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WEIGHTED NORM INEQUALITIES FOR A FAMILY OF ONE-SIDED MINIMAL OPERATORS

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1. Introduction

Given $\mu > 0$ and a real-valued, non-negative function f on $\mathbb{R}$, we define the one sided μ-minimal function of f, $m_\mu^+ f$, by

$$m_\mu^+ f(x) = \inf \frac{1}{|J|} \int_J f\,dy,$$

where the infimum is taken over all intervals J that lie to the right of x with the property that $0 \le \operatorname{dist}(x, J) < \mu|J|$. The minimal function $m_\mu^- f$ is defined similarly. Following our work in [2], [3], the purpose of this paper is to study the weighted norm inequalities that m_μ^+ satisfies.

Our motivation for considering these operators came from the analogous maximal operator, a variant of which was introduced by Martín-Reyes and de la Torre [7]. Specifically, for $\mu > 0$, define

$$M_\mu^+ f(x) = \sup \frac{1}{|J|} \int_J f\,dy,$$

where, as before, the supremum is taken over all intervals J to the right of x satisfying $0 \le \operatorname{dist}(x, J) < \mu|J|$. Clearly, $M_{\mu_1}^+ f(x) \le M_{\mu_2}^+ f(x)$ for $\mu_1 < \mu_2$. On the other hand, if $J = (a, b)$ is an interval to the right of x such that $0 \le \operatorname{dist}(x, J) < \mu_2|J|$, and if we define $J^* = (a^*, b)$ where $x \le a^* \le a$ is such that $a^* - x < \mu_1(b - a^*)$, then

$$\frac{1}{|J|} \int_J f\,dy \le \frac{(1+\mu_2)}{|J^*|} \int_{J^*} f\,dy.$$

Hence $M_{\mu_2}^+ f(x) \le (1 + \mu_2) M_{\mu_1}^+ f(x)$. Almost identical arguments show that each operator M_μ^+ is equivalent to the one-sided Hardy-Littlewood maximal operator, M^+.

The one-sided μ-minimal operators, on the other hand, while satisfying $m_{\mu_2}^+ f(x) \le m_{\mu_1}^+ f(x)$ for $\mu_1 < \mu_2$, are not equivalent: there exist functions f such that $m_{\mu_2}^+ f(x) \ll m_{\mu_1}^+ f(x)$. For example, let $f(x) = 2|x|^{-3} e^{-x^{-2}} \chi_{[-1,0]} + \chi_{(0,\infty)}$. Then for $x_k = -(1 + \mu_2)2^{-k}$,

$$\frac{m_{\mu_1}^+ f(x_k)}{m_{\mu_2}^+ f(x_k)} \to \infty \text{ as } k \to \infty.$$

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We will briefly examine the limit of m_μ^+ as $\mu \to 0$ in Section 2.

To explore the weak-type norm inequalities for $m_\mu^+ f$, we first define the class $W_{p,\mu}^+$.

DEFINITION. *A pair of non-negative weights (u, v) is in the class $W_{p,\mu}^+$, $p > 0$, $\mu > 0$, if given any pair of adjacent intervals I and J, I to the left of J with $|I| = \mu|J|$, then*

$$\frac{1}{|I|}\int_I u\,dx \le C\left(\frac{1}{|J|}\int_J v^{1/(p+1)}\,dx\right)^{p+1},$$

where C is independent of the choice of I and J.

It is easy to see that $W_{p,\mu_2}^+ \subset W_{p,\mu_1}^+$ for $\mu_1 < \mu_2$. We will give a simple example in Section 2 to show that the reverse inclusion is not true. We will also examine the $W_{p,\mu}^+$ classes in the single weight case $u = v$.

To study the strong-type norm inequalities for m_μ^+, we now define the class $(W_{p,\mu}^+)^*$. Throughout the paper, we use the notation $\sigma = v^{1/(p+1)}$.

DEFINITION. *A pair of non-negative weights (u, v) is in the class $(W_{p,\mu}^+)^*$, $p > 0$, $\mu > 0$, if given any pair of adjacent intervals I and J, I to the left of J with $|I| = \mu|J|$, then*

$$\int_{I\cup J} \frac{u}{m_\mu^+(\sigma/\chi_J)^p}\,dx \le C\int_J \sigma\,dx,$$

where C is independent of the choice of I and J.

As was the case in our previous work, a surprising result is that the strong and weak type inequalities are actually equivalent.

THEOREM 1. *Given $p > 0$, $\mu > 0$, the following are equivalent.*

(a) Weak-type inequality: there is a constant $C > 0$ independent of $f \ge 0$ with $1/f \in L^p(v)$ such that

$$u\{x\colon m_\mu^+ f(x) < 1/t\} \le \frac{C}{t^p}\int \frac{v}{f^p}\,dx;$$

(b) $(u, v) \in W_{p,\mu}^+$;

(c) Strong-type inequality: there is a constant $C > 0$ independent of $f \ge 0$ with $1/f \in L^p(v)$ such that

$$\int \frac{u}{(m_\mu^+ f)^p}\,dx \le C\int \frac{v}{f^p}\,dx;$$

(d) $(u, v) \in (W_{p,\mu}^+)^*$.

The material in this paper is organized as follows. In Section 2, we give some preliminary results concerning the limiting case of m_μ^+ as $\mu \to 0$, the inclusion properties of the $W_{p,\mu}^+$ classes, and the one weight case $u = v$.

