

CONFORMAL INVARIANTS OF SMOOTH DOMAINS AND EXTREMAL QUASICONFORMAL MAPPINGS OF ELLIPSES

SHANSHUANG YANG

1. Introduction

Let Ω be a domain in the complex plane \mathbb{C} . For the study of a class of domains (called QED domains) introduced by Gehring and Martio in connection with quasiconformal mappings, the following so-called *quasiextremal distance constant* (or QED constant) $M(\Omega)$ was introduced in [Y1]:

$$(1.1) \quad M(\Omega) = \sup_{A, B} \frac{\text{mod}(A, B; \mathbb{C})}{\text{mod}(A, B; \Omega)},$$

where the supremum is taken over all pairs of disjoint nondegenerate continua A and B in $\bar{\Omega}$, and $\text{mod}(A, B; \Omega)$ denotes the *modulus* of the family $\Gamma(A, B; \Omega)$ of curves that join A and B in Ω . The modulus $\text{mod}(A, B; \Omega)$ is also referred to as the *conformal module* of the quadrilateral with Ω as its domain and A, B as its one pair of opposite sides (for definitions, see [LV, Chapter 1]).

A domain Ω is a *QED domain* if its QED constant $M(\Omega)$ is finite. QED domains were introduced by Gehring and Martio [GM] as a useful class of domains in the study of quasiconformal mappings. In this paper we will concentrate on Jordan domains whose QED constants are finite. The QED constant is called a conformal invariant because it is invariant under Möbius transformations (or conformal mappings of the extended plane $\bar{\mathbb{C}}$). It is determined by the geometry of a domain and measures how far a domain is from being a disk. For example, it was shown in [Y1] that for a Jordan domain Ω , $M(\Omega) = 2$ if and only if it is a disk or half plane. Another closely related conformal invariant which is also determined by the geometry of a domain is called the *quasiconformal reflection constant* and defined as

$$(1.2) \quad R(\Omega) = \inf_f K(f),$$

where the infimum is taken over all homeomorphic reflections f in the boundary $\partial\Omega$ and where $K(f)$ denotes the maximal dilatation of f . A *homeomorphic reflection* in a Jordan curve is a homeomorphism of $\bar{\mathbb{C}}$ that interchanges the two components of

Received July 23, 1996

1991 Mathematics Subject Classification. Primary 30C62.

This research was supported in part by the University Research Committee of Emory University and by a grant from the National Science Foundation.

the complement of the curve taken with respect to the extended plane. For definitions of the maximal dilatation $K(f)$ and basic properties of quasiconformal mappings in the plane, we refer the reader to [LV]. It is well known that $R(\Omega)$ is finite if and only if $\partial\Omega$ is a *quasicircle* (or, equivalently, $\partial\Omega$ admits a quasiconformal reflection). The two invariants $R(\Omega)$ and $M(\Omega)$ are closely related to one another. For example it follows from [Y1, Theorem 5.1] that

$$(1.3) \quad M(\Omega) \leq R(\Omega) + 1$$

and it was conjectured in [GY] that the equality in (1.3) holds for all Jordan domains. We refer the reader to [Y1] and [Y2] for more properties and estimates of the QED constants and quasiconformal reflection constants of domains in the plane and in space. The purpose of this paper is to show that the above conjecture is not true by using ellipses.

On the basis of special examples and observations, a conjecture stronger than the one mentioned above was offered in [Y3]. Let h be an orientation preserving homeomorphism of the unit circle. The *maximal dilatation* of h is defined by

$$K_h = \sup \frac{\text{mod}(h(A), h(B); D)}{\text{mod}(A, B; D)},$$

where D is the unit disk and the supremum is taken over all pairs of disjoint nondegenerate continua A and B on the unit circle. We say that h is *quasiconformal* if its maximal dilatation is finite. One should notice that in literature a one-dimensional quasiconformal homeomorphism is often referred to as being *quasisymmetric*. It is clear that if h is the boundary function of a quasiconformal self-mapping f of D , then h is quasiconformal. In this case f is said to be a *quasiconformal extension* of h . By the well-known Beurling-Ahlfors extension theorem [BA] and the Riemann mapping theorem, the converse is also true; that is, if $h: \partial D \rightarrow \partial D$ is quasiconformal then it has a quasiconformal extension. For such an h we define

$$K_h^* = \inf\{K(f): f \text{ is a quasiconformal extension of } h\},$$

where $K(f)$ is the maximal dilatation of f . A quasiconformal extension f^* of h is said to be *extremal* if $K(f^*) = K_h^*$. The existence of f^* follows from the compactness of the family of all quasiconformal extensions of h . But in general such a quasiconformal extension is not unique (see [S1]). It was conjectured in [Y3] that for any homeomorphism h ,

$$(1.4) \quad K_h = K_h^*.$$

It is easy to see that this conjecture implies the conjecture that $M(\Omega) = R(\Omega) + 1$. In fact, a weaker version of (1.4) would also imply the same conjecture. We say that a homeomorphism h of the unit circle is *induced* by Ω if $h = f_1 \circ f_2$, where f_1 and f_2 are the boundary maps induced by conformal maps of D onto Ω and D^* onto Ω^*

(the exterior of Ω), respectively. One can show that if (1.4) holds for h induced by Ω , then (1.3) holds with equality.

Recently Anderson and Hinkkanen [AH] showed that conjecture (1.4) is not true by using affine maps and parallelograms. In this note we show that the weaker conjecture that $M(\Omega) = R(\Omega) + 1$ is not true for ellipses. Hence conjecture (1.4) is not true even for homeomorphisms induced by ellipses.

