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A RESULT ON CYCLES ALGEBRAICALLY EQUIVALENT TO ZERO

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0. Statement of the theorem

The inspiration for this paper comes in a theorem proven in [Sch] that implies for a geometric generic hypersurface $X_{|\mathbf{C}}$ of degree d in $\mathbf{P}^{n+1}$, with $n+2 \leq d \leq 2n-2$, there exist two lines on $X_{|\mathbf{C}}$ whose difference has infinite order in $CH_1(X_{|\mathbf{C}})_{alg}$. (This follows from [Sch, Thm 0.7.] and a connectedness result in [Bo, Thm 4.1.].) The argument involves a deformation of lines to a singular fiber, where some information is known. A different proof of this result, based on Roitman's theorem on zero cycles on varieties of non-zero genus, can be found in [P]. Alberto Collino [Co] has also indicated another proof, in a similar spirit to [P]. We would like to arrive at a general result which will have a broader scope of application. The proof will involve a combination of a deformation argument, together with some of Roitman's results on dimensions of orbits. If $H = H_{\mathbf{Q}}$ is a finite dimensional Hodge structure with Hodge decomposition $H_{\mathbf{C}} = \oplus_{p,q} H^{p,q}$, we define

$$\mathrm{Level}(H) = \begin{cases} \max\{p-q \mid H^{p,q} \neq 0\} & \text{if } H \neq 0 \\ -\infty & \text{if } H = 0. \end{cases}$$

We introduce the following:

(0.1) (i) Let $\{E_c\}_{c\in\Omega}$ be a flat family of k-dimensional (irreducible) subvarieties in some $\mathbf{P}^N$.

(ii) Let $\{X_t\}_{t\in W}$ be a flat family of subvarieties in $\mathbf{P}^N$, with generic member smooth.

(iii) $P = \{(c,t) \in \Omega \times W \mid E_c \subset X_t\}$, with projection diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \pi\ } & W \\ \big\downarrow{\scriptstyle \rho} & & \\ \Omega & & \end{array}$$

(iv) Assume W, Ω, P are smooth varieties, π, ρ are surjective with connected fibers, and that ρ is a smooth morphism. Also, we will set $\Omega_{X_t} = \rho(\pi^{-1}(t))$, and let $\delta = \dim \Omega_{X_t}$ for general $t \in W$.

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(v) Fix a closed point $t_0 \in W$, and an integer $\ell \geq 2$. Assume that there is an irreducible component $\Omega_{t_0} \subset \Omega_{X_{t_0}}$ of dimension $m \geq \ell$, with desingularization $\tilde{\Omega}_{t_0}$, such that the corresponding cylinder homomorphism $H_\ell(\tilde{\Omega}_{t_0}, \mathbf{Q}) \to W_{-2k-\ell}H_{2k+\ell}(X_{t_0}, \mathbf{Q})$ has image Hodge level ℓ. Finally, assume $\delta \geq (m - \ell) + 1$.

(0.2) THEOREM. *Given the setting in* (0.1) *above. Then, for general $t \in W$, there are* (*an uncountable number of*) *non-torsion classes in $CH_k(X_t)_{alg}$.*

1. Notation

1. All varieties are irreducible complex projective varieties.

2. $CH_k(X)$ is the Chow group of subvarieties of dimension k in X, modulo rational equivalence. $CH_k(X)_{alg}$ is the subgroup of cycles, algebraically equivalent to zero.

3. A c-closed subset of a projective variety Z, is a countable union of Zariski closed subsets of Z. The dimension of a c-closed set is taken to mean the maximum among the dimensions of its irreducible components. A c-open set is the complement of a c-closed set. A general point of a projective variety is a point in a c-open subset (defined by satisfying certain predetermined conditions).

4. For a mixed Hodge structure H with weight filtration $W.H$, the graded piece is given by $Gr_W^\ell H = W_\ell H / W_{\ell-1} H$.

