ILLINOIS JOURNAL OF MATHEMATICS
Volume 41, Number 1, Spring 1997

HARMONIC ENDOMORPHISM FIELDS

EDUARDO GARCÍA-RÍO[1], LIEVEN VANHECKE AND M. ELENA VÁZQUEZ-ABAL[1]

1. Introduction

A $(1, 1)$–tensor field φ on a pseudo-Riemannian manifold (M, g) determines a map $\varphi\colon TM \to TM$, where TM denotes the tangent bundle of M. The main purpose of this paper is to determine a necessary and sufficient condition for $\varphi\colon (TM, g^C) \to (TM, g^C)$ to be *harmonic*, where g^C is the (pseudo-Riemannian) complete lift metric as introduced by Yano and Ishihara in [18]. Our main result is that the harmonicity condition is equivalent to $\nabla^*\varphi = 0$, where ∇^* is the formal adjoint of the Levi Civita connection ∇ of (M, g).

In the remaining part we illustrate this result by means of several natural examples of endomorphism fields. In particular we consider the Ricci operator of (M, g) and also the shape operator of a hypersurface. The harmonicity of the corresponding map is equivalent to the constancy of the scalar curvature or, when (M, g) is Einsteinian, to the constancy of the mean curvature of the hypersurface. From these results we derive characterizations of harmonic manifolds and manifolds of constant sectional curvature by using geodesic spheres or tubes about geodesics as hypersurfaces. Further we consider the structure J on an almost Hermitian manifold (M, g, J) and we also treat the case of almost product structures. We finish by looking at these two kind of structures on the tangent bundle TM of (M, g). In this way we provide examples of harmonic maps of some special pseudo-Riemannian manifolds.

We refer to [17] where the above notion of a harmonic φ is used to define harmonic foliations.

2. Harmonic endomorphism fields

Let (M, g) be a connected, smooth pseudo-Riemannian manifold, ∇ its Levi Civita connection and R the corresponding Riemannian curvature tensor defined by $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ for smooth vector fields X, Y on M. Let ρ and Q, respectively, denote the Ricci tensor of type $(0, 2)$ and $(1, 1)$, respectively and let τ denote the scalar curvature.

Next, let $\varphi\colon M \to N$ be a smooth map between two pseudo-Riemannian manifolds with metric g and h, respectively and let $\varphi^{-1}(TN)$ be the pull-back bundle. The Levi

Received November 3, 1995.
1991 Mathematics Subject Classification. Primary 53C20, 53C25, 58E20.
Supported by projects XUGA 20701B93 and DGICYT PB94-0633-C02-01.

Civita connections on TM and TN induce a connection ∇ in the bundle of one-forms on M with values in $\varphi^{-1}(TN)$. Then $\alpha_\varphi = \nabla d\varphi$ is a symmetric bilinear form on TM which is called the *second fundamental form* of φ. The trace of α_φ with respect to g is called the *tension field* of φ, and denoted by $\tau(\varphi)$. The map φ is said to be harmonic if $\tau(\varphi) = 0$. (See [7], [8], [9] for more details and references.) Now, let $U \subset M$ be a domain with coordinates $(x^1, \dots, x^m)$, $m = \dim M$ and $V \subset N$ be a domain with coordinates $(z^1, \dots, z^n)$, $n = \dim N$, such that $\varphi(U) \subset V$ and suppose that φ is locally represented by $z^\alpha = \varphi^\alpha(x^1, \dots, x^m)$, $\alpha = 1, \dots, n$. Then we have

$$(1) \qquad (\alpha_\varphi)^\gamma_{ij} = \frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial \varphi^\gamma}{\partial x^k} + {}^N\Gamma^\gamma_{\alpha\beta}(\varphi)\frac{\partial \varphi^\alpha}{\partial x^i}\frac{\partial \varphi^\beta}{\partial x^j}.$$

Here ${}^M\Gamma^k_{ij}$ and ${}^N\Gamma^\gamma_{\alpha\beta}$ denote the Christoffel symbols of (M, g) and (N, h), respectively. So, φ is harmonic if and only if

$$(2) \qquad \tau(\varphi)^\gamma = g^{ij}(\alpha_\varphi)^\gamma_{ij} = 0$$

for $\gamma = 1, \dots, n$.

