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POINTED SIMPLICIAL COMPLEXES

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1. Introduction

For what follows, R is $k[x_1, \ldots, x_n]$, k is a field of characteristic p, I is a monomial ideal of R and $M = R/I$. The ranks of the free modules that appear in a minimal free resolution of M might depend on p. It is well known that the zeroth and nth betti number of M are independent of p. Bruns and Herzog, [BrHe95] show that if $n \leq 5$ all betti numbers of M are independent of p. In the same paper they show that for $i = 1, 2, n-1$, the ith betti number of M is always independent of p. Terai and Hibi [TeHia] show that the third and fourth betti numbers are independent of p when I is generated by monomials of degree 2 and also prove that the betti numbers are independent of k in some other cases as well [TeHia], [TeHib]. The most familiar classes of monomial ideals whose betti numbers are independent of p include (a) monomial ideals which are generated by R-sequences, (b) stable monomial ideals [ElKe90], and (c) squarefree stable ideals [ArHeHi95], [ChEv93].

A significant link between commutative algebra and topology comes from the Stanley-Reisner rings. First, to any monomial ideal J one can correspond a squarefree monomial ideal I. If Δ is the corresponding simplicial complex of I and $\tilde{C}_*$ is the augmented chain complex, Hochster's formula may be used to compute the betti numbers of I (and of J) from the k dimensions of the homology groups of Δ and its subcomplexes Δ/T.

In this paper we show that for certain simplicial complexes we can find a vertex y, such that $0 \longrightarrow \tilde{H}_i(\Delta/\{y\}) \longrightarrow \tilde{H}_i(\Delta) \longrightarrow \tilde{H}_{i-1}(\text{link } y) \longrightarrow 0$ is short exact for all i. We call complexes with this property *pointed* complexes. Examples of pointed complexes include the complexes and subcomplexes that correspond to the ideals of (a), (b) and (c). It is clear that if the reduced homology groups of both ends of the short exact sequence are free Z-modules then $\tilde{H}_i(\Delta)$ is a free Z-module. In practice, whenever the *associated* ideals of Δ/y and link y are of the same kind as the ideal of Δ, one can use induction on the total degree of the minimal generators of the ideals to conclude that the reduced homology groups are free Z-modules. This explains from a topological point of view why the ideals with pointed complexes and subcomplexes have betti numbers that do not depend on p.

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2. Notations and definitions

A simplicial complex Δ with vertex set $V_\Delta = \{x_1, \ldots, x_n\}$ is a family of subsets of V_Δ such that the next two conditions hold: (a) $\{x_i\}$ is an element of Δ, $\forall i$ and (b) if σ is in Δ and τ is a subset of σ, then τ is in Δ. If $\{x_{i1}, \ldots, x_{i(r+1)}\}$ is an element of Δ we call it an r-face of Δ. To any subset $F = \{x_{i_1}, \ldots, x_{i_s}\}$ of V_Δ we correspond a monomial $m_F = x_{i_1} \cdots x_{i_s}$ of $R = k[x_1, \ldots, x_n]$ and vice versa. To Δ we associate a squarefree ideal I of R whose generators are the monomials m_F such that $F \notin \Delta$. We call I the *associated* ideal of Δ. Conversely whenever I is a squarefree monomial ideal of R whose generators are of degree strictly bigger than 1, the associated simplicial complex Δ_I has vertex set $V = \{x_1, \ldots, x_n\}$ and faces the subsets F of V for which $m_F \notin I$. If I is a squarefree monomial ideal minimally generated by $x_{i1}, \ldots, x_{it}, m_1, \ldots, m_s$ and the degree of the monomials m_i is bigger than 1, then we set $\Delta_I := \Delta_{I'}$, the simplicial complex whose associated ideal is $I' = (m_1, \ldots, m_s)$. The vertex set of Δ_I is $\{x_1, \ldots, x_n\} \setminus \{x_{i1}, \ldots, x_{it}\}$.