2. Proof of the theorem

The proof is divided into three steps, the first of which is a deformation argument, the second involves Roitman's results, and the third is a specialization argument (which follows from Fulton's work).

Step 1. Choose a general point $t_1 \in W$, and also choose any points $c_0 \in \Omega_{X_{t_0}}$, $c_1 \in \Omega_{X_{t_1}}$. Since Ω is smooth and $\dim \Omega \geq 1$, one can always find a smooth and irreducible curve $C \subset \Omega$ such that $c_0, c_1 \in C$. Then $\rho^{-1}(C)$ is likewise smooth and irreducible, and one can find a smooth and irreducible curve $D \subset \rho^{-1}(C)$ such that $(c_0, t_0), (c_1, t_1) \in D$. There is a pullback diagram

$$\begin{array}{ccc} P_D & \xrightarrow{\pi'} & P \\ \rho_D \downarrow & & \downarrow \pi \\ D & \xrightarrow{\pi|_D} & W, \end{array}$$

where P_D is the unique irreducible component of $D \times_W P$ mapping onto D. Note that for each $(c, t) \in D$, $\dim \rho_D^{-1}((c, t)) = \delta$, with choice of subvariety $E_c \subset X_{\pi((c,t))=t}$.

We will write $(\tilde{c}, t) \in \rho_D^{-1}((c, t))$, to really mean $((c, t), (\tilde{c}, t)) \in \rho_D^{-1}(c, t)) \subset D \times_W P$. Then we note that since $(c, t) \in \rho_D^{-1}((c, t))$ for general (c, t), and since D is the closure of its general points, it follows that $(c, t) \in \rho_D^{-1}((c, t))$ for all $(c, t) \in D$. Now choose any points $e_0, e_1 \in \Omega$ subject to the condition that $(e_0, t_0) \in \rho_D^{-1}((c_0, t_0))$, $(e_1, t_1) \in \rho_D^{-1}((c_1, t_1))$. Now let $E \subset P_D$ be an irreducible curve passing through (e_0, t_0) and (e_1, t_1), which we can assume to be smooth, by passing to a desingularization. One can further pullback P_D to E, with pullback diagram

$$\begin{array}{ccc} P_E & \longrightarrow & P \\ {\scriptstyle\rho_E}\downarrow & & \downarrow{\scriptstyle\pi} \\ E & \longrightarrow & W, \end{array}$$

Moreover, for each point $(e, t) \in E$, there are corresponding subvarieties E_e, $E_{\rho\circ\rho_D(e,t)=c} \subset X_t$. Now let $\mathcal{X} \overset{def}{=} \coprod_{t\in W} X_t \to W$ be the flat morphism describing our family $\{X_t\}_{t\in W}$, and pullback this family to $X_E = E \times_W \mathcal{X} \xrightarrow{\lambda} E$, (and write $X_\xi = \lambda^{-1}(\xi)$). By construction, we have the following:

(2.0) (i) For any point $\xi \in E$, we have k-dimensional subvarieties $E_{e(\xi)}$, $E_{c(\xi)} \subset X_\xi$.

(ii) There are points $\xi_0, \xi_1 \in E$ such that (with respect to reduced scheme structure) $X_{\xi_0} = X_{t_0}$, $X_{\xi_1} = X_{t_1}$, $E_{e(\xi_0)} = E_{e_0} \subset X_{t_0}$, $E_{c(\xi_0)} = E_{c_0} \subset X_{t_0}$, $E_{e(\xi_1)} = E_{e_1} \subset X_{t_1}$, $E_{c(\xi_1)} = E_{c_1} \subset X_{t_1}$.