Section 3 gives the proof of $(d) \Rightarrow (c)$. The converse implication is proved by inserting the function $f = \sigma/\chi_J$ into the strong-type inequality. Similarly, the proof of $(a) \Rightarrow (b)$ is gotten by substituting $f = \sigma/\chi_J$ and $1/t = |J|^{-1} \int_J \sigma\, dy$ into the weak-type inequality. Surprisingly, a direct proof of the converse implication has not been found. The implication $(c) \Rightarrow (a)$ is easily proved using Chebyshev's inequality.

In Section 4, we prove the equivalence of $W_{p,\mu}^+$ and $(W_{p,\mu}^+)^*$.

In Section 5, we give an application of the one-sided μ-minimal operators to the problem of convergence of convolution operators $T_\epsilon f(x) = \phi_\epsilon * f(x)$, where $\phi_\epsilon(x) = \epsilon^{-1}\phi(\epsilon^{-1}x)$ for suitably defined $\phi \geq 0$. We study the type of convergence of functions $\{g_k\}$ to f so that the exceptional set E_f of convergence, i.e.,

$$E_f = \left\{x\colon \limsup_{\epsilon \to 0} |T_\epsilon f(x) - f(x)| > 0\right\},$$

is controlled by the E_{g_k}'s in the sense that if $|E_{g_k}| \leq M < \infty$ for $k \geq 0$, then $|E_f| \leq M$. Similar to our work in [2], we introduce a Muckenhoupt-type A_2 condition relative to the ϕ_ϵ's.

Throughout the paper, all functions are assumed to be measurable and notation is standard or defined as necessary. Given a function $g\colon \mathbb{R} \to \mathbb{R}$ and a measurable set E, $g(E)$ denotes $\int_E g\, dx$. The weights u and v satisfy $0 < u(I), v(I) < \infty$ for all finite intervals I. By g/χ_I we denote the function equal to g on I and infinity elsewhere. The letter C denotes a positive constant whose value may be different at each appearance.

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2. Preliminary results

If we define $m_*^+ f(x) = \lim_{\mu \to 0} m_\mu^+ f(x)$, then clearly $m_*^+ f(x) \leq m^+ f(x)$, where $m^+ f$ is the one-sided minimal operator defined as (see [3])

$$m^+ f(x) = \inf_{h>0} \frac{1}{h} \int_x^{x+h} f(y)\, dy.$$

It is worthwhile to note that in many cases, $m_*^+ f(x) = m^+ f(x)$:

(i) If f is locally integrable on a finite interval I, then $m_*^+(f/\chi_I)(x) = m^+(f/\chi_I)(x)$ almost everywhere. For x not in the closure of I, $m_*^+(f/\chi_I)(x) = m^+(f/\chi_I)(x) = \infty$. Let

$$E = \{x \in I\colon (f/\chi_I)(x) \geq m^+(f/\chi_I)(x)\}.$$

By the Lebesgue differentiation theorem, $|I\backslash E| = 0$. Fix any Lebesgue point $x \in E$ and suppose that $m_*^+(f/\chi_I)(x) < m^+(f/\chi_I)(x)$. For each $\mu > 0$, there exists an interval $J_\mu \subset I$ to the right of x so that $0 \le \text{dist}(x, J_\mu) < \mu|J_\mu|$ and

$$m_\mu^+(f/\chi_I)(x) \ge \frac{1}{|J_\mu|}\int_{J_\mu} f\,dy - \mu.$$

The intervals J_μ cannot converge to a non-empty interval J that has x as its left end point. For if they did, then by the Lebesgue dominated convergence theorem,

$$m_*^+(f/\chi_I)(x) \ge \lim_{\mu\to 0}\left[\frac{1}{|J_\mu|}\int_{J_\mu} f\,dy - \mu\right] = \frac{1}{|J|}\int_J f/\chi_J\,dy \ge m^+(f/\chi_I)(x)$$

which is a contradiction. Therefore, $|J_\mu| \to 0$ as $\mu \to 0$. Since $\text{dist}(x, J_\mu) \to 0$, $J_\mu \to \{x\}$, and so

$$\frac{1}{|J_\mu|}\int_{J_\mu} f\,dy \to (f/\chi_I)(x).$$

Therefore,

$$m_*^+(f/\chi_I)(x) \ge \lim_{\mu\to 0}\left[\frac{1}{|J_\mu|}\int_{J_\mu} f\,dy - \mu\right] = (f/\chi_I)(x) \ge m^+(f/\chi_I)(x),$$

which again is a contradiction.

(ii) If $f \in L^1_{\text{loc}}$ and $1/f \in L^p$, then $m_*^+ f(x) = m^+ f(x)$ for a.e. x.

Since $1/f \in L^p$, given any $\epsilon > 0$, there is $N_\epsilon > 0$ such that if I is an interval with $|I| > N_\epsilon$,

$$\frac{1}{|I|}\int_I f\,dy > \frac{1}{\epsilon}.$$

Since $m^+ f(x)$ is upper semi-continuous, on $I_k = [-k, k]$, $m^+ f(x)$ is bounded by some number R_k. For a fixed k, let ϵ be such that $1/\epsilon > R_k$ and let $L = k + 2N_\epsilon$. Then, for a.e. $x \in I_k$, $m^+ f(x) = m^+(f/\chi_{[-L,L]})(x)$ and for $\mu \le 1$, $m_\mu^+ f(x) = m_\mu^+(f/\chi_{[-L,L]})(x)$. By applying the first remark and letting k tend to ∞, we get the result.