We give the elements of V a linear order and construct the simplicial chain complex, $\tilde{C}_*(\Delta)$,with coefficients in Z:

$$0 \longrightarrow C_s(\Delta) \longrightarrow \cdots \longrightarrow C_0(\Delta) \longrightarrow C_{-1}(\Delta) \longrightarrow 0.$$

Here $C_r(\Delta)$ is the free Z module on the ordered r-faces of Δ $[x_{i1}, \ldots, x_{i(r+1)}]$, $(i1 < i2 < \cdots < i(r+1))$, and the differentiation is the map θ that sends $[x_{i1}, \ldots, x_{i(r+1)}]$ to $\sum_{j=1}^{j=r+1} (-1)^{j+1} [x_{i1}, \ldots, \widehat{x_{ij}}, \ldots, x_{i(r+1)}]$. For more details on simplicial complexes consult [St83] or [BrHe93]. We set $[x_{\sigma(i1)}, \ldots, x_{\sigma(ik)}] = \text{signum}(\sigma)[x_{i1}, \ldots, x_{ik}]$.

$\tilde{C}_*(1)$ stands for the complex $\tilde{C}$ shifted by 1 to the left: $\tilde{C}_i(1) = \tilde{C}_{i-1}$. By $\tilde{H}_i(\Delta)$ we mean the homology of $C_*(\Delta)$ at the ith place and by $\tilde{H}_i(\Delta, k)$ we mean the homology of $\tilde{C}_*(\Delta) \otimes k$. If $\tilde{H}_i(\Delta)$ is a free Z-module then $\tilde{H}_i(\Delta, k)$ has k-dimension equal to the rank of $\tilde{H}_i(\Delta)$ as a Z-module. If Δ contains the maximum face $\{x_1, \ldots, x_n\}$ then $\tilde{H}_i(\Delta) = 0$ for all i.

We briefly recall the connection between the betti numbers of I and the ranks of the homology groups of the associated simplicial complex. If I is a squarefree monomial ideal whose generators all have degree greater than 1 and Δ is the corresponding simplicial complex, then we can compute the betti numbers of I from the following formula due to Hochster [Ho77]: $b_q^R(R/I) = \sum \dim_k \tilde{H}_i(\Delta/T, k)$ where T varies among all subsets of V with $|T|+q = (n-1)-i$. Here Δ/T stands for the subcomplex of Δ consisting of all faces with vertices outside T. If I is a monomial squarefree ideal minimally generated by $x_{i1}, \ldots, x_{it}$ and the monomials $m_1, \ldots, m_s$ whose degree is bigger than 1, then we can compute the betti numbers of $I' = (m_1, \ldots, m_s)$ by the above formula and the betti numbers of I by shifting and adding successively (t-times) the betti numbers of I'.

Let y be a vertex of V_Δ. By $\text{link}_\Delta\ y$ we mean the faces G such that $G \cup y \in \Delta, y \cap G = \emptyset$. Let $F = [x_{i1}, \ldots, x_{it}]$ be an oriented face of Δ. We will consider the diminution $[\hat{y}, F]$ of F: we define $[\hat{y}, F] = (-1)^{s-1}[x_{i1}, \ldots, \hat{x}_{is}, \ldots, x_{it}]$

if $y = x_{is}$ for some s, otherwise we let $[\hat{y}, F] = 0$. We will also consider the augmentation $[y, F]$ of F by y: $[y, F] = 0$ if $y = x_{is}$ for some s, otherwise $[y, F] = [y, x_{i1}, \dots, x_{it}]$.

Let $T = \{x_{i1}, \dots, x_{ik}\}$. The associated ideal of Δ_I/T is the ideal $I \cap R'$ of $R' = k[x_{j1}, \dots, x_{js}]$, $(jt \neq il)$. It has the same generators as I except we omit these generators which are divisible by the variables in T. Finally we remark that $\Delta_I/T = \Delta_{(I+(x_{i1},\dots,x_{ik}))}$.

3. Pointed simplicial complexes

Let y be a vertex of Δ_I. We are going to consider the relations among the homology groups of Δ_I, $\Delta_{(I,y)}$ and $\Delta_{(I:y)}$.

Remark. (i) Let I be a squarefree ideal, Δ_I the corresponding simplicial complex and y a vertex of Δ_I. Then $\operatorname{link}_{\Delta_I} y = \Delta_{(I:y)}$.

(ii) Let I be a squarefree ideal, Δ_I the corresponding simplicial complex, y a vertex of Δ_I, $T = \{y\}$. Then $\Delta_I/T = \Delta_{(I,y)}$.

The proof of these statements is straightforward once we notice that the corresponding simplicial complexes have the same vertex set. Next we define the maps $e : C_*(\Delta_{(I,y)}) \longrightarrow C_*(\Delta_I)$ and $p_y C_*(\Delta_I) \longrightarrow C_*(\Delta_{(I:y)})(1)$ by $e(z) = z$ and $p_y(\sum a_i F_i) = \sum a_i[\hat{y}, F_i]$. Note that p_y is not a homomorphism of complexes; see Lemma 1.