It follows easily from (0.1) (iv) that $E_{e(\xi)} \sim_{alg} E_{c(\xi)}$ in X_ξ for all $\xi \in E$. The reader can easily verify from our construction of c_0, e_0 above, that in addition to (i) and (ii) above, the following can be arranged (we refer to (0.1)):

(iii) Let $S \subset \tilde{\Omega}_{t_0}$ be a general subvariety of dimension ℓ cut out by $m - \ell$ general hyperplane sections of $\tilde{\Omega}_{t_0}$. Then c_0 can be chosen to correspond to a general point $\tilde{c}_0$ of S, and e_0 can be chosen to correspond to a general point of a certain subvariety in S of dimension ≥ 1 passing through $\tilde{c}_0$. This follows from the fact that the fibers of ρ_E are of dimension δ, and that $\delta \geq (m - \ell) + 1$ ((0.1) (v)).

Step 2. Let S be given as in (iii) above, and consider the cycle class map $\kappa\colon S \to CH_k(X_{t_0})$. One could argue, using Chow variety arguments, that the fibers of κ are c-closed subsets of S. We would like to argue that for general $p \in S$, p is a component of the fiber $\kappa^{-1}(\kappa(p))$. This will essentially be a by-product of Roitman's work. By (0.1)(v) and the weak Lefschetz theorem, the image of the cylinder homomorphism $H_\ell(S, \mathbf{Q}) \to W_{-2k-\ell}H_{2k+\ell}(X_{t_0}, \mathbf{Q})$ has Hodge level ℓ. Therefore there is a non-zero holomorphic ℓ-form w in the image of the dual map $Gr_W^{\ell+2k}H^{\ell+2k}(X_{t_0}, \mathbf{C})^{\ell+k,k} \to$

$H^{\ell,0}(S,\mathbf{C})$, viz. $\eta \mapsto w$. Let $\mathcal{S}^N(S)$ be the N^{th}-symmetric product of S. The form $\sum_{i=1}^N Pr_i^*(w)$ induces a corresponding "form" w_N on $\mathcal{S}^N(S)$. More specifically, w_N will be regular outside of the singular set of $\mathcal{S}^{(N)}(S)$. There is a corresponding cycle class map $\kappa_N\colon \mathcal{S}^N(S) \to CH_k(X_{t_0})$. Since $\ell \geq 2$ ((0.1)(v)), we can use the following result found in [R, Section 3]:

(2.1) PROPOSITION. *Let $\Sigma \subset \mathcal{S}^N(S)$ be an irreducible subvariety passing through a general point of $\mathcal{S}^N(S)$, and suppose that $w_{N|\Sigma_{ns}} \equiv 0$, where Σ_{ns} is the non-singular part of Σ. Then* $\dim \Sigma \leq N\ell - N$.

Now let $\xi \in \mathcal{S}^N(S)$ and define $V_\xi^N = \{p \in S \mid \exists \mu \in \mathcal{S}^{N-1}(S), \kappa_N(\xi) = \kappa_N(\mu + p)\}$. We need the following:

(2.2) LEMMA. *Suppose that for all $p \in S$, there exists a subvariety $\Sigma_p \subset S$ of dimension ≥ 1 through p, such that $\kappa(\Sigma_p) = \kappa(p)$. Then* $\dim V_\xi^N \geq 1$ *for all ξ and N.*

Proof. Let $\xi \in \mathcal{S}^N(S)$ be given and choose $q \in |\xi|$. Then $\xi = q + \xi'$ for some $\xi' \in \mathcal{S}^{N-1}(S)$. Then it follows that $\Sigma_q \subset V_\xi^N$. [Note: Similarly, if $\dim V_\xi^1 \geq 1$ for all $\xi \in \mathcal{S}^1(S) = S$, then $\dim V_\xi^N \geq 1$ for all ξ and N.]