In Section 2 we find some extremal quasiconformal mappings associated with ellipses that will help us to compute the reflection constant of ellipses. Section 3 is devoted to the estimate of the boundary QED constant of ellipses. This section is largely influenced by the paper of Anderson and Hinkkanen [AH] and I would like to thank them for making the preprint of their paper available to me at an early stage. In Section 4 we study the relations among the above mentioned invariants of conformal extension domains. As a corollary we show that the ellipse does provide a counterexample to the Garnett-Yang conjecture [GY].

2. Extremal quasiconformal mappings for the ellipse

In this section we construct extremal quasiconformal reflections about ellipses and extremal quasiconformal mappings of $\bar{\mathbb{C}}$ that maps the unit circle onto ellipses. First we fix some notation which will be used in the paper. For each $a \geq 1$ let E_a denote the interior domain of the ellipse defined by the equation

$$\frac{u^2}{a^2} + v^2 = 1$$

in the w -plane, where $w = u + iv$. The unit disk will be denoted by D . For any Jordan domain Ω , the exterior domain $\bar{\mathbb{C}} \setminus \bar{\Omega}$ is denoted by Ω^* .

2.1 THEOREM. *The boundary of E_a admits a unique extremal quasiconformal reflection and $R(E_a) = a$.*

Proof. Define a map $w = f(z)$ by

$$(2.2) \quad f(z) = \begin{cases} F_a(z), & z \in \bar{D} \\ \frac{1}{2}[(a+1)z + (a-1)\frac{1}{z}], & z \in D^* \end{cases}$$

where F_a is the affine map determined by $F_a(x + iy) = ax + iy$. One can verify that $w = f(z)$ is a homeomorphism of the extended plane $\bar{\mathbb{C}}$ which maps D quasiconformally onto E_a and maps D^* conformally onto E_a^* . Thus the map

$$(2.3) \quad f_r(w) = f \circ J \circ f^{-1}(w)$$

is a quasiconformal reflection about ∂E_a with maximal dilatation $K(f_r) = a$, where J is the conformal reflection about the unit circle defined as $J(z) = 1/\bar{z}$.

We claim that $f_r(w)$ is the unique extremal quasiconformal reflection about ∂E_a . Otherwise, the conformal map of D^* onto E_a^* defined in (2.2) has an extremal quasiconformal extension onto D other than the affine map. This contradicts the well known fact that the affine map F_a is the unique extremal map of D onto E_a with the given boundary values (see, for example, [S2]). This contradiction proves our claim. Therefore $f_r(w)$ is the desired map and the quasiconformal reflection constant is given by $R(E_a) = K(f_r) = a$. \square

It is a well-known fact that any Jordan curve Γ that admits quasiconformal reflections is a *quasicircle*, the image of the unit circle under a quasiconformal map of $\bar{\mathbb{C}}$. By [LV, Theorem 5.1, p. 73], the collection of all K -quasiconformal maps of $\bar{\mathbb{C}}$ such that $f^{-1}(\Gamma)$ is the unit circle form a normal family. Thus, by the convergence theory of quasiconformal maps (see, for example, [LV, Theorem 5.3, p. 74]), the minimum of the maximal dilatations of all such quasiconformal maps exists and is called the *quasicircle constant* of the curve Γ . In particular, ellipses are quasicircles. Their quasicircle constants and their extremal quasiconformal maps are given in the following result.

2.4. THEOREM. *There is an extremal quasiconformal homeomorphism of $\bar{\mathbb{C}}$ which maps the ellipse ∂E_a onto the unit circle, whose maximal dilatation is \sqrt{a} .*

Proof. Let g be a homeomorphism of $\bar{E}_{\sqrt{a}}$ onto \bar{D} that is conformal on $E_{\sqrt{a}}$. Define a map $g_a(w)$ by

$$(2.5) \quad g_a(w) = \begin{cases} g \circ F_{\sqrt{a}}^{-1}(w), & w \in \bar{E}_a \\ J \circ g \circ F_{\sqrt{a}}^{-1} \circ f_r(w), & w \in E_a^* \end{cases}$$

where f_r is the reflection defined in (2.3) and J , as before, is the reflection about the unit circle. It is easy to see that $z = g_a(w)$ is a homeomorphism of $\bar{\mathbb{C}}$ which maps ∂E_a onto the unit circle and that g_a is K -quasiconformal in E_a with $K = \sqrt{a}$. Simple calculation reveals that

$$F_{\sqrt{a}}^{-1} \circ f_r(w) = F_{\sqrt{a}}^{-1} \circ f \circ J \circ f^{-1}(w) = F_{\sqrt{a}} \circ J \circ f^{-1}(w)$$

when $w \in E_a^*$, where f is the map defined in (2.2). Since f^{-1} is conformal in E_a^* , it follows that in E_a^* the maximal dilatation of g_a is also \sqrt{a} . Therefore, in light of Theorem 2.1, g_a is a desired extremal quasiconformal map with maximal dilatation $K(g_a) = \sqrt{a}$. \square

2.6. REMARKS. Since the affine map F_a maps the unit circle onto the ellipse ∂E_a and affine maps are usually extremal maps with the given boundary values, one might guess that the quasicircle constant of ∂E_a is a . But, according to Theorem 2.4, it is obviously not the case. It is the specific structure (not just the existence) of the extremal map constructed in (2.5) that will be needed in the next section.