Now, let TM denote the tangent bundle of M. This $2m$–dimensional manifold may be equipped with the pseudo-Riemannian complete lift metric g^C, of signature (m, m), defined by

$$(3) \qquad \begin{cases} g^C(X^H, Y^H) = g^C(X^V, Y^V) = 0, \\ g^C(X^H, Y^V) = g^C(X^V, Y^H) = g(X, Y)^V. \end{cases}$$

Here, the horizontal and vertical lifts of tangent vector fields X, Y on M refer to the decomposition of the tangent space of TM at every point in horizontal vectors with respect to ∇ and canonical vertical vectors. For vector fields X, Y on M the function $g(X, Y)^V$ on TM is the pull-back of $g(X, Y)$ under the projection $TM \to M$. For local coordinates $(x^1, \dots, x^{2m}) = (x^1, \dots, x^m; x^{\bar 1}, \dots, x^{\bar m})$, where $\bar i = i + m$, $i = 1, \dots, m$, we have the local expression

$$(4) \qquad g^C = \begin{pmatrix} x^{\bar k}\frac{\partial g_{ij}}{\partial x^k} & g_{ij} \\ g_{ij} & 0 \end{pmatrix},$$

$i, j = 1, \dots, m$ with respect to $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^{\bar 1}}, \dots, \frac{\partial}{\partial x^{\bar m}})$. We refer to [18] for more details.

Finally, let φ be a tensor field of type $(1, 1)$ on M. Then φ determines a map $\varphi\colon TM \to TM$.

DEFINITION. (i) The endomorphism field φ (or $(1, 1)$-tensor field φ) on (M, g) is said to be *harmonic* if the map $\varphi\colon (TM, g^C) \to (TM, g^C)$ is a harmonic map.

(ii) The $(0, 2)$-tensor field Φ on (M, g) determined by $\Phi(X, Y) = g(\varphi X, Y)$ for all tangent vectors X, Y on (M, g) is called *harmonic* if φ is harmonic on (M, g).

Following [2, p. 34], we denote by ∇^* the formal adjoint of the Levi Civita connection ∇ on (M, g). Then with respect to local coordinates we have

$$(\nabla^*\varphi)^k = -g^{ij}(\nabla_i\varphi)^k_j. \tag{5}$$

Now we are ready to state and prove the

MAIN THEOREM. *The endomorphism field φ on (M, g) is harmonic if and only if $\nabla^*\varphi = 0$.*

Proof. In local coordinates, $(x, y) = (x^1, \dots, x^m, x^{\bar{1}}, \dots, x^{\bar{m}})$, the map $\varphi: TM \to TM$ is given by

$$\varphi(x, y) = (x^1, \dots, x^m, \varphi^1_k(x)x^{\bar{k}}, \dots, \varphi^m_k(x)x^{\bar{k}}).$$

Moreover, at (x, y) the Christoffel symbols ${}^{TM}\Gamma^\alpha_{\beta\gamma}$ of the Levi Civita connection ∇^C of g^C, where $\alpha, \beta, \gamma = 1, \dots, 2m$, are given by

$${}^{TM}\Gamma^k = \begin{pmatrix} \Gamma^k_{ij} & 0 \\ 0 & 0 \end{pmatrix}, \quad {}^{TM}\Gamma^{\bar{k}} = \begin{pmatrix} x^{\bar{l}}\frac{\partial\Gamma^k_{ij}}{\partial x^l} & \Gamma^k_{ij} \\ \Gamma^k_{ij} & 0 \end{pmatrix}$$