(iii) It is easy to see that if $1/f \notin L^p$, it may happen that $m_*^+ f(x) < m^+ f(x)$ on a set of positive measure. Specifically, let a_n be a sequence of positive numbers such that $a_n \to \infty$ and $a_{n+1} > a_n^2$. Let $J_n = (a_n, a_n^2)$ and define $f = 0$ on each J_n. Putting $\mu_n = 1/(a_n - 1)$, we see that $m_{\mu_n}^+ f = 0$ on $[0, a_n]$ but $m^+ f$ can be made as large as desired by defining f appropriately on $\mathbb{R}\backslash \cup J_n$.

We now give an example to show that the inclusion $W_{p,\mu_2}^+ \subset W_{p,\mu_1}^+$ for $\mu_1 < \mu_2$ is proper. Specifically, we will find a pair of weights $(u, v) \in W_{1,1}^+\backslash W_{1,\mu}^+$ for $\mu > 1$.

Consider $I_i = (e^i - 2, e^i - 1)$, $J_i = (1 + e^{i-1}, e^i)$ for $i \geq 2$. Define u and v by

$$u(x) = \sum_i e^i \chi_{I_i} + \chi_{\mathbb{R}\setminus \cup I_i},$$
$$v(x) = \sum_i e^{2i} \chi_{J_i} + \chi_{\mathbb{R}\setminus \cup J_i}.$$

It is not difficult to see that $(u, v) \in W^+_{1,1}$ and the fact that $(u, v) \notin W^+_{1,\mu}$ for $\mu > 1$ follows immediately by taking $I = (e^i - \mu, e^i)$ and $J = (e^i, 1 + e^i)$.

In the single weight case $u = v$, the classes $W^+_{p,\mu}$ all collapse to the single class A^+_∞—the union of the classes A^+_p, $p > 1$, which govern the weighted norm inequalities for M^+. For if $(\omega, \omega) \in W^+_{p,\mu}$, then for any adjacent intervals I and J, I to the left of J, and $|I| = \mu|J|$,

$$\frac{1}{|I|}\int_I \omega\, dx \leq C\left(\frac{1}{|I \cup J|}\int_{I\cup J} \omega^{1/(p+1)}\, dx\right)^{p+1}. \tag{1}$$

In [3], we showed that this "one-sided" reverse Hölder inequality is equivalent to ω being in A^+_∞. Conversely, if $w \in A^+_\infty$ then (1) holds, and by the geometric characterization of A^+_∞ (see [3] or [9]), $\omega^{1/(p+1)}$ satisfies the one-sided doubling condition

$$\int_I \omega^{1/(p+1)}\, dx \leq C \int_J \omega^{1/(p+1)}\, dx,$$

where the constant C depends only on ω and μ. Hence $(\omega, \omega) \in W^+_{p,\mu}$.

3. Proof of (d) ⇒ (c)

We first state a preliminary lemma that will be used repeatedly throughout the paper. It is a technical result due to Muckenhoupt [10] generalized to arbitrary regular measures. With the appropriate substitutions, the proof is identical to his proof for Lebesgue measure and so is omitted.

LEMMA 2. *Given a function f, a regular measure ν and an interval I, let $\{I_\alpha\}$ be a collection of intervals contained in I such that, for each α,*

$$\int_{I_\alpha} f\, d\nu \geq N\nu(I_\alpha).$$

If $J = \cup_\alpha I_\alpha$, then

$$\int_J f\, d\nu \geq (N/2)\nu(J).$$

To prove that $(d) \Rightarrow (c)$, we will first consider the special case where f is such that $1/f$ has compact support. For each $k \in \mathbb{Z}$ define

$$A_k = \left\{x \colon 2^{-k-1} \le m_\mu^+ f(x) < 2^{-k}\right\},$$

and let K_k be an arbitrary compact subset of A_k. For each $x \in A_k$ there is an open interval $J_{x,k}$ to the right of x such that $0 \le \operatorname{dist}(x, J_{x,k}) < \mu |J_{x,k}|$ and

$$\frac{1}{|J_{x,k}|} \int_{J_{x,k}} f\,dy < 2^{-k}.$$

Note that $\cup J_{x,k} \subset T$, where T is some interval containing the support of $1/f$. This will be important later in the proof when we apply Lemma 2. Let $I_{x,k}$ be the interval that is adjacent to $J_{x,k}$, to the left of $J_{x,k}$ and $|I_{x,k}| = \mu |J_{x,k}|$; then $A_k \subset \cup I_{x,k}$. Therefore, by compactness, for each k we can find a finite collection $\{I_{j,k}\}_{j=1}^{m_k} \subset \{I_{x,k}\}$ that covers the set K_k. In fact,

$$K_k = \bigcup_{j=1}^{m_k} E_{j,k},$$

where the $E_{j,k}$'s are the disjoint sets defined inductively by $E_{1,k} = I_{1,k} \cap K_k$, $E_{2,k} = (I_{2,k} \setminus I_{1,k}) \cap K_k, \ldots$.

For an arbitrary positive integer N we have

$$\begin{aligned}
(2) \qquad \int_{\cup_{k=-N}^{N} K_k} \frac{u}{\left(m_\mu^+ f\right)^p}\,dx &= \sum_{k=-N}^{N} \sum_{j=1}^{m_k} \int_{E_{j,k}} \frac{u}{\left(m_\mu^+ f\right)^p}\,dx \\
&\le 2^p \sum_k \sum_j u(E_{j,k}) \cdot 2^{kp} \\
&\le 2^p \sum_k \sum_j u(E_{j,k}) |J_{j,k}|^p \left(\int_{J_{j,k}} f\,dy\right)^{-p} \\
&= 2^p \sum_k \sum_j u(E_{j,k}) \frac{|J_{j,k}|^p}{\sigma\left(J_{j,k}\right)^p} \\
&\quad \cdot \left(\frac{1}{\sigma(J_{j,k})} \int_{J_{j,k}} \frac{f}{\sigma} \cdot \sigma\,dy\right)^{-p}.
\end{aligned}$$