3. Estimates for QED constants of E_a

In this section we work with a version of QED constant slightly different from the one defined in (1.1) and prove one of our main results.

The *boundary* QED constant of a Jordan domain Ω , denoted by $M_b(\Omega)$, is obtained by taking the supremum in (1.1) over all pairs of disjoint nondegenerate continua A and B on the boundary $\partial\Omega$. The success of finding the exact values for the QED constants of some special domains is mainly based on the relation that $M_b(\Omega) \leq M(\Omega) \leq R(\Omega) + 1$ (see [Y1], [GY]). In this section we show, with the help of the extremal quasiconformal maps found in the previous section, that the simple relation that $M_b(\Omega) = R(\Omega) + 1$ suggested by those examples is not true for ellipses.

3.1. THEOREM. *For any $a > 1$ we have*

$$(3.2) \quad M_b(E_a) < a + 1.$$

Proof. According to the definition,

$$(3.3) \quad M_b(E_a) = \sup_{A, B} \frac{\text{mod}(A, B; \mathbb{C})}{\text{mod}(A, B; E_a)},$$

where the supremum is taken over all pairs of disjoint nondegenerate continua A and B on the ellipse ∂E_a . We shall consider two cases, namely, the case when the supremum in (3.3) is attained for some nondegenerate continua and the case when the supremum is not attained.

The attained supremum case. In this case there are disjoint nondegenerate continua $A, B \subset \partial E_a$ such that

$$M_b(E_a) = \frac{\text{mod}(A, B; \mathbb{C})}{\text{mod}(A, B; E_a)}.$$

Let h be a conformal map of E_a onto D and let

$$f_1(w) = \begin{cases} h(w), & w \in \bar{E}_a \\ J \circ h \circ f_r(w), & w \in E_a^* \end{cases}$$

where f_r is the reflection defined in (2.3). Then f_1 is a homeomorphism of $\bar{\mathbb{C}}$ that maps E_a conformally onto D and maps E_a^* quasiconformally onto D^* . Let $A' = f_1(A)$, $B' = f_1(B)$. Since the modulus $\text{mod}(A', B'; \mathbb{C})$ is equal to the conformal capacity of the ring domain $\bar{\mathbb{C}} \setminus (A' \cup B')$ (see, for example, [Ge] or [Ah, Chapter 4]), it follows that

$$\text{mod}(A', B'; \mathbb{C}) = \int_{\mathbb{C}} |\nabla u|^2 dm = 2 \int_D |\nabla u|^2 dm,$$

where u is a real-valued continuous function on $\bar{\mathbb{C}}$ which is harmonic in $\mathbb{C} \setminus (A' \cup B')$ with constant value 0 on A' and constant value 1 on B' . Thus

$$(3.4) \quad \text{mod}(A, B; \mathbb{C}) \leq \int_{\mathbb{C}} |\nabla(u \circ f_1)|^2 dm.$$

Suppose (3.2) does not hold. Then

$$\text{mod}(A, B; \mathbb{C}) = (a + 1) \text{mod}(A, B; E_a).$$

Since f_1 is conformal in E_a and K -quasiconformal in E_a^* with $K = a$, it follows from (3.4) that

$$\begin{aligned} (a + 1) \text{mod}(A, B; E_a) &\leq \int_{E_a} |\nabla(u \circ f_1)|^2 dm + \int_{E_a^*} |\nabla(u \circ f_1)|^2 dm \\ &= \int_D |\nabla u|^2 dm + \int_{E_a^*} |\nabla(u \circ f_1)|^2 dm \\ &\leq \text{mod}(A, B; E_a) + a \int_{D^*} |\nabla u|^2 dm. \end{aligned}$$

This yields

$$\text{mod}(A, B; E_a) \leq \int_D |\nabla u|^2 dm = \text{mod}(A, B; D) = \text{mod}(A, B; E_a).$$

Therefore, all the equalities in the above inequalities hold. In particular the equality in (3.4) holds. Hence, by the uniqueness of the minimizer for Dirichlet integrals, we conclude that $u \circ f_1$ is harmonic in E_a^* .

By the definition of f_r , we see that for $w \in E_a^*$,

$$f_1(w) = J \circ h \circ F_a \circ J \circ f^{-1}(w),$$

where f is the conformal map of D onto E_a defined in (2.2). It is easy to verify that if u is harmonic and if g is conformal, then both $u \circ g$ and $u \circ J$ are harmonic. Therefore, the conclusion that $u \circ f_1$ is harmonic in E_a^* implies that $U \circ F_a$ is harmonic in D , where $U = u \circ J \circ h$ is harmonic itself in E_a . Thus, by comparing the Laplacians of U and $U \circ F_a$, one can show that U must be a linear function, which contradicts the fact that U takes constant values 0 and 1 on two nondegenerate continua on ∂E_a respectively. This proves that (3.2) holds in the attained supremum case.

The degenerate case. In this case, for each $n \geq 1$, there are four distinct points $z_{1,n}, z_{2,n}, z_{3,n}, z_{4,n}$ oriented counterclockwise on ∂E_a such that

$$M_b(E_a) = \lim_{n \rightarrow \infty} \frac{\text{mod}(A_n, B_n; \mathbb{C})}{\text{mod}(A_n, B_n; E_a)},$$

where A_n and B_n are disjoint subarcs on the ellipse ∂E_a joining $z_{1,n}$ to $z_{2,n}$ and joining $z_{3,n}$ to $z_{4,n}$, respectively. We may assume, by passing to subsequences if necessary,

that the limit points z_j ($j = 1, 2, 3, 4$) of $\{z_{j,n}\}$ exist as $n \rightarrow \infty$. Since we are dealing with the degenerate case, at least two of the limit points coincide.