for $i, j, k = 1, \dots, m$ [18]. So from (1), (2), (3), (4) for the second fundamental form $\nabla(d\varphi)$ and the tension field $\tau(\varphi)$ we have

$$\begin{aligned}
\nabla(d\varphi)^\gamma_{\alpha\beta}(x, y) &= \frac{\partial^2\varphi^\gamma}{\partial x^\alpha \partial x^\beta}(x) - {}^{TM}\Gamma^\delta_{\alpha\beta}(x, y)\frac{\partial\varphi^\gamma}{\partial x^\delta}(x) \\
&\quad + {}^{TM}\Gamma^\gamma_{\lambda\mu}(\varphi(x, y))\frac{\partial\varphi^\lambda}{\partial x^\alpha}(x)\frac{\partial\varphi^\mu}{\partial x^\beta}(x), \\
\tau(\varphi)^\gamma(x, y) &= (g^C)^{\alpha\beta}(x, y)\nabla(d\varphi)^\gamma_{\alpha\beta}(x, y),
\end{aligned}$$

where at (x, y), in terms of the Christoffel symbols of ∇

$$\begin{aligned}
\nabla(d\varphi)^k_{ij} &= 0, \\
\nabla(d\varphi)^k_{\bar{i}j} &= 0, \\
\nabla(d\varphi)^k_{\bar{i}\bar{j}} &= 0, \\
\nabla(d\varphi)^{\bar{k}}_{ij} &= x^{\bar{l}}\left(\frac{\partial^2\varphi^k_l}{\partial x^i \partial x^j} - \Gamma^a_{ij}\frac{\partial\varphi^k_l}{\partial x^a} - \frac{\partial\Gamma^a_{ij}}{\partial x^l}\varphi^k_a + \frac{\partial\Gamma^k_{ij}}{\partial x^a}\varphi^a_l + \Gamma^k_{ia}\frac{\partial\varphi^a_l}{\partial x^j} + \Gamma^k_{aj}\frac{\partial\varphi^a_l}{\partial x^i}\right), \\
\nabla(d\varphi)^{\bar{k}}_{\bar{i}j} &= (\nabla_j\varphi)^k_i, \\
\nabla(d\varphi)^{\bar{k}}_{\bar{i}\bar{j}} &= 0.
\end{aligned}$$

Hence, by means of (5),

$$\tau(\varphi)^k = 0, \quad \tau(\varphi)^{\bar{k}} = 2g^{ij}(\nabla_i\varphi)_j^k = -2(\nabla^*\varphi)^k.$$

This yields the required result. □

Remark. It follows from the main result and [2, pp. 34-35] that when Φ is skew-symmetric or symmetric, respectively, then φ (or Φ) is harmonic if and only if Φ is coclosed (i.e., $\delta\Phi = 0$) or has vanishing divergence, respectively.

3. Examples and applications

In what follows we shall give several examples and applications of the notion and result considered in Section 2. In this way we provide examples of harmonic maps of some special pseudo-Riemannian manifolds.

We start by noting that in [5] the authors also considered the notion of harmonic symmetric $(0, 2)$-tensors. They first considered the notion of a harmonic Riemannian metric g' on a Riemannian manifold (M, g) and called g' a harmonic metric with respect to g if $\mathrm{id}_M\colon (M, g) \to (M, g')$ is a harmonic map. The analytic expression of this condition then led the authors to the definition of a harmonic symmetric $(0, 2)$-tensor Φ. It turns out that Φ is harmonic in the sense of [5] if and only if

$$\Phi' = \Phi - \frac{1}{2}(\operatorname{tr}\Phi)g$$

is harmonic in the sense of Section 2. Hence, the examples given in [5] yield examples of harmonic endomorphism fields.

Before getting more results, we consider the Ricci tensor ρ on a Riemannian (M, g) and note that

$$(\operatorname{div}\rho)(X) = -\sum_{i=1}^{m}(\nabla_{e_i}\rho)(e_i, X) = -\frac{1}{2}\nabla_X\tau,$$

where $(e_1, \dots, e_m)$ is an arbitrary orthonormal basis of T_pM at each $p \in M$. So, we get at once.