We have the following commutative diagram with exact columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_i(\Delta_{(I,y)}) & \xrightarrow{e} & B_i(\Delta_I) & \xrightarrow{p_y} & B_{i-1}(\Delta_{(I:y)}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_i(\Delta_{(I,y)}) & \xrightarrow{e} & Z_i(\Delta_I) & \xrightarrow{p_y} & Z_{i-1}(\Delta_{(I:y)}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{H}_i(\Delta_{(I,y)}) & \longrightarrow & \tilde{H}_i(\Delta_I) & \longrightarrow & \tilde{H}_{i-1}(\Delta_{(I:y)}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The first two rows are not exact in general. Below we record some of the properties of the maps e and p_y.

LEMMA 1. *Let $F \in \Delta_I$ and y a vertex of Δ_I. Then $p_y(\theta(F)) = -\theta(p_y(F))$.*

Proof. If y is not among the vertices of F then both sides are zero. Otherwise $F = [y, x_{i1}, \ldots, x_{is}]$. We let $F' = [x_{i1}, \ldots, x_{is}]$ and suppose that $\theta(F') = \sum a_i F_i$, $a_i = + - 1$. Then $p_y(\theta(F)) = p_y((F' - \sum a_i[y, F_i])) = -\sum a_i F_i = -\theta(F') = -\theta(p_y(F))$. □

LEMMA 2. $0 \longrightarrow B_i(\Delta_{(I,y)}) \xrightarrow{e} B_i(\Delta_I) \xrightarrow{p_y} B_{i-1}(\Delta_{(I:y)}) \longrightarrow 0$ *is a complex, e is injective and p_y is surjective.*

Proof. It is clear that the first map is injective and that the image of e is in the kernel of p_y. To show that p_y is surjective it is enough to show that whenever F is an i face of $\Delta_{(I:y)}$ then $\theta(F)$ has a preimage. If $F \in \Delta_{(I:y)}$ then $[y, F]$ is an $i+1$ face of Δ_I so that $p_y([y, F]) = F$ and $p_y(\theta(-[y, F])) = \theta(F)$ by the previous lemma. □

LEMMA 3. $0 \longrightarrow Z_i(\Delta_{(I,y)}) \xrightarrow{e} Z_i(\Delta_I) \xrightarrow{p_y} Z_{i-1}(\Delta_{(I:y)})$ *is exact.*

Proof. Lemma 1 shows that the image of p_y consists of cycles. e is clearly injective and its image is contained in the kernel of p_y. Moreover let $c = \sum a_t F_t$ be an element in the kernel of p_y. Suppose that y is a vertex of $F_1, \ldots, F_s$ and $F_j = [y, F_j']$ for $j = 1, \ldots, s$. Since $\sum a_t p_y(F_t) = 0$, $\sum a_j F_j' = 0$ $(j = 1, \ldots, s)$ and $c' = a_1 F_1 + \cdots + a_s F_s = 0$. Thus $c = e(c - c')$. □

Whenever the kernel of p_y is equal to the image of e (on the first row) and p_y is surjective (on the second row) an easy diagram chase shows that the third row of our commutative diagram is exact, see also the 3×3 Lemma [Ro79]. In this case if $\tilde{H}_i(\Delta_{(I,y)})$ and $\tilde{H}_{i-1}(\Delta_{(I:y)})$ are free Z-modules, then $\tilde{H}_i(\Delta_I)$, is also a free Z-module. The following condition guarantees that the first two rows are exact.

Definition. The simplicial complex Δ is *i-pointed* with respect to y if there exists a vertex $z \neq y$ with the property that whenever F is an $(i-1)$-face of Δ and F is in the link of y then $z \cup F$ is a face of Δ.

For example the 1-skeleton of a triangle or a square are 1-pointed with respect to any vertex. The triangulation of the projective plane is not 2-pointed for any vertex.