Let g_a be the quasiconformal map of \mathbb{C} defined in (2.5). According to Theorem 2.4, g_a maps ∂E_a onto the unit circle and $K(g_a) = \sqrt{a}$. By the construction of g_a , it has the form

$$g_a(w) = g \circ F_{\sqrt{a}}^{-1}(w)$$

for $w \in \bar{E}_a$, where g is a conformal map of $E_{\sqrt{a}}$ onto D and $F_{\sqrt{a}}$ is an affine map. Let

$$A'_n = F_{\sqrt{a}}^{-1}(A_n), \quad B'_n = F_{\sqrt{a}}^{-1}(B_n), \quad A''_n = g(A'_n), \quad B''_n = g(B'_n).$$

Then, by the conformal invariance of modulus, it follows that

$$\begin{aligned} \text{mod}(A_n, B_n; \mathbb{C}) &\leq \sqrt{a} \cdot \text{mod}(A''_n, B''_n; \mathbb{C}) \\ &= 2\sqrt{a} \cdot \text{mod}(A''_n, B''_n; D) = 2\sqrt{a} \cdot \text{mod}(A'_n, B'_n; E_{\sqrt{a}}). \end{aligned}$$

Therefore,

$$\frac{\text{mod}(A_n, B_n; \mathbb{C})}{\text{mod}(A_n, B_n; E_a)} \leq 2\sqrt{a} \frac{\text{mod}(A'_n, B'_n; E_{\sqrt{a}})}{\text{mod}(A_n, B_n; E_a)}.$$

By the following lemma, the right hand side of the last inequality approaches $2\sqrt{a}$ as $n \rightarrow \infty$. This yields

$$M_b(E_a) \leq 2\sqrt{a} < a + 1$$

as desired. \square

It remains to prove the lemma which is used in the proof of Theorem 3.1. This lemma is also interesting in its own right.

3.5. LEMMA. *Let F be an affine map which maps the interior E of an ellipse onto the interior E' of another ellipse. For each $n \geq 1$ let $z_{1,n}, z_{2,n}, z_{3,n}, z_{4,n}$ be four distinct points oriented counterclockwise on the boundary of E . Suppose that the limit points z_j ($j = 1, 2, 3, 4$) of $\{z_{j,n}\}$ exist as $n \rightarrow \infty$ and that at least two of the four limit points coincide. Then*

$$(3.6) \quad \lim_{n \rightarrow \infty} \frac{\text{mod}(A_n, B_n; E)}{\text{mod}(F(A_n), F(B_n); E')} = 1,$$

where A_n and B_n are disjoint subarcs on the ellipse ∂E joining $z_{1,n}$ to $z_{2,n}$ and joining $z_{3,n}$ to $z_{4,n}$, respectively.

Proof. Without loss of generality we may assume that F is given by $F(x + iy) = ax + iy$ with $a \geq 1$ and that E is the unit disk D . Then $E' = F(E) = E_a$, the ellipse defined in Section 1. To prove (3.6) we first observe that

$$(3.7) \quad \text{mod}(A_n, B_n; D) = \frac{\pi}{\log \Psi([z_{1,n}, z_{2,n}, z_{3,n}, z_{4,n}])},$$

where $[z_{1,n}, z_{2,n}, z_{3,n}, z_{4,n}]$ is the *cross ratio* determined by

$$[z_{1,n}, z_{2,n}, z_{3,n}, z_{4,n}] = \frac{|z_{3,n} - z_{2,n}| \cdot |z_{4,n} - z_{1,n}|}{|z_{2,n} - z_{1,n}| \cdot |z_{4,n} - z_{3,n}|},$$

and $\Psi(t)$ is the so-called Teichmüller function determined by the modulus of the Teichmüller ring on the plane which has the important property that

$$(3.8) \quad \lim_{t \rightarrow \infty} \frac{\Psi(t)}{t} = 16.$$

For more details about the Teichmüller function and Teichmüller ring, we refer the reader to [LV, Chapter 2].

Next we fix a conformal map $\zeta = \phi(w)$ of E_a onto the upper half plane such that

$$\zeta_j = \phi(w_j) \neq \infty, \quad j = 1, 2, 3, 4,$$

where

$$w_j = \lim_{n \rightarrow \infty} w_{n,j}, \quad w_{n,j} = F(z_{n,j}).$$

By the Schwarz reflection principle for analytic functions, $\phi(w)$ can be extended analytically to a neighborhood of w_1 . Thus, near w_1 , we have

$$(3.9) \quad \zeta = \phi(w) = \phi(w_1) + \phi'(w_1)(w - w_1) + O((w - w_1)^2).$$

In order to establish (3.6) we need to consider the following four cases.

Case 1. $z_1 = z_2$ and z_2, z_3, z_4 are distinct.

Case 2. $z_1 = z_2 \neq z_3 = z_4$.

Case 3. $z_1 = z_2 = z_3 \neq z_4$.

Case 4. $z_1 = z_2 = z_3 = z_4$.