PROPOSITION 3.1. *The Ricci endomorphism field on a Riemannian manifold is harmonic if and only if the scalar curvature is constant. Moreover, the Einstein tensor* $G = \rho - \frac{1}{2}\tau g$ *is always harmonic.*

Note that ρ is always harmonic in the sense of [5]. Our notion of harmonicity for ρ is thus more restrictive.

Using this result we may give another characterization of harmonic manifolds (M, g) with $\dim M > 2$. Indeed, in [6] it is proved that a Riemannian (M^m, g),

$m > 2$, is a harmonic manifold if and only if every sufficiently small geodesic sphere has constant scalar curvature, i.e., the scalar curvature only depends on the radius of the sphere. Hence we have:

PROPOSITION 3.2. *Let (M^m, g), $m \geq 3$, be a Riemannian manifold. Then (M, g) is a harmonic space if and only if the Ricci endomorphism field of any sufficiently small geodesic sphere is harmonic.*

Instead of geodesic spheres one may also consider tubes of sufficiently small radius about geodesics. Then, in [10], an (M, g) is said to be scalar curvature harmonic with respect to geodesics γ if the scalar curvature for all small tubes about all γ only depends on the radius. It is proved in [10] that such an (M, g) is a real space form. So, we obtain:

PROPOSITION 3.3. *A Riemannian manifold (M^m, g), $m \geq 3$, is a real space form if and only if the Ricci endomorphism field of every small tube about all geodesics is harmonic.*

The result in Proposition 3.2 implies that all small geodesic spheres in harmonic spaces provide examples of harmonic maps by means of the Ricci tensor of type $(1, 1)$. We refer to [1] for the known examples of harmonic spaces. In what follows we will show that the same holds when we consider the shape operator of these geodesic spheres.

To prove this we turn to submanifold theory. Let $\overline{M}$ be an oriented hypersurface in a Riemannian (M, g). Let ξ denote a unit normal vector field on $\overline{M}$ and let S be the shape operator of $\overline{M}$ defined by $\nabla_X \xi = -SX$ for X tangent to $\overline{M}$. S is related to the second fundamental form σ by $g(SX, Y) = g(\sigma(X, Y), \xi)$. Then the Codazzi equation reads

$$(R_{XY}Z)^{\perp} = (\nabla_Y \sigma)(X, Z) - (\nabla_X \sigma)(Y, Z) \tag{6}$$

for X, Y, Z tangent to $\overline{M}$. (See [4] for more details.) The mean curvature h is given by $h = \operatorname{tr} S$. So, from (6) we have:

LEMMA 3.1. *Let $\overline{M}$ be an oriented hypersurface in (M, g) with unit normal vector field ξ. Then we have*

$$\rho(X, \xi) = (\operatorname{div} \sigma)(X) + Xh$$

for any X tangent to $\overline{M}$.

Using this lemma we immediately have:

PROPOSITION 3.4. *An oriented hypersurface $\overline{M}$ in an Einstein manifold (M, g) has constant mean curvature if and only if the shape operator is harmonic.*

Since harmonic spaces, which are Einstein spaces, may be defined as (M, g) all of whose sufficiently small geodesic spheres have constant mean curvature (see, for example, [6]), we have:

PROPOSITION 3.5. *An Einstein manifold (M, g) is a harmonic space if and only if the shape operator of each sufficiently small geodesic sphere is harmonic.*

Using the notion of harmonicity with respect to geodesics as developed in [12] by means of the mean curvature of tubes about geodesics, we have in a similar way as for Proposition 3.3:

PROPOSITION 3.6. *An Einstein manifold (M, g) is a real space form if and only if the shape operator of every small tube about all geodesics is harmonic.*

Remark. A $(0, 2)$-tensor field Φ is a Killing tensor field if and only if for all X we have $(\nabla_X \Phi)(X, X) = 0$. It follows that a symmetric Killing tensor field of type $(0, 2)$ is harmonic if and only if it has constant trace.