We now prove the following:

(2.3) PROPOSITION. Dim $V_\xi^N = 0$ *for some ξ and N.*

Proof (sketch only). The proof given here is a slight variation of one appearing in [R]. We assume to the contrary that $\dim V_\xi^N \geq 1$ for all ξ and N. Let $Y \subset S$ be a general hyperplane section. Then $Y \cap V_\xi^N \neq \emptyset$ for all ξ. We define the c-closed set

$$W_Y = \{(p,\xi) \in S \times \mathcal{S}^N(S) \mid \kappa(p) = \kappa_N(\xi) \text{ modulo the image } CH_0(Y) \to CH_k(X_{t_0})\}.$$

Then it is easy to see that the projection $Pr_2\colon W_Y \to \mathcal{S}^N(S)$ is onto. [Proof: We have $\kappa_{N-1}(\mu) = \kappa_N(\xi)$ modulo $CH_0(Y) \to CH_k(X_{t_0})$ for some $\mu \in \mathcal{S}^{N-1}(S)$. Now proceed by downward induction on N.] Thus $\dim W_Y \geq N\ell$. The fibers of the projection $Pr_1\colon W_Y \to S$ have therefore dimension $\geq N\ell - \ell > N\ell - N$ for $N > \ell$. Now define the c-closed set

$$W_{Y,\xi} = \{\nu \in \mathcal{S}^N(S) \mid \kappa_N(\nu) = \kappa_N(\xi) \text{ modulo } CH_0(Y) \to CH_k(X_{t_0})\}.$$

Note that $\kappa_N(\xi) = \kappa(p)$ modulo $CH_0(Y) \to CH_k(X_{t_0})$ for some $p \in S$. Thus $\dim W_{Y,\xi} > N\ell - N$ for $N > \ell$, and we can assume $W_{Y,\xi}$ is irreducible (with $\xi \in W_{Y,\xi}$) by restricting to irreducible components. Note that for dimension reasons alone $w_{|Y} \equiv 0$; moreover if $\tilde{\kappa}_\xi^r\colon \mathcal{S}^r(Y) \times \mathcal{S}^r(Y) \to CH_k(X)$ is the map given by $\tilde{\kappa}_\xi^r(A,B) =$

$\kappa_N(\xi) + \kappa_r(A) - \kappa_r(B)$, then one easily checks that the (well defined) pullback $\tilde{\kappa}_\xi^{r,*}(\eta) = \underline{Pr}_1^*(w_r) - \underline{Pr}_2^*(w_r) = 0$ on $\mathcal{S}^r(Y) \times \mathcal{S}^r(Y)$, where $w_r = \sum_i Pr_i^*(w)$ is the obvious "form" and $\underline{Pr}_j$: $\mathcal{S}^r(Y) \times \mathcal{S}^r(Y) \to \mathcal{S}^r(Y)$ is the j^{th}-projection. Next, we observe that $\kappa_N(W_{Y,\xi}) \subset \cup_r Im(\tilde{\kappa}_\xi^r)$. It follows from some standard c-closed arguments that $\kappa_N(W_{Y,\xi}) \subset \tilde{\kappa}_\xi^r(\mathcal{S}^r(Y) \times \mathcal{S}^r(Y))$ for some r, and so there exist an irreducible component V of the c-closed set $\{(a,b) \in W_{Y,\xi} \times \mathcal{S}^r(Y) \times \mathcal{S}^r(Y) \mid \kappa_N(a) = \tilde{\kappa}_\xi^r(b)\}$ such that the projection Pr_1 is dominant in the commutative diagram below:

$$\begin{array}{ccc} V & \xrightarrow{Pr_2} & \mathcal{S}^r(Y) \times \mathcal{S}^r(Y) \\ {\scriptstyle Pr_1}\downarrow & & \downarrow{\scriptstyle \tilde{\kappa}_\xi^r} \\ W_{Y,\xi} & \xrightarrow{\kappa_N} & CH_k(X_{t_0}). \end{array}$$

Then as a generalization of [Sa, Prop. 2.5] or [R, Section 3] (see [Le-1, Section 3]), there are well defined pullbacks which agree, viz. $(\kappa_N \circ Pr_1)^*(\eta) = (\tilde{\kappa}_\xi^r \circ Pr_2)^*(\eta)$. But $(\tilde{\kappa}_\xi^r \circ Pr_2)^*(\eta) = Pr_2^* \circ \tilde{\kappa}_\xi^{r,*}(\eta) = 0$, hence $0 = (\kappa_N \circ Pr_1)^*(\eta) = Pr_1^* \circ \kappa_N^*(\eta)$. Since Pr_1 is dominating, we deduce that $w_N|_{(W_{Y,\xi})_{ns}} = \kappa_N^*(\eta)|_{(W_{Y,\xi})_{ns}} = 0$. This contradicts (2.1) for general choice of ξ, since $\dim W_{Y,\xi} > N\ell - N$ for $N > \ell$.