Define the measure ω on $X = \mathbb{Z} \times \mathbb{N}$ by

$$\omega(k, j) = \frac{u(E_{j,k}) |J_{j,k}|^p}{\sigma(J_{j,k})^p} \qquad \text{for } 1 \le j \le m_k,$$

and $\omega(k, j) = 0$ for $j > m_k$. Further, for $h \in L^2(\sigma)$ define

$$Sh(k, j) = \frac{\sigma(J_{j,k})}{\int_{J_{j,k}} h\sigma\,dy} \qquad \text{and} \qquad Th(k, j) = \frac{\int_{J_{j,k}} h\sigma\,dy}{\sigma(J_{j,k})}.$$

By Hölder's inequality, $Sh(k,j) \le T(h^{1-r'})(k,j)^{r-1}$ for $r > 1$. Putting $r = 1 + \frac{2}{p}$ and rewriting (2), we get

$$\int_{\bigcup_{k=-N}^{N} K_k} \frac{u}{(m_\mu^+ f)^p}\,dx \le 2^p \int_X S\left(\frac{f}{\sigma}\right)^p d\omega \le 2^p \int_X T\left(\frac{\sigma^{r'-1}}{f^{r'-1}}\right)^2 d\omega.$$

If T were a bounded operator from $L^2(\sigma) \to L^2(X, d\omega)$, then

$$\int_{\bigcup_{k=-N}^{N} K_k} \frac{u}{(m_\mu^+ f)^p}\,dx \le C \int_{\mathbb{R}} \frac{\sigma^p}{f^p}\sigma\,dx = C \int_{\mathbb{R}} \frac{v}{f^p}\,dx.$$

By taking nested compact sets $K_{i,k} \subset K_{i+1,k}$ that increase monotonically to A_k (modulo a set of measure zero), the monotone convergence theorem yields

$$\int_{\bigcup_{k=-N}^{N} A_k} \frac{u}{(m_\mu^+ f)^p}\,dx \le C \int_{\mathbb{R}} \frac{v}{f^p}\,dx.$$

Letting $N \to \infty$ gives the desired result.

Therefore, it remains to show that T: $L^2(\sigma) \to L^2(X, d\omega)$ is bounded. Since T is clearly bounded in L^∞, by Marcinkiewicz interpolation it will suffice to show that T is weak (1,1): for all $\lambda > 0$,

$$\int_{\{Th > \lambda\}} d\omega \le \frac{C}{\lambda} \int_{\mathbb{R}} h\sigma\,dx.$$

To prove this, define the set

$$G(\lambda) = \{(k,j)\colon Th(k,j) > \lambda\} = \left\{(k,j)\colon \frac{1}{\sigma(J_{j,k})}\int_{J_{j,k}} h\sigma\,dx > \lambda\right\}$$

and let

$$G = \bigcup_{(k,j)\in G(\lambda)} J_{j,k}.$$

The open set G is the countable union of disjoint open intervals J_i. Therefore by Lemma 2, we have

$$\frac{1}{\sigma(J_i)}\int_{J_i} h\sigma\,dx \ge \frac{\lambda}{2}. \tag{3}$$

If $J_{j,k} \subset G$, then $J_{j,k} \subset J_i$ for exactly one i. Hence, if $x \in E_{j,k}$ and $J_{j,k} \subset J_i$, then

$$m_\mu^+(\sigma/\chi_{J_i})(x) \le \frac{1}{|J_{j,k}|}\int_{J_{j,k}} \sigma\,dy.$$

That is,

$$\frac{|J_{j,k}|}{\sigma(J_{j,k})} \leq \inf_{x\in E_{j,k}} \left[m_\mu^+\left(\sigma/\chi_{J_i}\right)(x)\right]^{-1}.$$

Therefore,

$$\begin{aligned}\int_{\{Th>\lambda\}} d\omega &= \sum_{(k,j)\in G(\lambda)} \frac{u\left(E_{j,k}\right)\left|J_{j,k}\right|^p}{\sigma\left(J_{j,k}\right)^p} \\ &= \sum_i \sum_{(k,j)\in G(\lambda):\ J_{j,k}\subset J_i} \frac{u\left(E_{j,k}\right)\left|J_{j,k}\right|^p}{\sigma\left(J_{j,k}\right)^p} \\ &\leq \sum_i \sum_{(k,j)\in G(\lambda):\ J_{j,k}\subset J_i} u\left(E_{j,k}\right) \inf_{x\in E_{j,k}} \left[m_\mu^+\left(\sigma/\chi_{J_i}\right)(x)\right]^{-p} \\ &\leq \sum_i \sum_{(k,j)\in G(\lambda):\ J_{j,k}\subset J_i} \int_{E_{j,k}} \frac{u}{m_\mu^+(\sigma/\chi_{J_i})^p}\, dx. \qquad (4)\end{aligned}$$

Now let I_i be the interval adjacent to J_i on the left such that $|I_i| = \mu|J_i|$. If $J_{j,k} \subset J_i$, then $E_{j,k} \subset I_i \cup J_i$; hence, since the $E_{j,k}$'s are disjoint, (4) is bounded by

$$\sum_i \int_{I_i\cup J_i} \frac{u}{m_\mu^+(\sigma/\chi_{J_i})^p}\, dx \leq C\sum_i \sigma(J_i) \leq \frac{2C}{\lambda}\sum_i \int_{J_i} h\sigma\, dx \leq \frac{2C}{\lambda}\int_{\mathbb{R}} h\sigma\, dx.$$

The first inequality follows from the $(W_{p,\mu}^+)^*$ condition, the second from (3) and the third since the J_i's are disjoint.

