ILLINOIS JOURNAL OF MATHEMATICS
Volume 42, Number 4, Winter 1998

INTERPOLATION OF HERZ-TYPE HARDY SPACES

EUGENIO HERNÁNDEZ AND DACHUN YANG

1. Introduction

To study convolution algebras, Beurling in [4] first introduced some spaces of functions which are now called the Beurling algebras. Later Herz in [19] generalized these function spaces to further study the properties of functions. These generalized spaces of functions are just the prototype of Herz spaces. Since then, the theory of Herz spaces has been significantly developed and these spaces have turned out to be very useful in analysis. An interesting account with many applications for the generalized Herz spaces in some particular cases is given in [2]. In particular, in [18] the authors of this paper characterized the intermediate spaces obtained by the complex method of interpolation for the families of Herz spaces and gave many interesting applications.

On the other hand, in recent years, a theory of Herz-type Hardy spaces has been developed (see [7], [14], [15], [22]–[25]). These new Hardy spaces are a sort of local version of the ordinary Hardy spaces and are good substitutes for the latter when considering, for example, the boundedness of non-translation invariant singular integral operators (see [26]). In this paper, we are going to characterize the intermediate spaces obtained by applying the complex method of interpolation to the families of Herz-type Hardy spaces and to the mixed couples of Herz spaces and Herz-type Hardy spaces.

Let us first introduce some notation. Let $B_k = \{x \in \mathbb{R}^n\colon |x| \le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Let $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, $\widetilde{\chi}_k = \chi_k$ if $k \in \mathbb{N}$ and $\widetilde{\chi}_0 = \chi_{B_0}$, where χ_{C_k} is the characteristic function of the set C_k.

Definition 1.1 [19]. Let $\alpha \in \mathbb{R}$, $0 < p \le \infty$ and $0 < q \le \infty$.

(a) The homogeneous Herz space $K_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})\colon \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)}\right\}^{1/p}.$$

Received May 20, 1997.
1991 Mathematics Subject Classification. Primary 41A05, 42B30.
The first-named author was supported by grant PB 90-187 from the DGICYT (Ministerio de Educación y Ciencia-Spain) and by a grant from Southwestern Bell. The second-named author was supported by the SEDF and the NNSF of China.

(b) The non-homogeneous Herz space $K_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)\colon \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} = \left\{\sum_{k=0}^{\infty} 2^{k\alpha p}\|f\widetilde{\chi}_k\|^p_{L^q(\mathbb{R}^n)}\right\}^{1/p}.$$

(Here the usual modifications are made when $p = \infty$ or $q = \infty$.)

Obviously, $\dot{K}_p^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) = K_p^{0,p}(\mathbb{R}^n)$ for any $p \in (0, \infty)$. The spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ are quasi-Banach spaces and if $p, q \geq 1$, $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ are Banach spaces.

The spaces $K_q^{n(1-1/q),1}(\mathbb{R}^n) \equiv A^q(\mathbb{R}^n)$ with $1 < q < \infty$ are called Beurling algebras and were introduced by Beurling in [4] with different, but equivalent norms. The equivalence of the norms was proved by Feichtinger in [12]. The spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ were introduced by Herz in [19] also with different norms. Flett in [13] gave a characterization of these spaces which is easily seen to be equivalent to Definition 1.1 (see also [2]).

Let $\phi^*(f)$ be the vertical maximal function of f defined by

$$\phi^*(f)(x) = \sup_{t>0} |(f * \phi_t)(x)|,$$

where $\phi_t(x) = \frac{1}{t^n}\phi(\frac{x}{t})$ for $t > 0$, $\phi \in C_0^{\infty}(\mathbb{R}^n)$, $\operatorname{supp}\phi \subseteq B(0, 1)$ and

$$\int_{\mathbb{R}^n} \phi(x)\,dx \neq 0.$$

In the following, we will use $\mathcal{S}'(\mathbb{R}^n)$ to denote the class of tempered distributions on $\mathbb{R}^n$.

Definition 1.2. Let $0 < p \leq \infty$, $1 < q < \infty$, $\alpha \in \mathbb{R}$ and ϕ be as above.