For Case 1 it follows from (3.9) that

$$(3.10) \quad [\zeta_{1,n}, \zeta_{2,n}, \zeta_{3,n}, \zeta_{4,n}] = \frac{|\phi(w_{3,n}) - \phi(w_{2,n})| \cdot |\phi(w_{4,n}) - \phi(w_{1,n})|}{|\phi(w_{2,n}) - \phi(w_{1,n})| \cdot |\phi(w_{4,n}) - \phi(w_{3,n})|} \\ \sim \frac{|\zeta_3 - \zeta_2| \cdot |\zeta_4 - \zeta_1|}{|\phi'(w_1)(w_{2,n} - w_{1,n})| \cdot |\zeta_4 - \zeta_3|} \\ \sim \frac{C}{|w_{2,n} - w_{1,n}|}$$

as $n \rightarrow \infty$, where C is a nonzero constant independent of n . Here and in what follows the notation $A \sim B$ means that $A/B \rightarrow 1$ as $n \rightarrow \infty$. Therefore, by the conformal invariance of modulus, it follows from (3.7), (3.8) and (3.10) that

$$(3.11) \quad \frac{\text{mod}(A_n, B_n; D)}{\text{mod}(A'_n, B'_n; E_a)} = \frac{\pi / \log \Psi([z_{1,n}, z_{2,n}, z_{3,n}, z_{4,n}])}{\pi / \log \Psi([\zeta_{1,n}, \zeta_{2,n}, \zeta_{3,n}, \zeta_{4,n}])} \\ \sim \frac{\log \Psi(\frac{C_1}{|w_{2,n} - w_{1,n}|})}{\log \Psi(\frac{C_2}{|z_{2,n} - z_{1,n}|})} \\ \sim \frac{\log |w_{2,n} - w_{1,n}|}{\log |z_{2,n} - z_{1,n}|}$$

as $n \rightarrow \infty$, where C_1 and C_2 are nonzero constants independent of n . Since

$$\frac{1}{a} \leq \frac{|w_{2,n} - w_{1,n}|}{|z_{2,n} - z_{1,n}|} \leq a$$

and $|z_{2,n} - z_{1,n}| \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} \frac{\log |w_{2,n} - w_{1,n}|}{\log |z_{2,n} - z_{1,n}|} = 1.$$

This together with (3.11) yields (3.6).

For Case 2, reasoning as in Case 1 gives

$$\frac{\text{mod}(A_n, B_n; D)}{\text{mod}(A'_n, B'_n; E_a)} \sim \frac{\log(|w_{2,n} - w_{1,n}| \cdot |w_{4,n} - w_{3,n}|)}{\log(|z_{2,n} - z_{1,n}| \cdot |z_{4,n} - z_{3,n}|)} \rightarrow 1$$

as $n \rightarrow \infty$. This proves (3.6).

For the triple degeneracy case, namely $z_1 = z_2 = z_3 \neq z_4$, we need to consider two subcases separately.

Subcase I. $[z_{1,n}, z_{2,n}, z_{3,n}, z_{4,n}] \rightarrow \infty$ or 0. We first note that, by considering the complementary components of $A_n \cup B_n$ on the unit circle if necessary, we may assume that $[z_{1,n}, z_{2,n}, z_{3,n}, z_{4,n}] \rightarrow \infty$ as $n \rightarrow \infty$. Then as in Case 1 we have

$$\frac{\text{mod}(A_n, B_n; D)}{\text{mod}(A'_n, B'_n; E_a)} \sim \frac{\log([w_{1,n}, w_{2,n}, w_{3,n}, w_{4,n}])}{\log([z_{1,n}, z_{2,n}, z_{3,n}, z_{4,n}])} \rightarrow 1$$

as $n \rightarrow \infty$ as desired.

Subcase II. $[z_{1,n}, z_{2,n}, z_{3,n}, z_{4,n}] \rightarrow \lambda \neq 0, \infty$. In this case we have

$$\lim_{n \rightarrow \infty} [z_{1,n}, z_{2,n}, z_{3,n}, z_{4,n}] = \lim_{n \rightarrow \infty} \frac{|z_{3,n} - z_{2,n}|}{|z_{2,n} - z_{1,n}|} = \lambda.$$

Since $z_{j,n}$ ($j = 1, 2, 3, 4$) are on the unit circle and $w_{j,n} = F_a(z_{j,n})$, elementary calculations yield

$$\lim_{n \rightarrow \infty} [w_{1,n}, w_{2,n}, w_{3,n}, w_{4,n}] = \lim_{n \rightarrow \infty} \frac{|w_{3,n} - w_{2,n}|}{|w_{2,n} - w_{1,n}|} = \lambda.$$

Therefore it follows from (3.7) and (3.9) that

$$\begin{aligned} \frac{\text{mod}(A_n, B_n; D)}{\text{mod}(A'_n, B'_n; E_a)} &= \frac{\log \Psi([\zeta_{1,n}, \zeta_{2,n}, \zeta_{3,n}, \zeta_{4,n}])}{\log \Psi([z_{1,n}, z_{2,n}, z_{3,n}, z_{4,n}])} \\ &\sim \frac{\log([w_{1,n}, w_{2,n}, w_{3,n}, w_{4,n}])}{\log([z_{1,n}, z_{2,n}, z_{3,n}, z_{4,n}])} \rightarrow \frac{\log \Psi(\lambda)}{\log \Psi(\lambda)} \end{aligned}$$

as $n \rightarrow \infty$.