Finally, there are the following results, the first of which is a simple generalization of [R, p. 591], and the second a consequence of rigidity ([Lec], also see [Sch, Lemma 4.2]), or a generalization of [R, Sect. 2]:

(1) For any integer $N \geq 1$, codim V_ξ^N, as a function of $\xi \in \mathcal{S}^N(S)$, attains its maximum on a c-open subset of $\mathcal{S}^N(S)$.
(2) The torsion subgroup $CH_k(X_{t_0})_{tor}$ is countable.

As a consequence of the above, there is the following:

(2.4) COROLLARY. *Let $c_0 \in S$ be a general point, $\Sigma \subset S$ a subvariety of dimension ≥ 1 through c_0, and e_0 a general point in Σ. Then $\kappa(c_0 - e_0) \in CH_k(X_{t_0})_{alg}$ is a non-torsion class.*

Step 3. We now refer to the setting of (2.0)(i), (ii) and (iii). By step 2, we can now assume that $\{E_{c(\xi_0)} - E_{e(\xi_0)}\} \in CH_k(X_{t_0})_{alg}$ and that $\{E_{c(\xi_0)} - E_{e(\xi_0)}\} \in CH_k(X_{t_0})_{alg}$ is a non-torsion class. Let η be the generic point of E. Then $E_{c(\xi_0)}$, $E_{e(\xi_0)}$ are specializations of the cycles $E_{c(\eta)}, E_{e(\eta)} \in CH_k((X_K)$, where X_K is the generic fiber (over η). Let R be the local ring of E at t_0 [Note that $K = \mathrm{Quot}(R)$]. Since E is smooth, R is a discrete valuation ring. According to [F, Section 4.4.], there is a commutative diagram and specialization map below:

$$\begin{array}{ccccc}
\{\overline{E_{c(\eta)} - E_{e(\eta)}}\} & \longmapsto & & & \{E_{c(\eta)} - E_{e(\eta)}\} \\
 & & CH_{k+1}(X_R) & \longrightarrow & CH_k(X_K) \\
\downarrow & & \downarrow & \swarrow h & \\
\{E_{c(\xi_0)} - E_{e(\xi_0)}\} \in & & CH_k(X_{t_0}) & &
\end{array}$$

We conclude therefore that $M(E_{c(\eta)} - E_{e(\eta)})$ is not rationally equivalent to zero for any integer $M \geq 1$, and therefore $E_{c(\xi)} - E_{e(\xi)}$ is a non-torsion class in $CH_k(X_\xi)$ for sufficiently general $\xi \in E$. This implies that $CH_k(X_t)_{alg}$ has non-torsion classes for general $t \in W$. In fact, there is the following result:

(2.5) PROPOSITION. *Let Z be a smooth projective variety, and k an integer. If $CH_k(Z)_{alg} \neq 0$, then $CH_k(Z)_{alg}$ is uncountable.* [*Hence by the countability of $CH_k(Z)_{tor}$, there must exist non-torsion classes.*]

Proof. See [Sch, Thm 0.8.]. Alternatively, if $CH_k(Z)_{alg} \neq 0$, then there exist an abelian variety A and a non-trivial cycle induced homomorphism $A \to CH_k(Z)_{alg}$ with c-closed fibers. The connected component of zero in the kernel will be an abelian variety B; thus factoring out by B, the corresponding induced map $A/B \to CH_k(Z)_{alg}$ has countable fibers, hence uncountable image.