Since a manifold (M, g) whose Ricci tensor ρ is a Killing tensor has automatically constant scalar curvature, we get that such ρ is harmonic. It follows from the general theory that any Riemannian space with volume–preserving geodesic symmetries (up to sign), i.e., D'Atri spaces, has a harmonic Ricci tensor. The same result holds for the C-spaces, i.e., spaces such that the Jacobi operator field has constant eigenvalues along geodesics. For both cases we refer to [1] for more information and examples.

Now, we proceed the construction of examples by considering manifolds (M, g) which are equipped with an additional structure given by an endomorphism field. We start by looking at an almost Hermitian manifold (M, g, J) and denote by Ω its Kähler form defined by $\Omega(X, Y) = g(X, JY)$ for all tangent X, Y. Then we have

$$(\delta\Omega)(X) = -\sum_{i=1}^{m} (\nabla_{e_i}\Omega)(e_i, X) = -\sum_{i=1}^{m} g(X, (\nabla_{e_i} J)e_i).$$

Since an almost Hermitian manifold is said to be semi-Kählerian if Ω is coclosed (see for example [11]) we easily obtain:

PROPOSITION 3.7. *Let (M, g, J) be an almost Hermitian manifold. Then J is harmonic if and only if (M, g, J) is semi–Kählerian.*

Next, let P be an almost product metric structure on a Riemannian manifold (M, g), i.e., P is a $(1, 1)$-tensor field such that $P^2 = \text{id.}$ and $g(PX, PY) = g(X, Y)$ for all tangent X, Y (see for example [19]). Then the $(0, 2)$-tensor φ defined by $\varphi(X, Y) = g(X, PY)$ determines a pseudo-Riemannian metric on (M, g). Note that conversely, any pseudo-Riemannian metric on (M, g) gives rise to an almost product metric structure. Here we have:

PROPOSITION 3.8. *The pseudo-Riemannian metric tensor φ on an almost product metric manifold (M, g, P) is harmonic if and only if P is harmonic.*

Note that the eigenspaces of P determine two complementary and orthogonal distributions D, D' on M. When these distributions are integrable and determine foliations $\mathcal{D}$, $\mathcal{D}'$ with minimal leaves, then P is certainly harmonic since $\nabla^* P = -2(\alpha_{\mathcal{D}} - \alpha_{\mathcal{D}'})$, where $\alpha_{\mathcal{D}}$ and $\alpha_{\mathcal{D}'}$ are the mean curvature vectors of the leaves of $\mathcal{D}$ and $\mathcal{D}'$, respectively [16]. We refer to [3] for examples of such manifolds.

We finish this short list of examples by considering the tangent bundle TM of a Riemannian manifold (M, g). As is well known, TM may be equipped with a Riemannian metric g^S, called the Sasaki metric, which is defined by

$$g^S(X^H, Y^H) = g^S(X^V, Y^V) = g(X, Y)^V, \quad g^S(X^H, Y^V) = 0$$

for X, Y tangent to M [14], [15]. Moreover, the endomorphism field J on TM defined by

$$JX^V = -X^H, \quad JX^H = X^V$$

determines an almost Hermitian structure on TM and the endomorphism field P determined by

$$PX^V = X^H, \quad PX^H = X^V$$

defines an almost product metric structure on TM. It follows that also $Q = PJ = -JP$ is an almost product metric structure.