LEMMA 4. *If Δ is $(i+1)$-pointed with respect to y then the top row of the diagram is exact.*

Proof. Let $c = \sum a_t \theta(F_t) \in B_i(\Delta_I)$ be in the kernel of p_y. Without loss of generality we can assume that y is a vertex of $F_1, \ldots, F_s$ and that $F_t = [y, F_t']$ for

$t = 1, \ldots, s$. Let z be the vertex of the definition. Then $G_t = [z, F_t']$ is a face of Δ_I. We claim that $c = e(\sum a_t \theta(G_t))$ where $G_t = [z, F_t']$ for $t = 1, \ldots, s$ and $G_t = F_t$ for all other t. Indeed if $\theta(F_t') = \sum F_{tj}'$ then $\theta(F_t) = F_t' - \sum [y, F_{tj}']$ and $\theta(G_t) = F_t' - \sum [z, F_{tj}']$. Since $p_y(c) = 0$, $\sum \sum a_t F_{tj}' = 0$, $\sum \sum a_t [y, F_{tj}'] = \sum \sum a_t [z, F_{tj}'] = 0$ and $\sum a_t \theta(G_t) = \sum a_t F_t'$ where in this sum t varies from $1, \ldots, s$. Finally $e(\sum a_t \theta(G_t)) = \sum a_t \theta(G_t) = \sum a_t \theta(F_t)$. □

LEMMA 5. *If Δ is i-pointed with respect to y then the second row of the diagram is exact.*

Proof. We show that $p_y(Z_i(\Delta_I)) = Z_{i-1}(\Delta_{(I:y)})$. The proof is similar to the previous one. Let $c' = \sum a_t F_t$ be a cycle in $Z_{i-1}(\Delta_{(I:y)})$ so that F_t are $(i-1)$ faces in $\Delta_{(I:y)}$ and $y \cup F_t$ is in Δ_I for all t. Then $z \cup F_t$ is in Δ_I. The element $c = \sum a_t [y, F_t] - \sum a_t [z, F_t]$ is a cycle in $Z_i(\Delta_I)$, and $p_y(c) = c'$. □

COROLLARY 6. *If Δ is i and $(i+1)$-pointed with respect to y then*

$$0 \longrightarrow \tilde{H}_i(\Delta_{(I,y)}) \longrightarrow \tilde{H}_i(\Delta_I) \longrightarrow \tilde{H}_{i-1}(\Delta_{(I:y)}) \longrightarrow 0$$

is short exact.

4. Examples

4.1. *Monomial complete intersections.* Let I be an a monomial squarefree ideal whose generators form an R-sequence. In this case I is a monomial complete intersection.

Example. If I is a squarefree monomial ideal generated by an R-sequence, Δ is the corresponding simplicial complex, T is any subset of the vertex set of Δ then Δ/T is i-pointed with respect to any of its vertices.

Proof. Let T be a subset of the vertex set of Δ and consider the subcomplex $\Delta' = \Delta/T$. Let I' be the associated ideal of Δ'. I' is generated by a subset of the generators of I and is a monomial complete intersection.

Let $y = x_l$ be a variable that divides a generator of I'. We can take z to be any other vertex of $\Delta_{I'}$ that divides the same generator of I as x_l. Let F be in the link of y. If zm_F is in I then a monomial generator r of I divides zm_F. Since m_F does not involve y, y does not divide r and r must divide m_F, a contradiction. □

If x_l is a vertex of Δ_I then $(I : x_l)$ and (I, x_l) are ideals which are generated by R-sequences. An easy induction on the total degree of the generators implies that $\tilde{H}_i(\Delta_I/T)$ is a free Z-module.

4.2. *Shifted complexes.* Let Δ be a simplicial complex. We write the subsets of V_Δ in an ascending order of indices: $F = \{x_{i_1}, \ldots, x_{i_l}\}$ where $i_1 < i_2 < \cdots < i_l$. We give the i-faces of Δ a partial order: $F = \{x_{l_1}, \ldots, x_{l_{i+1}}\} \leq G = \{x_{j_1}, \ldots, x_{j_{i+1}}\}$ iff $l_1 \leq j_1, l_2 \leq j_2, \ldots, l_{i+1} \leq j_{i+1}$.

Definition. Δ is a *shifted* simplicial complex if whenever $F \in \Delta$ and $G < F$ then $G \in \Delta$.

Shifted complexes were considered by Kalai [Ka93]. The corresponding ideals are squarefree strongly stable ideals and their minimal resolution is given in [ArHeHi95] (see also cite ChEv93).

We remark the following:

(i) If Δ is a shifted simplicial complex and T is any subset of the vertex set of Δ then Δ/T is a shifted simplicial complex,
(ii) If Δ is a shifted simplicial complex with vertex set $\{x_1, \ldots, x_n\}$ then the link of x_n is also a shifted complex.

