}(\zeta)\, d\sigma(\zeta).$$

We then have

$$hF(z) = \sum_{k=0}^{s+1} a_k \sum_{q=0}^{k} C_k^q h_{k,q} \overline{F}(z),$$

where $h_{k,q}F(z) = \sum_i \nu_i \delta(w_i)^{1+s} \sigma(B_i)(N_z^{k-q} S(\cdot, w_i)/F) N_z^q S(z, w_i)$. For $0 \leq k \leq s+1$ and $0 \leq q \leq k$, let

$$\begin{aligned} e_i(z) &= \sigma(B_i)^{1/2} \delta(w_i)^k N_z^q S(z, w_i), \\ f_i(z) &= \sigma(B_i)^{1/2} \delta(w_i)^{1+s-k} N_z^{k-q} S(z, w_i) \end{aligned}$$

and $\lambda_i = a_k C_k^q \nu_i$. It is immediate that (e_i) and (f_i) are N.W.O. families and that (λ) is in ℓ^p. This completes the proof of theorem. □

4. Remarks and problems

The theorem gives a sufficient condition for a Hankel operator h to belong to $\mathcal{S}_p$. Let $1 < p < +\infty$ and suppose that h, a Hankel operator defined as in (1.1), is in $\mathcal{S}_p$. Then there exists $C > 0$ such that

$$\sum_i |(h(e_i)/f_i)|^p < C\|h\|^p_{\mathcal{S}_p}, \tag{4.12}$$

where e_i and f_i are two N.W.O. families [9], [24]. Let $e_i(z) = \sigma(B_i)^{1/2} S(z, w_i)$ and $f_i(z) = \sigma(B_i)^{1/2} S(z, w_i)$. Then (4.12) gives

$$\sum_i \sigma(B_i)^p \left| \int_{\partial\Omega} S^2(w_i, \zeta) b(\zeta)\, d\sigma(\zeta) \right|^p < +\infty.$$

Let $Tb(w) = \int_{\partial\Omega} S^2(w, \zeta) b(\zeta)\, d\sigma(\zeta)$. Since (Q_i) is a Whitney covering we obtain

$$\int_\Omega |Tb(w)|^p (-r(w))^{-p} B(w, w)^{1-p}\, dV(w) < +\infty. \tag{4.13}$$

If Ω is the ball of $\mathbb{C}^n$, there exist real numbers $a_0, a_1, \ldots, a_{n-1}$ such that $S(w, \zeta)^2 = \sum_{k=0}^{n} a_k N_w^k S(w, \zeta)$. Then $Tb(w) = \sum_{k=0}^{n-1} a_k N_w^k b(w)$. Moreover $B(w, w) \simeq$

$\delta(w)^{-(n+1)}$, so it follows from the relation 4.13 that $(-r(\zeta))^n \nabla^n b$ in $L^p(\Omega,\ B(\zeta,\zeta)dV(\zeta))$ and hence the sufficient condition is also a necessary condition with $l = n$ [9]. The characterization of Tb remains an open problem in the general case.

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