(a) The homogeneous Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n)\colon \phi^*(f) \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)\}.$$

Moreover, we define $\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \|\phi^*(f)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$.

(b) The non-homogeneous Herz-type Hardy space $HK_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$HK_q^{\alpha,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n)\colon \phi^*(f) \in K_q^{\alpha,p}(\mathbb{R}^n)\}.$$

Moreover, we define $\|f\|_{HK_q^{\alpha,p}(\mathbb{R}^n)} = \|\phi^*(f)\|_{K_q^{\alpha,p}(\mathbb{R}^n)}$.

Clearly, $H\dot{K}_p^{0,p}(\mathbb{R}^n) = HK_p^{0,p}(\mathbb{R}^n) = H^p(\mathbb{R}^n)$ for $p \in (0,\infty)$, the standard Hardy spaces (see [11]). The spaces $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $HK_q^{\alpha,p}(\mathbb{R}^n)$ are quasi-Banach spaces and if $p, q \geq 1$, $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $HK_q^{\alpha,p}(\mathbb{R}^n)$ are Banach spaces.

The space $HK_q^{n(1-1/q),1}(\mathbb{R}^n)$ with $1 < q \leq 2$ and $n = 1$ was introduced by Chen and Lau in [7] and García-Cuerva in [14] generalized it to all $q \in (1,\infty)$ and n-dimensions. Then García-Cuerva and Herrero [15] and Lu and Yang [22] further generalized this theory to the spaces $H\dot{K}_q^{n(1/p-1/q),p}(\mathbb{R}^n)$ and $HK_q^{n(1/p-1/q),p}(\mathbb{R}^n)$ with $0 < p \leq 1 < q < \infty$ independently. Lu and Yang in [23]–[25] also studied the spaces $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $HK_q^{\alpha,p}(\mathbb{R}^n)$ with $0 < \alpha < \infty$, $0 < p \leq \infty$ and $1 < q < \infty$.

By the results in [22]–[25] (see also [15]), we know that the definitions of the spaces $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $HK_q^{\alpha,p}(\mathbb{R}^n)$ with $1 < q < \infty$, $\alpha > -n/q$ and $0 < p \leq \infty$ are independent of the choice of ϕ. In addition, the Herz spaces and the Herz-type Hardy spaces have the following well-known relationship (see [21] and [22]–[23]):

$$HK_q^{\alpha,p}(\mathbb{R}^n) \begin{cases} = K_q^{\alpha,p}(\mathbb{R}^n), & \text{if } 1 < q < \infty, -n/q < \alpha < n(1-1/q) \\ & \text{and } 0 < p \leq \infty \\ \subsetneqq K_q^{\alpha,p}(\mathbb{R}^n), & \text{if } 1 < q < \infty, -\infty < \alpha \leq -n/q \\ & \text{or } n(1-1/q) \leq \alpha < \infty, 0 < p \leq \infty, \end{cases} \tag{1.1}$$

and

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \begin{cases} = \dot{K}_q^{\alpha,p}(\mathbb{R}^n), & \text{if } 1 < q < \infty, -n/q < \alpha < n(1-1/q) \\ & \text{and } 0 < p \leq \infty \\ \neq \dot{K}_q^{\alpha,p}(\mathbb{R}^n), & \text{if } 1 < q < \infty, -\infty < \alpha \leq -n/q \\ & \text{or } n(1-1/q) \leq \alpha < \infty, 0 < p \leq \infty. \end{cases} \tag{1.2}$$

Moreover, if $0 \leq \alpha < \infty$, $1 \leq q \leq \infty$ and $0 < p \leq \infty$, then

$$K_q^{\alpha,p}(\mathbb{R}^n) = L^q(\mathbb{R}^n) \cap \dot{K}_q^{\alpha,p}(\mathbb{R}^n), \quad \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} \sim \|f\|_{L^q(\mathbb{R}^n)} + \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)},$$

and if $1 < q < \infty$, $n(1-1/q) \leq \alpha < \infty$ and $0 < p \leq \infty$, then

$$HK_q^{\alpha,p}(\mathbb{R}^n) = L^q(\mathbb{R}^n) \cap H\dot{K}_q^{\alpha,p}(\mathbb{R}^n), \quad \|f\|_{HK_q^{\alpha,p}(\mathbb{R}^n)} \sim \|f\|_{L^q(\mathbb{R}^n)} + \|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

It is well known that Coifman, Cwikel, Rochberg, Sagher and Weiss in [8] and [9] developed a theory of complex interpolation for the families of Banach spaces. Hernández in [16] and [17] introduced a theory of complex interpolation for the families of Banach lattices and used it to identify the intermediate spaces of many classical spaces, and Tabacco Vignati in [27]–[31] considered the case of quasi-Banach spaces.