Finally, the quadruple degeneracy case ($z_1 = z_2 = z_3 = z_4$) can be treated as in Case 3 by considering two subcases. The details are left to the reader. This completes the proof of Lemma 3.5. \square

3.12. REMARK. Let h be any homeomorphism of the unit circle induced by conformal maps of D and D^* onto E_a and E_a^* , respectively. It follows from Theorem 2.1 that the extremal quasiconformal extension of h has maximal dilatation $K_h^* = a$. Since for any disjoint nondegenerate continua A and B on ∂E_a we have

$$\text{mod}(A, B; \mathbb{C}) \geq \text{mod}(A, B; E_a) + \text{mod}(A, B; E_a^*),$$

the relation $K_h^* = K_h$ would imply that $M_b(E_a) = a + 1$ which contradicts Theorem 3.1. Therefore, conjecture (1.4) does not hold even for homeomorphisms induced by Jordan domains bounded by ellipses. This phenomenon indicates that the maximal dilatation of an extremal quasiconformal map can not be uniquely determined by the dilatation of the corresponding boundary homeomorphism.

4. QED constants of conformal extension domains

Because it is not known whether the QED constant $M(\Omega)$ defined in (1.1) and the boundary QED constant $M_b(\Omega)$ are the same for ellipses, Theorem 3.1 does not immediately give a counter example to the Garnett-Yang conjecture [GY] that $M(\Omega) = R(\Omega) + 1$, where $R(\Omega)$ is the QC reflection constant defined in (1.2). In order to apply Theorem 3.1 to obtain such a counter example, we establish a general result that will shed new light on the relations among these conformal invariants for smooth domains. We say that a Jordan domain Ω is a *conformal extension domain* if any conformal map between Ω and the unit disk D has a conformal extension to a neighborhood of Ω . By [Po, Proposition 3.1], a domain is a conformal extension domain if and only if its boundary is an analytic curve. Another main result of this paper is the following.

4.1. THEOREM. *Let Ω be a conformal extension domain. Then $M(\Omega) = M_b(\Omega)$ or $M(\Omega) < R(\Omega) + 1$. In the first case, the supremum in (1.1) is attained by a pair of disjoint nondegenerate continua on $\partial\Omega$.*

Proof. Observe that Ω is a quasidisk. Hence all the constants involved here are finite. For each $n \geq 1$, fix disjoint nondegenerate continua A_n and B_n on $\bar{\Omega}$ such that

$$(4.2) \quad M(\Omega) = \lim_{n \rightarrow \infty} \frac{\text{mod}(A_n, B_n; \mathbb{C})}{\text{mod}(A_n, B_n; \Omega)}$$

and that A_n and B_n converge in the Hausdorff metric to continua A and B , respectively.

Nondegenerate case. First we consider the nondegenerate case that A and B are disjoint nondegenerate continua. In this case, by the continuity of moduli, we have

$$(4.3) \quad M(\Omega) = \frac{\text{mod}(A, B; \mathbb{C})}{\text{mod}(A, B; \Omega)}.$$

We will show that either A, B are on the boundary of Ω (and hence $M_b(\Omega) = M(\Omega)$) or $M(\Omega) < R(\Omega) + 1$.

Choose a homeomorphism f of \mathbb{C} such that $f: \Omega \rightarrow D$ is conformal and that $f: \Omega^* \rightarrow D^*$ is K -quasiconformal with $K = R(\Omega)$. Let $E = f(A)$ and $F = f(B)$, and denote the reflection images of E, F about the unit circle by E^*, F^* , respectively. Then, as in the attained supremum case in Theorem 3.1,

$$\text{mod}(E \cup E^*, F \cup F^*; \mathbb{C}) = \int_{\mathbb{C}} |\nabla u|^2 dm = 2 \int_D |\nabla u|^2 dm,$$

where u is a real-valued continuous function on $\bar{\mathbb{C}}$ which is harmonic in $\mathbb{C} \setminus (E \cup E^* \cup F \cup F^*)$ with constant value 0 on $E \cup E^*$ and constant value 1 on $F \cup F^*$. Since f is conformal in Ω and K -QC in Ω^* with $K = R(\Omega)$, it follows that

$$\begin{aligned} (4.4) \quad \text{mod}(A \cup A^*, B \cup B^*; \mathbb{C}) &\leq \int_{\Omega} |\nabla(u \circ f)|^2 dm + \int_{\Omega^*} |\nabla(u \circ f)|^2 dm \\ &\leq \int_D |\nabla u|^2 dm + R(\Omega) \int_{D^*} |\nabla u|^2 dm \\ &= (1 + R(\Omega)) \text{mod}(E, F; D) \\ &= (1 + R(\Omega)) \text{mod}(A, B; \Omega). \end{aligned}$$

If A or B are not on the boundary of Ω , then $A \neq A \cup A^*$ or $B \neq B \cup B^*$. Using the uniqueness of harmonic functions, one can show that

$$\text{mod}(A \cup A^*, B \cup B^*; \mathbb{C}) > \text{mod}(A, B; \mathbb{C}).$$

Thus, it follows from (4.3) and (4.4) that

$$M(\Omega) < 1 + R(\Omega).$$

This yields the desired conclusion in the nondegenerate case.

Degenerate cases. Depending on the sizes and relative positions of A and B , there are five degenerate cases to be considered:

Case 1. A is a single point, B is a nondegenerate continuum and $A \cap B = \emptyset$.

Case 2. A, B both are single points and $A \cap B = \emptyset$.

Case 3. A is a single point, B is nondegenerate and $A \cap B \neq \emptyset$.

Case 4. A, B both are single points and $A \cap B \neq \emptyset$.

Case 5. A, B both are nondegenerate and $A \cap B \neq \emptyset$.