A rather straightforward computation, which we omit here, using the expressions for the Riemannian connections of (TM, g^S) and (TM, g^C) (see [13], [18]) then yields the following result:

PROPOSITION 3.9. *Let (M, g) be a Riemannian manifold.*

(i) *The endomorphism fields J and Q are harmonic on (TM, g^S) and the endomorphism field P is harmonic on (TM, g^S) if and only if (M, g) is Ricci-flat.*
(ii) *The endomorphism field Q is harmonic on (TM, g^C) and the endomorphism fields J and P are harmonic on (TM, g^C) if and only if (M, g) is Ricci-flat.*

REFERENCES

[1] J. BERNDT, F. TRICERRI AND L. VANHECKE, *Generalized Heisenberg groups and Damek–Ricci harmonic spaces*, Lecture Notes in Math., no. 1598, Springer-Verlag, New York, 1995.

[2] A.L. BESSE, *Einstein manifolds*, Ergeb. Math. Grenzgeb. (3), Folge 10, Springer-Verlag, New York, 1982.

[3] A. BONOME, R. CASTRO, E. GARCÍA–RÍO, L. HERVELLA AND Y. MATSUSHITA, *Almost complex manifolds with holomorphic distributions*, Rend. Mat. **14** (1994), 567–589.

[4] B. Y. CHEN, *Geometry of submanifolds*, Pure Appl. Math., no. 22, Marcel Dekker, New York, 1973.

[5] B. Y. CHEN AND T. NAGANO, *Harmonic metrics, harmonic tensors, and Gauss maps*, J. Math. Soc. Japan **36** (1984), 295–313.

[6] B. Y. CHEN AND L. VANHECKE, *Differential geometry of geodesic spheres*, J. Reine Angew. Math. **325** (1981), 28–67.

[7] J. EELLS AND L. LEMAIRE, *A report on harmonic maps*, Bull. London Math. Soc. **10** (1978), 1–68.

[8] J. EELLS AND L. LEMAIRE, *Another report on harmonic maps*, Bull. London Math. Soc. **20** (1988), 385–524.

[9] J. EELLS AND J. H. SAMPSON, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.

[10] L. GHEYSENS AND L. VANHECKE, *Total scalar curvature of tubes about curves*, Math. Nachr. **103** (1981), 177–197.

[11] A. GRAY AND L. HERVELLA, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. **123** (1980), 35–58.

[12] A. GRAY AND L. VANHECKE, *The volumes of tubes about curves in a Riemannian manifold*, Proc. London Math. Soc. **44** (1982), 215–243.

[13] O. KOWALSKI, *Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold*, J. Reine Angew. Math. **250** (1971), 124–129.

[14] S. SASAKI, *On the differential geometry of tangent bundles of Riemannian manifolds*, Tôhoku Math. J. **10** (1958), 338–354.

[15] ———, *On the differential geometry of tangent bundles of Riemannian manifolds, Part II*, Tôhoku Math. J. **14** (1962), 146–155.

[16] PH. TONDEUR, *Foliations on Riemannian manifolds*, Universitext, Springer-Verlag, New York, 1988.

[17] PH. TONDEUR AND L. VANHECKE, *Harmonicity of a foliation and of an associated map*, Bull. Austral. Math. Soc. **54** (1996), 241–246.

[18] K. YANO AND S. ISHIHARA, *Tangent and cotangent bundles*, Pure Appl. Math., no. 16, Marcel Dekker, New York, 1973.

[19] K. YANO AND M. KON, *Structures on manifolds*, Series in Pure Math., no. 3, World Scientific Publ. Co., Singapore, 1984.

E. García-Río, Departamento de Análise Matemática, Facultade de Matemáticas, Universidade de Santiago de Compostela, 15706 Santiago de Compostela, Spain
eduardo@zmat.usc.edu

M. E. Vázquez-Abal, Departamento de Xeometría e Topoloxía, Facultade de Matemáticas, Universidade de Santiago de Compostela, Santiago de Compostela, 15706 Spain
meva@zmat.usc.es

L. Vanhecke, Department of Mathematics, Katholieke Universiteit Leuven, Celestijnelaan 200 B, 3001 Leuven, Belgium
fgaga@ccl.kleuven.ac.be