To complete the proof, fix an arbitrary f and define the sequence $f_n = f/\chi_{[-n,n]}$. Clearly the sequence decreases monotonically to f. The sequence $m_\mu^+(f_n)$ is also monotonically decreasing and $m_\mu^+ f \leq \lim_{n\to\infty} m_\mu^+(f_n)$. On the other hand, for a fixed x in $\mathbb{R}$ and $\epsilon > 0$, there is an interval J to the right of x with $0 \leq \operatorname{dist}(x, J) < \mu|J|$ so that for all n sufficiently large,

$$m_\mu^+ f(x) \geq \frac{1}{|J|}\int_J f\, dy - \epsilon \geq m_\mu^+(f_n)(x) - \epsilon.$$

Therefore, by the monotone convergence theorem, the strong-type inequality holds for all f and we are done.

4. Proof of (b) ⇒ (d)

We require the following well-known covering properties for $\mathbb{R}$. Their proofs can be found in [2].

LEMMA 3. *Let $\mathcal{F}$ be a collection of intervals in $\mathbb{R}$ of positive length. Then there exists a countable sub-collection $\mathcal{F}_0$ such that* $\cup\{I\colon\ I \in \mathcal{F}\} = \cup\{I\colon\ I \in \mathcal{F}_0\}$.

LEMMA 4. *Let $\mathcal{F}$ be a finite collection of intervals in $\mathbb{R}$. Then there exist two sub-collections $\mathcal{F}_1$ and $\mathcal{F}_2$ such that the intervals in $\mathcal{F}_i$ are disjoint, $i = 1, 2$, and* $\cup\{I\colon I \in \mathcal{F}\} = \cup\{I\colon I \in \mathcal{F}_1\} \cup \{I\colon I \in \mathcal{F}_2\}$.

To prove that $(b) \Rightarrow (d)$, first fix adjacent intervals I and J, I to the left of J, such that $|I| = \mu|J|$. We write

$$\int_{I\cup J} \frac{u}{m_\mu^+(\sigma/\chi_J)^p}\,dx = \int_I \frac{u}{m_\mu^+(\sigma/\chi_J)^p}\,dx + \int_J \frac{u}{m_\mu^+(\sigma/\chi_J)^p}\,dx$$

and estimate each integral separately.

Step 1. Show that $\displaystyle\int_I \frac{u}{m_\mu^+(\sigma/\chi_J)^p}\,dx \le C\sigma(J)$.

Fix $\epsilon > 0$ and define

$$E_t = \left\{x \in I\colon m_\mu^+(\sigma/\chi_J)(x) < 1/t\right\}.$$

Then for any $R > 0$,

$$\int_I \frac{u}{m_\mu^+(\sigma/\chi_J)^p}\,dx \le R^p u(I) + p\int_R^\infty t^{p-1} u(E_t)\,dt. \tag{5}$$

Note that if $x \in E_t$, then there is an interval $J_x^t \subset J$ to the right of x so that $0 \le \operatorname{dist}(x, J_x^t) < \mu|J_x^t|$ and

$$\frac{1}{|J_x^t|}\int_{J_x^t} \sigma\,dy < \frac{1}{t}. \tag{6}$$

Associate to J_x^t the adjacent interval I_x^t, where I_x^t is to the left of J_x^t and $|I_x^t| = \mu|J_x^t|$. Now for each $x \in E_t$, I_x^t contains the right endpoint of I since $J_x^t \subset J$. That is, for every pair of points x_1 and x_2 in E_t, the intervals $I_{x_1} \cap I$ and $I_{x_2} \cap I$ are such that one is contained in the other. Therefore, E_t is the union of nested intervals $I_x^t \cap I$, $x \in E_t$, so we can find a point x_t such that

$$u(E_t) \le u\left(I_{x_t}^t \cap I\right) + \epsilon_0(t),$$

where

$$\epsilon_0(t) = \frac{\epsilon}{2}\chi_{(0,1]} + \frac{\epsilon}{2p}\frac{1}{t^{p+1}}\chi_{(1,\infty)}.$$

Then by the $W_{p,\mu}^{+}$ condition and our choice of the J_x^t's,

$$u(E_t) \le u\left(I_{x_t}^t\right) + \epsilon_0(t) \le C\frac{\left|I_{x_t}^t\right| \sigma\left(J_{x_t}^t\right)^{p+1}}{\left|J_{x_t}^t\right|^{p+1}} + \epsilon_0(t)$$

$$\le C\frac{\left|I_{x_t}^t\right|}{t^{p+1}} + \epsilon_0(t) \le C\frac{|I|}{t^{p+1}} + \epsilon_0(t).$$

Therefore,

$$p\int_R^\infty t^{p-1}u(E_t)\,dt \le Cp\,|I|\int_R^\infty t^{-2}\,dt + p\int_R^\infty t^{p-1}\epsilon_0(t)\,dt \le C\frac{|I|}{R} + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, by combining this with (5) we see that

$$\int_I \frac{u}{m_\mu^+(\sigma/\chi_J)^p}\,dx \le R^p u(I) + C\frac{|I|}{R}.$$

Let $R^p = \dfrac{\sigma(J)}{u(I)}$; then the $W_{p,\mu}^{+}$ condition gives

$$\int_I \frac{u}{m_\mu^+(\sigma/\chi_J)^p}\,dx \le C\sigma(J),$$

which is what we wanted.