In Section 2 of this paper, we first establish an interpolation theorem for the log-subharmonic operators, which generalizes Theorem 2.3 of Tabacco Vignati in [29]. Using this theorem and a method that is different from Tabacco Vignati in [30] (or see [27]), we characterize the intermediate spaces obtained by the complex method of interpolation for the families of Herz-type Hardy spaces. As a corollary of this, we deduce the results obtained by García-Cuerva and Herrero in [15].

In Section 3 of this paper, along the lines of the method introduced by Calderón and Torchinsky [6] (also see [3], [20], [15]), we treat the complex interpolation of some mixed couples of Herz spaces and Herz-type Hardy spaces which are not covered by the theorems obtained in Section 2.

Finally, we like to remark that for non-homogeneous spaces, we have also obtained similar results to all of the theorems on homogeneous spaces. To limit the length of this paper, we only state our results for homogeneous spaces.

2. Interpolation for families of Herz-type Hardy spaces

In this section, we will characterize the intermediate spaces obtained by applying the complex method of interpolation to the families of Herz-type Hardy spaces. Let us first introduce some notation (see [27]–[31] for more details).

Let Δ denote the open unit disc in $\mathbb{C}$, the set of complex numbers, and T the boundary of Δ. For each $\theta \in T$, we consider a quasi-Banach space $(B(\theta), \|\cdot\|_{B(\theta)})$, and denote by $c(\theta)$ the constants in the quasi-triangle inequalities. We say that the family $\{B(\theta)\}_{\theta\in T}$ is an interpolation family of quasi-Banach spaces if each $B(\theta)$ is continuously embedded in a Hausdorff topological vector space $\mathcal{U}$, the function $\theta \to \|b\|_{B(\theta)}$ is measurable for each $b \in \cap_{\theta\in T} B(\theta)$, and $\log c(\theta) \in L^1(T)$; $\mathcal{U}$ is called the containing space of the given family $\{B(\theta)\}_{\theta\in T}$. We define

$$\beta = \left\{ b \in \cap_{\theta\in T} B(\theta) \;\middle|\; \int_T \log^+ \|b\|_{B(\theta)}\, d\theta < \infty \right\},$$

called the log-intersection space of the given family $\{B(\theta)\}_{\theta\in T}$. Let $\mathcal{G} = \mathcal{G}(\Delta, B(\cdot))$ be the space of all the β-valued analytic functions of the form

$$g(z) = \sum_{j=1}^{m} \psi_j(z) b_j$$

for which $\|g\|_\infty \equiv \sup_{\theta\in T} \|g(\theta)\|_{B(\theta)} < \infty$, where $m \in \mathbb{N}$, $\psi_j \in N^+(\Delta)$, the positive Nevalinna class for Δ (see [10], Chapter 2) and $b_j \in \beta$, $j = 1, \dots, m$. For every $a \in \beta$ and $z \in \Delta$, we define

$$\|a\|_z = \inf\{\|g\|_\infty \colon g \in \mathcal{G},\ g(z) = a\}.$$

If N_z denotes the set of functions of β such that $\|a\|_z = 0$, the completion $B(z)$ of $(\beta/N_z, \|\cdot\|_z)$ will be called the interpolation space at z of the family $\{B(\theta)\}_{\theta\in T}$. We also denote $B(z)$ by $[B(\theta)]_z$.