In all these cases we will show that $M(\Omega) \leq 2$. But, by [Y1, Theorem 4.6], for any Jordan domain Ω we have $M(\Omega) \geq 2$. Thus the degenerate cases cannot occur unless Ω is a disk. Since Ω is a conformal extension domain, we may choose $R > 1$ and a conformal map g of the disk $D_R = \{z : |z| < R\}$ onto a Jordan domain Ω' such that $g(D) = \Omega$. Let $f = g^{-1}$ and let $A'_n = f(A_n)$, $B'_n = f(B_n)$, where A_n and B_n are as in (4.2).

In Case 1, we have

$$\text{mod}(A_n, B_n; \Omega) = \text{mod}(A'_n, B'_n; D) \geq \frac{1}{2} \text{mod}(A'_n, B'_n; \mathbb{C}).$$

Thus

$$(4.5) \quad \frac{\text{mod}(A_n, B_n; \mathbb{C})}{\text{mod}(A_n, B_n; \Omega)} \leq 2 \frac{\text{mod}(A_n, B_n; \mathbb{C})}{\text{mod}(A'_n, B'_n; \mathbb{C})}.$$

On the other hand, we shall show that

$$(4.6) \quad \lim_{n \rightarrow \infty} \frac{\text{mod}(A_n, B_n; \mathbb{C})}{\text{mod}(A'_n, B'_n; \mathbb{C})} \leq 1.$$

To this end, choose $a_n, b_n \in A_n$ and $c_n, d_n \in B_n$ such that

$$|b_n - c_n| = d(A_n, B_n), \quad |b_n - a_n| = \max_{x \in A_n} |x - b_n|, \quad |d_n - c_n| = \max_{x \in B_n} |x - c_n|.$$

Using some basic properties of the modulus, we obtain that

$$\text{mod}(A'_n, B'_n; \mathbb{C}) \geq \frac{2\pi}{\log \Psi([f(a_n), f(b_n), f(c_n), f(d_n)])},$$

$$\text{mod}(A_n, B_n; \mathbb{C}) \leq \frac{2\pi}{\log \frac{|b_n - c_n|}{|b_n - a_n|}}.$$

Thus, by (3.8) and the fact that f is conformal in a domain containing A , it follows that

$$(4.7) \quad \frac{\text{mod}(A_n, B_n; \mathbb{C})}{\text{mod}(A'_n, B'_n; \mathbb{C})} \leq \frac{\log \Psi([f(a_n), f(b_n), f(c_n), f(d_n)])}{\log \frac{|b_n - c_n|}{|b_n - a_n|}} \\ \sim \frac{\log \frac{1}{|f(b_n) - f(a_n)|}}{\log \frac{1}{|b_n - a_n|}} \rightarrow 1$$

as n approaches infinity. This yields (4.6). Hence, by (4.5), $M(\Omega) \leq 2$ as desired.

Case 2 can be treated like Case 1. In this case, however, to establish (4.6) we need the following estimates instead of (4.7):

$$\begin{aligned} \frac{\text{mod}(A_n, B_n; \mathbb{C})}{\text{mod}(A'_n, B'_n; \mathbb{C})} &\leq \frac{\log \Psi([f(a_n), f(b_n), f(c_n), f(d_n)])}{\log \frac{|b_n - c_n|}{|b_n - a_n|} + \log \frac{|b_n - c_n|}{|d_n - c_n|}} \\ &\sim \frac{\log |f(b_n) - f(a_n)| + \log |f(d_n) - f(c_n)|}{\log |b_n - a_n| + \log |d_n - c_n|} \longrightarrow 1. \end{aligned}$$

For Case 3, choose $a_n, b_n \in A_n$ and $c_n, d_n \in B_n$ as in Case 1 above. We divide this case into two subcases. First we assume that

$$(4.8) \quad \lim_{n \rightarrow \infty} [a_n, b_n, c_n, d_n] \neq \infty.$$

In this subcase, we claim that

$$(4.9) \quad \lim_{n \rightarrow \infty} \frac{\text{mod}(A_n, B_n; \mathbb{C})}{\text{mod}(A_n, B_n; \Omega')} = 1.$$

Choose $\delta > 0$ such that $A_n \subset D(b_n, \delta) \subset \Omega'$ for large n . Then, it follows that

$$\text{mod}(A_n, B_n; \mathbb{C}) \leq \text{mod}(A_n, B_n; \Omega') + \frac{2\pi}{\log \frac{\delta}{|b_n - a_n|}},$$

$$\text{mod}(A_n, B_n; \mathbb{C}) \geq \frac{2\pi}{\log \Psi([a_n, b_n, c_n, d_n])}.$$

This together with (4.8) yields

$$\begin{aligned} 1 &\geq \frac{\text{mod}(A_n, B_n; \Omega')}{\text{mod}(A_n, B_n; \mathbb{C})} \\ &\geq \frac{\text{mod}(A_n, B_n; \mathbb{C}) - 2\pi / \log \frac{\delta}{|b_n - a_n|}}{\text{mod}(A_n, B_n; \mathbb{C})} \longrightarrow 1, \end{aligned}$$

which proves claim (4.9). Finally, since

$$\text{mod}(A_n, B_n; \Omega') = \text{mod}(A'_n, B'_n; D_R) \leq 2 \text{mod}(A_n, B_n; \Omega),$$

(4.9) yields $M(\Omega) \leq 2$ as desired.

Next we assume that the limit in (4.8) is infinity. In this subcase, (4.5) is valid. As in Case 1 one can show that (4.6) is also true. Thus it follows that $M(\Omega) \leq 2$. The detail adjustment in verifying (4.6) is left to the reader.