Theorem 7. Let Δ be a shifted simplicial complex, T any subset of the vertex set of Δ. Then Δ/T is i-pointed for all i.

Proof. Let I be the squarefree ideal associated to Δ. Since Δ/T is also a shifted complex it is enough to prove the claim for Δ. We choose y to be the vertex of highest index in the vertex set of Δ and z to be the vertex of Δ of immediate lower index. Let F be a cycle in the link of y. Since $F \cup \{y\} \in \Delta$ and z has index greater than or equal to any of the indices that appear in F it follows that either $F \cup \{z\} = F$ or $F \cup \{z\} < F \cup \{y\}$. In both cases $F \cup \{z\}$ is in Δ. □

As in the previous examples, it follows by induction that $\tilde{H}_i(\Delta/T)$ is a free Z-module.

4.3. *Polarizations of ideals.* We recall a technique that associates to every monomial ideal a squarefree ideal.

Let J be an ideal of $S = k[x_1, \ldots, x_n]$ minimally generated by monomials $m_i = \prod x_j^{a_{ji}}$. For each variable x_j we let b_j be the largest exponent such that x_j/m_i for some i. We will replace each occurence of x_j by new variables x_{jt} in a systematic way. For this we consider the ring $R = k[x_{11}, \ldots, x_{1b_1}, x_{21}, \ldots, x_{nb_n}]$. For each m_i we consider its polarization $pm_i = \prod_j \prod_{t=1}^{a_{ji}} x_{jt}$. By the *polarization* of J we mean the ideal I of R generated by the monomials pm_i. For example the polarization of the ideal J of $k[x_1, x_2, x_3]$ where $J = (x_1^3, x_1^2 x_2, x_1 x_2^3, x_3)$ is the ideal $I = (x_{11}x_{12}x_{13}, x_{11}x_{12}x_{21}, x_{11}x_{21}x_{22}x_{23}, x_{31})$ and the underlying ring is $k[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}]$.

The two ideals J and its polarization I are intimately related; see for example [Fr82]. The betti numbers of I and J are the same: one can get the minimal resolution of J from the minimal resolution of I by substituting the value x_j for each appearance of the variable x_{jt}. Thus the dimensions of the homology groups of Δ and Δ / T where Δ is the simplicial complex that corresponds to I determine the betti numbers of J. With the notation as above we have:

THEOREM 8. *Let J be a monomial ideal such that x_i^2 divides m_j. If I is the polarization of J then Δ_I is i-pointed with respect to x_{i1}.*

Proof. Let z be the vertex x_{i2}. Suppose that $x_{i1} \cup F \in \Delta_I$. If $x_{i2}m_F \in I$ it has to be divisible by one of the generators of I. The polarization technique guarantees that none of the generators of I can be divisible by x_{12} unless it is also divisible by x_{11}. It follows that m_F is divisible by a generator of I so that $F \in \Delta_I$ which is a contradiction. □

COROLLARY 9. *Let $\{m_j\}$ be a collection of monomials such that $x_{f(j)}^2 | m_j$ and let J be the ideal generated by the m_j. If I is the polarization of J and T is any subset of the vertex set of Δ then Δ_I / T is i-pointed with respect to some vertex x_{j1}.*

Proof. The generators of the associated ideal of Δ_I / T form a subset of the generating set of Δ_I, so one can apply the previous theorem. □

For example it is well known that the betti numbers of $I = (abe, abf, acf, acd, ade, bce, bcd, bdf, def, cef)$ depend on the characteristic of k, (I corresponds to the triangulation of the projective plane). Consider now the ideals $I_1 = (a^2be, a^2bf, a^2cf, a^2cd, a^2de, bce, bcd, bdf, def, cef)$ and $I_2 = (a^2be, a^2bf, a^2cf, a^2cd, a^2de, b^2ce, b^2cd, b^2df, d^2ef, c^2ef)$. With the notation as before, the polarization of I_1 is i-pointed for all i with respect to a_{11} and the associated ideal of the link of a_{11} is I. It follows that the betti numbers of I_1 also depend on the characteristic. On the other hand one can see that the betti numbers of I_2 are independent of characteristic by using Theorem 8 recursively until the associated ideals involve less than 6 variables.

Our final application examines the ideals which are the polarizations of stable ideals. First we recall the definitions of stable ideals.

Definition. Let J be a monomial ideal of $k[x_1, \dots, x_n]$. J is a stable ideal if for all monomials $m \in J$, then $\frac{m}{x_k}x_i \in J$ for all $i \leq k$ where x_k is the variable of largest index that divides m.