In order to characterize the intermediate spaces obtained by the complex method of interpolation for the families of Herz-type Hardy spaces, we need an interpolation theorem for log-subharmonic operators. Let us first introduce the definition of log-subharmonic operator (see [29] or [27]). Let $\mathcal{M}$ be the set of measurable complex-valued functions on some measure space (Y, ν). An operator M mapping $\mathcal{M}$ into the class $\mathcal{N}$ of non-negative-valued measurable functions on some other measure space (X, μ) is said to be of maximal-type provided it satisfies

$$M(\lambda a) = |\lambda| Ma, \quad \text{for all } \lambda \in \mathbb{C} \text{ and all } a \in \mathcal{M}; \tag{2.1}$$

$$M(a) = M(|a|), \quad \text{for all } a \in \mathcal{M}; \tag{2.2}$$

$$M(a)(x) \le M(b)(x), \quad \text{if } |a(y)| \le |b(y)|, a, b \in \mathcal{M}; \tag{2.3}$$

$$M\left[\int_T f(\cdot, \theta)\, d\theta\right](x) \le \int_T M(f(\cdot, \theta))(x)\, d\theta. \tag{2.4}$$

If $\{B(\theta)\}_{\theta \in T}$ is an interpolation family with containing space $\mathcal{U}$, we say that an operator $\widetilde{M}\colon \mathcal{U} \to \mathcal{N}$ is a log-subharmonic operator associated with the family $\{B(\theta)\}_{\theta \in T}$ if it can be expressed as the composition $M \cdot L$ of a linear operator L mapping $\mathcal{U}$ into $\mathcal{M}$ and a maximal-type operator M.

To establish the interpolation theorem for log-subharmonic operators, we need the following lemma.

LEMMA 2.1. *Let $\alpha(z) \in \mathbb{R}$, $0 < \ell(z), \gamma(z) \le 1$. Let $f(x) = \sum_{j=1}^N \beta_j \chi_{E_j}(x)$ with $N \in \mathbb{N}$, $\beta_j > 0$, $j = 1, \dots, N$, and $\{E_j\}_{j=1}^N$ pairwise disjoint sets of finite measure. Define*

$$g(x, z) = \sum_{k \in N_1} 2^{k\alpha(z)} f(x)^{\ell(z)} \chi_k(x) \|f\chi_k\|_{L^1(\mathbb{R}^n)}^{\gamma(z)-\ell(z)},$$

where $N_1 = \{k \in \mathbb{Z} \mid \|f\chi_k\|_{L^1(\mathbb{R}^n)} > 0\}$. Let $S(z)$ be the set of all the $g(x, z)$ corresponding to all the above different functions $f(x)$. Then $S(z)$ is dense in the Herz spaces $\dot{K}_{1/\ell(z)}^{-\alpha(z), 1/\gamma(z)}(\mathbb{R}^n)$.

Proof. Let $f_k(x) = \sum_{j=1}^N \beta_j \chi_{E_j}(x)$ with $N \in \mathbb{N}$, $\beta_j > 0$, $E_j \subseteq C_k$, $j = 1, \dots, N$, and $\{E_j\}_{j=1}^N$ pairwise disjoint sets. For such an f_k, we define

$$g_k(x, z) = \sum_{j=1}^N \beta_j \chi_{E_j}(x) \|f_k\|_{L^1(C_k)}^{\gamma(z)/\ell(z)-1}.$$

Let $S_k(z)$ be the set of all the $g_k(x, z)$. To prove the lemma, by the definitions, it suffices to prove that $S_k(z)$ is dense in $L^1(C_k)$. To do this, let $\widetilde{g}_k(x) = \sum_{j=1}^N d_j \chi_{E_j}(x)$; we only need to show that there exist $\{\beta_j\}_{j=1}^N$, such that $g_k(x, z) = \widetilde{g}_k(x)$. In other words, we need to prove that there exist $\{\beta_j\}_{j=1}^N$ such that

$$\begin{cases} \beta_1 &= d_1 \left(\sum_{j=1}^N \beta_j |E_j|\right)^{1-\gamma(z)/\ell(z)} \\ \vdots & \\ \beta_N &= d_N \left(\sum_{j=1}^N \beta_j |E_j|\right)^{1-\gamma(z)/\ell(z)}. \end{cases} \tag{2.5}$$