To deal with Case 4, we consider the same two subcases as in Case 3. If (4.8) holds, the argument in Case 3 is valid here as well. Thus $M(\Omega) \leq 2$.

For the subcase where $[a_n, b_n, c_n, d_n] \rightarrow \infty$ as $n \rightarrow \infty$, we need a different approach to prove (4.6). Observe that, since f is conformal in the domain Ω' containing A and B , there exist constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$\lambda_1|x - y| \leq |f(x) - f(y)| \leq \lambda_2|x - y|$$

for all $x, y \in A_n \cup B_n$ and sufficiently large n . Therefore, using the discrete form of an equivalent definition for capacity due to Bagby [Ba, Theorem 5], it is not difficult to show that, for large n ,

$$(4.10) \quad \frac{2\pi}{\text{mod}(A'_n, B'_n; \mathbb{C})} \leq \lambda(\lambda_1, \lambda_2) + \frac{2\pi}{\text{mod}(A_n, B_n; \mathbb{C})},$$

where λ is a constant depending only on λ_1 and λ_2 . Since $[a_n, b_n, c_n, d_n] \rightarrow \infty$ implies that $\text{mod}(A_n, B_n; \mathbb{C}) \rightarrow 0$, (4.6) follows from (4.10). Thus we have $M(\Omega) \leq 2$ by (4.5) and (4.6).

Finally, for Case 5, it follows that

$$(4.11) \quad \begin{aligned} \text{mod}(A_n, B_n; \mathbb{C}) &\leq \text{mod}(A_n, B_n; \Omega') + \text{mod}(\partial\Omega, \partial\Omega'; \mathbb{C}) \\ &\leq 2 \text{mod}(A_n, B_n; \Omega) + \text{mod}(\partial\Omega, \partial\Omega'; \mathbb{C}). \end{aligned}$$

We observe in this case that $\text{mod}(A_n, B_n; \mathbb{C}) \rightarrow \infty$ and that $\text{mod}(\partial\Omega, \partial\Omega'; \mathbb{C})$ is finite. Thus (4.11) yields that $M(\Omega) \leq 2$ as desired. This completes the proof of Theorem 4.1. \square

It is easy to see that the ellipse is an analytic curve. Thus, combining Theorems 3.1 and 4.1, we obtain the following counterexample for the Garnett-Yang conjecture that $M(\Omega) = R(\Omega) + 1$.

4.12. COROLLARY. *For any $a > 1$ we have*

$$M(E_a) < a + 1,$$

where E_a is the interior domain of the ellipse defined in Section 2.

4.13. REMARKS. In Theorem 4.1 we have actually shown that, for a conformal extension domain Ω , either

$$(4.14) \quad M(\Omega) < 1 + R(\Omega)$$

or there exist disjoint nondegenerate continua A and B on $\partial\Omega$ such that

$$(4.15) \quad M(\Omega) = M_b(\Omega) = \frac{\text{mod}(A, B; \mathbb{C})}{\text{mod}(A, B; \Omega)}.$$

We believe that one should be able to prove (4.14) for some other smooth domains by using this result together with the harmonic function technique used in Theorem 3.1. However, the question of whether $M(\Omega) = M_b(\Omega)$ still remains open, even for ellipses. Theorem 4.1 may shed some light on this problem by reducing the number of cases one needs to consider.

REFERENCES

- [Ah] L. V. Ahlfors, *Conformal invariants*, McGraw-Hill, New York, 1973.
- [AH] J. M. Anderson and A. Hinkkanen, *Quadrilaterals and extremal quasiconformal extensions*, Comment. Math. Helv. **70** (1995), 455–474.
- [Ba] T. Bagby, *The modulus of a plane condenser*, J. Math. Mech. **17**, 1967, 315–329.
- [BA] A. Beurling and L. V. Ahlfors, *The boundary correspondence under quasiconformal mappings*, Acta Math. **96** (1956), 125–142.
- [GY] J. B. Garnett and S. Yang, *Quasiextremal distance domains and integrability of derivatives of conformal mappings*, Mich. Math. J. **41** (1994), 389–406.
- [Ge] F. W. Gehring, *Quasiconformal mappings*, Complex analysis and its applications, vol. 2, International Atomic Energy Agency, Vienna, 1976, 213–268.
- [GM] F. W. Gehring and O. Martio, *Quasiextremal distance domains and extension of quasiconformal mappings*, J. Analyse Math. **45** (1985), 181–206.
- [LV] O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, New York, 1973.
- [Po] C. Pommerenke, *Boundary behaviour of conformal maps*, Springer-Verlag, Berlin, 1992.
- [S1] K. Strebel, *Zur Frage der Eindeutigkeit extremaler quasikonformer Abbildungen des Einheitskreises*, Comment. Math. Helv. **36** (1962), 306–323.
- [S2] ———, *Extremal quasiconformal mappings*, Results in Math. **10** (1986), 169–210.
- [Y1] S. Yang, *QED domains and NED sets in \mathbb{R}^n* , Trans. Amer. Math. Soc. **334** (1992), 97–120.
- [Y2] ———, *Extremal distance and quasiconformal reflection constants of domains in \mathbb{R}^n* , J. Analyse Math. **62** (1994), 1–28.
- [Y3] ———, *Harmonic and extremal quasiconformal extensions*, J. Hunan Univ. **22** (1995), No. 6, 42–51.

Department of Mathematics, Emory University, Atlanta, GA 30322
 syang@mathcs.emory.edu