In [ElKe90], Eliahou and Kervaire described the minimal resolution of stable ideals. Theorem 10 explains from a topological point of view why the betti numbers of these ideals do not depend on p. First we remark that if $J = (m_1, \dots, m_s)$ is a stable ideal of R and J contains the variable x_t then J must contain all variables of index less than t and $J = (x_1, \dots, x_t, m_{t+1}, \dots, m_s)$ where m_j is a monomial in $k[x_{t+1}, \dots, x_n]$. If $J' = (m_{t+1}, \dots, m_s)$, I is the polarization of J, I' the polarization of J' then J' is stable in $k[x_{t+1}, \dots, x_n]$ and $\Delta_{I'} = \Delta_I$.

THEOREM 10. *If I is an ideal which is the polarization of a stable ideal J, Δ_I is the corresponding simplicial complex, T is any subset of the vertex set of Δ and $\tilde{H}_i(\Delta_I/T) \neq 0$, then Δ_I/T is i-pointed with respect to some vertex x_{r1}.*

Proof. By the previous remark we can assume that I is the associated ideal of Δ_I. We can also assume that the sum of the degrees of the generators of I is strictly bigger than 2.

We first treat the case $T = \emptyset$. Since J is stable, x_1 must appear to a power of at least 2 in J and x_{12} is also a vertex of Δ_I. By Theorem 8, Δ_I is i-pointed with respect to x_{i1}.

Let T now be a nonempty subset of the vertex set of Δ_I and consider the subcomplex $\Delta' = \Delta/T$. We can assume that Δ' consists of more than one vertex. Let L be the associated ideal. If the vertex set $V_{\Delta'}$ of Δ' contains some vertex x_{fs} where $s \geq 2$ but not x_{fl} for some $l < s$ then Δ' is a cone with respect to x_{fs}. Indeed if F is in Δ' then $x_{fs}m_F$ cannot be in L, since none of the generators of L is divisible by x_{fs} but not by x_{fl}. Thus the homology of Δ' is zero for all i. So we can assume that if $x_{fs} \in V_{\Delta'}$ then $x_{fl} \in V_{\Delta'}$, $\forall l < s$. Let r be the smallest index such that x_{r1} is in $V_{\Delta'}$. We claim that Δ' with respect to $y = x_{r1}$ is i-pointed. Indeed if $x_{rt} \in V_{\Delta'}$ for $t > 1$ then we let $z = x_{rt}$ and the proof is the same as in Theorem 8. Suppose that $V_{\Delta'}$ does not contain x_{rt} for any $t > 1$. Let z be the vertex x_{ls} with the property that if x_{hk} is any other vertex of $V_{\Delta'}$ then either $l > h$ or $l = h$ and $s > k$. Let F be a face of Δ' which is in the link of x_{r1}. If $z \cup F \notin \Delta'$ then $x_{ls}m_F \in I$ and $x_{ls}m_F = mb$ for some generator m of I. Since m_F is not in I it follows that m is divisible by x_{ls}. Suppose that m is the polarization of the generator μ. Since J is stable, $\frac{\mu}{x_l}x_r$ is in J and the polarization of that element $\frac{m}{x_{ls}}x_{r1}$ is in I, (notice that x_r does not divide μ). Therefore $\frac{m}{x_{ls}}x_{r1} = x_{r1}m_F \in I$, a contradiction since F is in the link of x_{r1}. □

Remarks. An easy induction on the total degree of the generators of the associated ideals implies that $\tilde{H}_i(\Delta_I/T)$ is a free Z-module. Note that if I is a squarefree monomial ideal which is the polarization of J, then the polarization of $(J : x_i)$ is $\phi((I : x_{i1}))$ where ϕ is the isomorphism $k[x_{11} \dots, \widehat{x_{i1}}, \dots, x_{nb_n}] \longrightarrow k[x_{11} \dots, \widehat{x_{ib_i}}, \dots, x_{nb_n}]$ such that $\phi(x_{jl}) = x_{jl}$ if $j \neq i$ and $\phi(x_{il}) = x_{i(l-1)}$, $(l = 2, \dots, b_i)$. One can also show using the same techniques that a basis for the cycle space of $\tilde{C}_i(\Delta_I)$ consists of elements of the form $\theta(G)$ where G is an $i+1$-simplex, (not necessarily a face of Δ_I).

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