Step 2. Show that $\displaystyle\int_J \frac{u}{m_\mu^+(\sigma/\chi_J)^p}\,dx \le C\sigma(J).$

Fix $\lambda \in \mathbb{N}$ such that $2^{-\lambda} \le \mu$. Define the intervals J_i inductively as follows: let J_1 be the open interval that comprises the left half of the interval J. Let J_2 be the open interval adjacent to J_1 that comprises the left half of the interval $J\backslash J_1$. Let J_3 be the open interval adjacent to J_2 that comprises the left half of the interval $J\backslash(J_1 \cup J_2)$, etc. For each i, $|J_i| = 2|J_{i+1}|$. Now, divide each J_i into $\alpha_\lambda = 2^{\lambda+1}$ equal intervals, $J_{i,1} \ldots J_{i,\alpha_\lambda}$ and to each $J_{i,j}$ adjoin to the right an interval $J'_{i,j} \subset J$ with $|J'_{i,j}| = |J_{i+1}| = 2^{-i-1}|J|$. The intervals $J'_{i,j}$ are of bounded overlap:

$$\sum_i \sum_{j=1}^{\alpha_\lambda} \chi_{J'_{i,j}} \le 2^{\lambda+2}.$$

Fix a pair (i, j) and let

$$E_t = E_{i,j,t} = \{x \in J_{i,j}\colon m_\mu^+(\sigma/\chi_J)(x) < 1/t\}.$$

If $x \in E_t$, then there exists $J_x^t = J_{i,j,x}^t \subset J$ to the right of x such that $0 \leq \operatorname{dist}(x, J_x^t) < \mu |J_x^t|$ and

$$\frac{1}{|J_x^t|} \int_{J_x^t} \sigma \, dy < \frac{1}{t}. \tag{7}$$

Associate to J_x^t the adjacent interval I_x^t, where I_x^t is to the left of J_x^t and $|I_x^t| = \mu |J_x^t|$. By Lemma 3, there exists a countable collection $\{I_k^t\}_{k\in\mathbb{N}} \subset \{I_x^t\}_{x\in E_t}$ such that

$$E_t = \bigcup_{x\in E_t} \left(I_x^t \cap J_{i,j}\right) = \bigcup_k \left(I_k^t \cap J_{i,j}\right).$$

Let

$$E_{t,n} = \bigcup_{k=1}^{n} \left(I_k^t \cap J_{i,j}\right).$$

By Lemma 4, there exists a disjoint sub-collection $\{I_{k,n}^t \cap J_{i,j}\}_{k=1}^{m_{t,n}} \subset \{I_k^t \cap J_{i,j}\}_{k=1}^{n}$ such that

$$u\left(E_{t,n}\right) \leq 2 \sum_{k=1}^{m_{t,n}} u\left(I_{k,n}^t \cap J_{i,j}\right). \tag{8}$$

Among the set $\{I_{k,n}^t \cap J_{i,j}\}_{k=1}^{m_{t,n}}$, there is at most one interval, call it $I_{k_1,n}^t$, that contains the right hand endpoint of $J_{i,j}$. Similarly, there is at most one $I_{k,n}^t$, call it $I_{k_2,n}^t$, that contains the left hand endpoint of $J_{i,j}$. All of the other intervals are properly contained in $J_{i,j}$. A simple geometrical argument shows that $\left|J_{k_1,n}^t\right| \leq 2\,|J_i| = 2^{k+2}\left|J_{i,j}\right|$; hence $\left|I_{k_1,n}^t\right| = \mu \left|J_{k_1,n}^t\right| \leq 2^{\lambda+2}\mu \left|J_{i,j}\right|$. Similarly, $\left|I_{k_2,n}^t\right| = \mu \left|J_{k_2,n}^t\right| \leq 2^{\lambda+2}\mu \left|J_{i,j}\right|$. Therefore

$$\sum_{k=1}^{m_{t,n}} \left|I_{k,n}^t\right| \leq (1 + 2^{\lambda+3}\mu) \left|J_{i,j}\right|. \tag{9}$$

Then by (8), the $W_{p,\mu}^+$ condition, (7), and (9),

$$u\left(E_{t,n}\right) \leq 2 \sum_{k=1}^{m_{t,n}} u\left(I_{k,n}^t\right) \leq C \sum_{k=1}^{m_{t,n}} \frac{\left|I_{k,n}^t\right| \sigma\left(J_{k,n}^t\right)^{p+1}}{\left|J_{k,n}^t\right|^{p+1}} \leq \frac{C}{t^{p+1}} \sum_{k=1}^{m_{t,n}} \left|I_{k,n}^t\right| \leq \frac{C\left|J_{i,j}\right|}{t^{p+1}}.$$

Since the right hand side of the above inequality is independent of n, we may take the limit as n tends to infinity to get

$$u\left(E_t\right) \leq \frac{C\left|J_{i,j}\right|}{t^{p+1}}.$$

Reasoning exactly as in step 1, we see that

$$\int_{J_{i,j}} \frac{u}{m_\mu^+(\sigma/\chi_J)^p}\, dx \leq u\left(J_{i,j}\right) R^p + C\frac{\left|J_{i,j}\right|}{R}$$

for each pair (i, j) and $R > 0$. Let $R^p = \dfrac{\sigma\left(J'_{i,j}\right)}{u\left(J_{i,j}\right)}$; since $|J_{i,j}| = 2^{-\lambda}|J'_{i,j}|$ and $2^{-\lambda} \leq \mu$, $(u, v) \in W^+_{p,2^{-\lambda}}$, so

$$\int_{J_{i,j}} \frac{u}{m^+_\mu(\sigma/\chi_J)^p}\, dx \leq C\sigma\left(J'_{i,j}\right).$$

Finally, since the intervals $\{J'_{i,j}\}$ have bounded overlap, we sum over (i, j) to get the desired inequality.