From (2.5), we easily deduce that $\sum_{j=1}^{N} \beta_j |E_j| = (\sum_{j=1}^{N} d_j |E_j|)^{\ell(z)/\gamma(z)}$. Hence,

$$\beta_j = d_j \left(\sum_{j=1}^{N} d_j |E_j| \right)^{\ell(z)/\gamma(z)-1}$$

This finishes the proof of Lemma 2.1. $\square$

THEOREM 2.1. *Let $\widetilde{M}$ be a log-subharmonic operator associated with an interpolation family of quasi-Banach spaces $\{B(\theta)\}_{\theta \in T}$. Suppose that*

$$\|\widetilde{M}a\|_{\dot{K}_{q(\theta)}^{\alpha(\theta),p(\theta)}(\mathbb{R}^n)} \leq \eta(\theta)\|a\|_{B(\theta)}, \quad \text{for } a \in \beta,$$

where $\alpha(\theta) \in \mathbb{R}, 0 < p(\theta), q(\theta) \leq \infty, \alpha(\theta), 1/p(\theta)$ and $1/q(\theta) \in L^1(T)$.

If $\log \eta \in L^1(T)$, then for all $a \in \beta$,

$$\|\widetilde{M}a\|_{\dot{K}_{q(z)}^{\alpha(z),p(z)}(\mathbb{R}^n)} \leq \eta(z)\|a\|_z,$$

where $\alpha(z) = \int_T \alpha(\theta) P_z(\theta)\, d\theta$,

$$\frac{1}{p(z)} = \int_T \frac{1}{p(\theta)} P_z(\theta)\, d\theta, \quad \frac{1}{q(z)} = \int_T \frac{1}{q(\theta)} P_z(\theta)\, d\theta,$$

$\eta(z) = \exp \int_T \log \eta(\theta) P_z(\theta)\, d\theta$ and $P_z(\theta)$ is the Poisson kernel for evaluation at z.

Remark 2.1. If $\alpha(\theta) = 0$ and $0 < p(\theta) = q(\theta) \leq \infty$, Theorem 2.1 is just Theorem 2.3 in [29].

Proof of Theorem 2.1. Let $a \in \beta$ and $z_0 \in \Delta$. For any given $\varepsilon > 0$, there exists $f(z) = \sum_{j=1}^{m} \varphi_j(z) a_j \in \mathcal{G}(\Delta, B(\cdot))$ such that $f(z_0) = a$ and $\|f\|_\infty \leq \|a\|_{z_0} + \varepsilon$, where $m \in \mathbb{N}$, $\varphi_j \in N^+(\Delta)$ and $a_j \in \beta$, $j = 1, \ldots, m$. To prove the theorem, we claim that it is enough to show that the function $z \to \log\|\widetilde{M}(f(z))(\cdot)\|_{\dot{K}_{q(z)}^{\alpha(z),p(z)}(\mathbb{R}^n)}$ is subharmonic in the unit disk. If this is true, we have

$$\|\widetilde{M}(f(z_0))(\cdot)\|_{\dot{K}_{q(z_0)}^{\alpha(z_0),p(z_0)}(\mathbb{R}^n)} \leq \exp \int_T \log\|\widetilde{M}(f(\theta))(\cdot)\|_{\dot{K}_{q(\theta)}^{\alpha(\theta),p(\theta)}(\mathbb{R}^n)} P_{z_0}(\theta)\, d\theta.$$

Therefore,

$$\begin{aligned}
\|\widetilde{M}a\|_{\dot{K}_{q(z_0)}^{\alpha(z_0),p(z_0)}(\mathbb{R}^n)} &= \|\widetilde{M}(f(z_0))(\cdot)\|_{\dot{K}_{q(z_0)}^{\alpha(z_0),p(z_0)}(\mathbb{R}^n)} \\
&\leq \exp \int_T \log \left(\eta(\theta) \|f(\theta)\|_{B(\theta)} \right) P_{z_0}(\theta)\, d\theta \\
&\leq \eta(z_0)\|f\|_\infty \leq \eta(z_0)(\|a\|_{z_0} + \varepsilon).
\end{aligned}$$

Thus, letting $\varepsilon \to 0$, we obtain

$$\|\widetilde{M}a\|_{\dot{K}_{q(z_0)}^{\alpha(z_0),p(z_0)}(\mathbb{R}^n)} \leq \eta(z_0)\|a\|_{z_0},$$

which is the desired result.