3. Applications of the theorem

Using [Bo] and [Le-2], there is the following result:

(3.0) THEOREM (See [Bo, Cor. 2.2.] and [Le-2, Cor. 3.8.]). *Assume given a smooth general hypersurface $Z \subset \mathbf{P}^{n+1}$ of degree $d_0 \geq 3$. Let $k = \left[\frac{n+1}{d_0}\right]$, and $\Omega_Z \stackrel{\text{def}}{=} \{\mathbf{P}^k$'s $\subset Z\}$. Then Ω_Z is smooth and of dimension m, where $m = (k+1)(n+1-k) - \binom{d_0+k}{k}$ (and provided $m \geq 0$); moreover if $m \geq \ell$, where $\ell = n - 2k$, then the cylinder homomorphism $H_{n-2k}(\Omega_Z, \mathbf{Q}) \to H_n(Z, \mathbf{Q})$ is surjective.* [*Also, if in addition $\ell \geq 2$, then $CH_k(Z)_{alg}$ is infinite dimensional.*]

Now let $X \subset \mathbf{P}^{n+1}$ be a general hypersurface of degree d, and k an integer ≥ 0. If we set Ω_X to be the variety of k-planes on X, then according to [Bo], $\dim \Omega_X = \delta$, where $\delta = (k+1)(n+1-k) - \binom{d+k}{k}$. The role of δ, and of m, ℓ in (3.0), will be the same as for δ, m, ℓ in (0.1) above. As for choices of W, Ω, X_{t_0} in (0.1), we will set W to be the projective space of hypersurfaces of degree d in $\mathbf{P}^{n+1}$, Ω to be the Grassmannian of $\mathbf{P}^k$'s in $\mathbf{P}^{n+1}$, and $X_{t_0} = Z \cup M$, where Z is given in (3.0),

M is a smooth hypersurface of degree $d - d_0$ (where we assume $d \geq d_0 \geq 3$), and where Z meets M transversally in a smooth variety K. Since $d_0 \geq 3$ it follows that Level$(H_n(Z, \mathbf{Q})) = \ell$. Now let $\tilde{\Omega}_{t_0} = \Omega_Z$, and assume $\ell \geq 2$. Then the image of the cylinder homomorphism $H_\ell(\tilde{\Omega}_{t_0}, \mathbf{Q}) \to W_{-2k-\ell}H_{2k+\ell=n}(X_{t_0}, \mathbf{Q})$ has level ℓ. [This is easily seen from the Mayer-Vietoris description $W_{-n}H_n(X_{t_0}) \simeq \{H_n(Z, \mathbf{Q}) \oplus H_n(M, \mathbf{Q})\}/H_n(K, \mathbf{Q})$, where $\dim H_n(K, \mathbf{Q}) \leq 1$ by the Lefschetz theorem.] Now in order to satisy the conditions of Theorem (0.2), viz. (0.1), we are going to require $d_0 \leq d, k = \left[\frac{n+1}{d_0}\right], \ell \geq 2, m \geq \ell$, and $\delta \geq (m - \ell) + 1$. In this case, Ω_{X_t} is connected for all $t \in W$ (see [Bo, Thm 4.1.]), and $\rho\colon P \to \Omega$ is a projective bundle. A reformulation of these conditions appears in (3.1) below. In particular, we deduce:

(3.1) COROLLARY. *Let $X \subset \mathbf{P}^{n+1}$ be a general hypersurface of degree $d \geq 3$. Assume given positive integers d_0, ℓ, k satisfying*

(i) $k = \left[\frac{n+1}{d_0}\right]$,
(ii) $n - 2k \geq 2$,
(iii) $k(n + 2 - k) + 1 - \binom{d_0+k}{k} \geq 0$,
(iv) $0 \leq \binom{d+k}{k} - \binom{d_0+k}{k} \leq n - 2k - 1$.

Then $CH_k(X)_{alg}$ is uncountable. In particular $CH_k(X)_{alg}$ contains non-torsion classes.

Example. If we choose $d_0 = n + 1$, so that $k = 1$, then Schoen's result as stated in Section 0 follows.

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