5. Application to convolution operators

Throughout this section let ϕ be a non-negative function of compact support such that $\|\phi\|_1 = 1$. Define the family of convolution operators $T_\epsilon f(x) = \phi_\epsilon * f(x)$, $\epsilon > 0$, where $\phi_\epsilon(x) = \epsilon^{-1}\phi\left(\epsilon^{-1}x\right)$; then it is well known that $T_\epsilon f \to f$ in L^p, $1 \leq p < \infty$. Further, if the associated maximal operator

$$T^* f(x) = \sup_{\epsilon>0} |\phi_\epsilon * f(x)|$$

is dominated by the Hardy-Littlewood maximal function, Mf, then $T_\epsilon f(x) \to f(x)$ for a.e. x. However, the estimate $T^* f(x) \leq CMf(x)$ places a significant restriction on ϕ: for example,

$$\psi(x) = \sup_{|t|\geq|x|} \phi(t) \in L^1.$$

(See [5] or [13].) If $\psi \notin L^1$ then there may exist $f \in L^1$ such that

$$\limsup_{\epsilon\to 0} T_\epsilon f(x) = \infty$$

almost everywhere. For the convenience of the reader we sketch a simple example: define the sequence $\{\alpha_n\}_{n=1}^\infty$ such that the intervals $I_n = [1/n, 1/n + \alpha_n]$ are disjoint and

$$\phi(x) = \frac{1}{x^2}\chi_{\cup I_n}(x)$$

is in L^1. Let $\phi_j(x) = j\phi(jx)$; then $T^* f(x) = \sup |\phi_j * f(x)|$ is not weak-type $(1, 1)$ since $\sup x\phi(x) = \infty$. (See [5, p. 296].) This implies that there exists a function $f \in L^1$ for which $T^* f(x) = \infty$ a.e. (See Proposition 1 in [13, p. 441].)

Define the exceptional set for the pointwise convergence of the T_ϵ's by

$$E_f = \{x\colon \limsup_{\epsilon\to 0} |T_\epsilon f(x) - f(x)| > 0\}.$$

The question we are interested in is the following: Given a sequence $\{g_k\}$ converging pointwise to a function f, under what additional hypotheses is E_f controlled by the

E_{g_k}'s — that is, if $|E_{g_k}| \leq M < \infty$ for all k, then $|E_f| \leq M$. As the previous example shows, L^1 convergence is not sufficient: there exist $g_k \in C_c$ such that $g_k \to f$ in L^1, and clearly E_{g_k} is empty for continuous g_k.

To give the correct condition, we need to assume that $1/f \in L^p$ for some $p > 0$. We can do this with no loss of generality since given f, we can replace f by $F(x) = f(x) + e^{|x|}$. Then $1/F \in L^p$ and $E_f = E_F$.

We now define the minimal operator associated to the T_ϵ's:

$$T_* f(x) = \inf_{\epsilon>0} \phi_\epsilon * f(x).$$

The following result may be thought of as a Harnack inequality for the T_ϵ's.

LEMMA 5. *Suppose for some $h_0 > 0$ the set $\{x\colon \phi(x) > h_0\}$ contains a non-empty open interval $I_0 \subset (-\infty, 0)$. Then there exist constants $\mu = \mu_\phi > 0$ and $c = c_\phi > 0$ such that for every $x \in \mathbb{R}$,*

$$T_* f(x) \geq c m_\mu^+ f(x).$$

Remark. If $I_0 \subset (0, \infty)$, then m_μ^+ is replaced by m_μ^-.

Proof. Suppose $I_0 = (a, b), b \leq 0$. Define $\mu = -b/(b-a)$ and $\phi_0 = h_0 \cdot \chi_{I_0}$. Then $0 \leq \phi_0 \leq \phi$, and so for $x \in \mathbb{R}$,

$$\begin{aligned} T_\epsilon f(x) &\geq \frac{1}{\epsilon} \int_{\epsilon a}^{\epsilon b} \phi_0(t/\epsilon) f(x-t)\, dt \\ &= \frac{h_0}{\epsilon} \int_{\epsilon a}^{\epsilon b} f(x-t)\, dt \\ &= h_0 |I_0| \frac{1}{\epsilon |I_0|} \int_{x-\epsilon b}^{x-\epsilon a} f(t)\, dt \\ &\geq h_0 |I_0| m_\mu^+ f(x). \end{aligned}$$

Let $c = h_0 |I_0|$ and we are done.

COROLLARY 6. *Suppose for some $h_0 > 0$ the set $\{x\colon \phi(x) > h_0\}$ contains a non-empty open interval I_0 such that $I_0 \subset (-\infty, 0)$. If $0 < p < \infty$ and $(u, v) \in W_{p,\mu}^+$, then*

$$\int_{\mathbb{R}} \frac{u}{(T_* f)^p}\, dx \leq c \int_{\mathbb{R}} \frac{v}{f^p}\, dx.$$

Proof. This follows from Lemma 5 and Theorem 1.

For general ϕ, the set $\{x\colon \phi(x) > h\}$ may not contain an interval for any $h > 0$, and thus Lemma 5 is not applicable. We can avoid this by replacing ϕ with

$\tilde{\phi} = (\phi + \chi_{[-1,0]})/2$. Then, apart from a set of measure 0, $E_f = \tilde{E}_f$, where $\tilde{E}_f = \{x\colon \limsup |\tilde{\phi}_\epsilon * f(x) - f(x)| > 0\}$. This follows at once from the fact that if f is locally integrable, then $(\chi_{[-1,0]})_\epsilon * f(x) \to f(x)$ for a.e. x.

We now define the Muckenhoupt-type A_2 condition that plays a key role in controlling the sets E_f. For $w \geq 0$, define

$$A_2(w) = \sup_I \frac{1}{|I|} \int_I w\, dy \cdot \frac{1}{|I|} \int_I 1/w\, dy + \sup_{x,\epsilon>0} T_\epsilon w(x) \cdot T_\epsilon(1/w)(x).$$

The first term is the usual A_2-condition and the second term can be viewed as an A_2-condition relative to $\{\phi_\epsilon\}$. Note that if for some $h > 0$, the set $\{x\colon \phi(x) > h\}$ contains an interval, the first term is dominated by the second and can thus be dropped.

LEMMA 7. *Let f, g be non-negative functions such that $A_2(|f - g|) = c_0 < \infty$. Then*

$$\left| \frac{1}{T_\epsilon f(x)} - \frac{1}{T_\epsilon g(x)} \right| \leq \frac{c_0}{\{T_\epsilon F(x)\}^3}, \text{ where } F = \left(\frac{fg}{|f - g|} \right)^{1/3}.$$

Proof. Apply Hölder's inequality with respect to the measure $\phi_\epsilon(t)\, dt$ to get

$$\begin{aligned}\{T_\epsilon F(x)\}^3 \cdot T_\epsilon(|f - g|)(x) &\leq T_\epsilon f(x) \cdot T_\epsilon g(x) \cdot T_\epsilon \left(\frac{1}{|f - g|} \right)(x) \cdot T_\epsilon(|f - g|)(x) \\ &\leq c_0 T_\epsilon f(x) \cdot T_\epsilon g(x).\end{aligned}$$

THEOREM 8. *Let (u, v) be a pair of weights and fix $p > 0$. Let f be a non-negative, locally integrable function such that $1/f \in L^p(v)$. Then there exists $\mu > 0$ such that, if $(u, v) \in W^+_{3p,\mu}$, the following holds*:

If $\{g_k\}$ is a sequence of non-negative functions satisfying

$$\frac{1}{g_k} \to \frac{1}{f} \text{ in } L^p(v) \quad \text{and} \quad A_2(|g_k - f|) \leq c < \infty \text{ for all } k,$$

then given any $\lambda < u(E_f)$ and $\eta > 0$, there exists $k = k(\lambda, \eta)$ such that $u(E_{g_k}) > \lambda - \eta$.

Proof. Since the measure udx is absolutely continuous, by the comment following Corollary 6, we may assume that the set $\{x\colon \phi(x) > h\}$ contains an open interval contained in $(-\infty, 0)$ for some h. Therefore Lemma 5 applies, so fix μ as in that result.

Suppose now that $(u, v) \in W^+_{3p,\mu}$. Then $u \leq cv$, so $1/f \in L^p(u)$. Hence $f(x) > 0$ for almost every x (with respect to udx). Further, since f is locally integrable, $f(x) < \infty$ a.e.. Therefore $u(E_f) = u(D)$, where

$$D = \left\{ x\colon \limsup_{\epsilon \to 0} \left| \frac{1}{T_\epsilon f(x)} - \frac{1}{f(x)} \right| > 0 \right\}.$$

Now let λ and η be as in the statement of the theorem, and let $D_i \subset D$ be the set where the given limit supremum is larger than $1/i$. Then we can find i sufficiently large so that $u(D_i) > \lambda$. Now for each k let

$$F_k = \left(\frac{f g_k}{|f - g_k|}\right)^{1/3}.$$

Then by Lemma 7,

$$\left|\frac{1}{T_\epsilon f(x)} - \frac{1}{f(x)}\right| \leq \frac{c}{T_* F_k(x)^3} + \left|\frac{1}{T_\epsilon g_k(x)} - \frac{1}{g_k(x)}\right| + \left|\frac{1}{g_k(x)} - \frac{1}{f(x)}\right|.$$

Hence, taking the limit supremum as ϵ tends to 0,

$$D_i \subset \left\{x\colon\ T_* F_k(x)^3 < 3ci\right\} \cup D_{g_k} \cup \left\{x\colon\ \left|\frac{1}{g_k(x)} - \frac{1}{f(x)}\right| > \frac{1}{3i}\right\},$$

where D_{g_k} is defined as D with f replaced by g_k. As before, $u(D_{g_k}) = u(E_{g_k})$. Since $(u, v) \in W^+_{3p,\mu}$, by Corollary 6 and Theorem 1,

$$u(D_i) < ci^p \int_{\mathbb{R}} \left|\frac{1}{g_k} - \frac{1}{f}\right|^p v\,dx + u(E_{g_k}).$$

Now choose k so large that the first term is $\leq \eta$ and we are done.

COROLLARY 9. *With the same hypotheses as above, if $u(E_{g_k}) \leq M < \infty$ for all k, then $u(E_f) \leq M$.*

Remarks. (i) If $u = v = 1$, (that is, the unweighted case) we trivially have $(u, v) \in W^+_{p,\mu}$ for all p and μ. Given a non-negative u, then $(u, e^u) \in W^+_{p,\mu}$ for all p and μ.

(ii) We can replace the norm convergence of $1/g_k$ to $1/f$ by the stronger hypothesis that the g_k's decrease monotonically to f. In this case, Theorem 8 can be thought of as a Harnack principle for the T_ϵ's.

(iii) The question of extending the convergence results given above to $\mathbb{R}^n$ for $n > 1$ remains open. It is unclear what the appropriate substitute for m^+_μ should be.

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