To verify our claim, we notice that since the functions $p(z), q(z)$ are strictly positive and continuous on Δ, for any $\rho > 0$ such that

$$\bar{B}_\rho(z_0) = \{z\colon\ |z - z_0| \leq \rho\} \subset \Delta,$$

we can find $r > 0$ such that $0 < r < p(z), q(z)$ when $z \in \bar{B}_\rho(z_0)$. Moreover, since subharmonicity is a local property, it suffices to show that

$$\begin{aligned}
&\log\|\widetilde{M}(f(z_0))(\cdot)\|_{\dot{K}_{q(z_0)}^{\alpha(z_0),p(z_0)}(\mathbb{R}^n)} \\
&\quad\leq \frac{1}{2\pi}\int_0^{2\pi} \log\|\widetilde{M}(f(z_0+\rho e^{i\theta}))(\cdot)\|_{\dot{K}_{q(z_0+\rho e^{i\theta})}^{\alpha(z_0+\rho e^{i\theta}),p(z_0+\rho e^{i\theta})}(\mathbb{R}^n)}\, d\theta
\end{aligned}$$

for any such $\rho > 0$.

Define $1/\ell(z) = (q(z)/r)'$, $1/\gamma(z) = (p(z)/r)'$, where we denote the conjugate index of p by p'; that is, $1/p + 1/p' = 1$. Let f_1 be a simple and positive function on $\mathbb{R}^n$ of the form given in Lemma 2.1. That is, $f_1(x) = \sum_{j=1}^N \beta_j \chi_{E_j}(x)$, where $N \in \mathbb{N}$, $\beta_j > 0$, $j = 1, \ldots, N$ and $\{E_j\}_{j=1}^N$ are pairwise disjoint sets of finite measure. Define

$$g(x,z) = \sum_{k\in N_1} 2^{k\alpha(z)r} f_1(x)^{\ell(z)} \chi_k(x) \|f_1\chi_k\|_{L^1(\mathbb{R}^n)}^{\gamma(z)-\ell(z)},$$

where $N_1 = \{k \in \mathbb{Z} \mid \|f_1\chi_k\|_{L^1(\mathbb{R}^n)} > 0\}$. Then we have

$$\begin{aligned}
I(z) &\equiv \int_{\mathbb{R}^n} g(x,z)(\widetilde{M}(f(z)))^r(x)\, dx \\
&= \sum_{k\in N_1}\sum_{j=1}^N 2^{k\alpha(z)r}(\beta_j)^{\ell(z)}\|f_1\chi_k\|_{L^1(\mathbb{R}^n)}^{\gamma(z)-\ell(z)} \int_{E_j\cap C_k} (\widetilde{M}(f(z)))^r(x)\, dx \\
&\equiv \sum_{k\in N_1}\sum_{j=1}^N \beta_{j,k}(z).
\end{aligned}$$

As in [29] (pp. 328, 329) (also see [27]), we can prove that $\beta_{j,k}(z)$ is log-subharmonic and therefore so is $I(z)$. Thus

$$\begin{aligned}
&\log \int_{\mathbb{R}^n} g(x,z_0)(\widetilde{M}(f(z_0)))^r(x)\, dx \\
&= \log I(z_0) \leq \int_0^{2\pi} \log I(z_0+\rho e^{i\theta})\, \frac{d\theta}{2\pi}
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \log\left\{\int_{\mathbb{R}^n} g(x, z_0+\rho e^{i\theta})(\widetilde{M}(f(z_0+\rho e^{i\theta})))^r(x)\,dx\right\}\frac{d\theta}{2\pi}\\
&\le \int_0^{2\pi} \log\left\{\|g(\cdot, z_0+\rho e^{i\theta})\|_{\dot{K}^{-r\alpha(z_0+\rho e^{i\theta}),1/\gamma(z_0+\rho e^{i\theta})}_{1/\ell(z_0+\rho e^{i\theta})}(\mathbb{R}^n)}\right.\\
&\quad\left.\times \|(\widetilde{M}(f(z_0+\rho e^{i\theta})))^r(\cdot)\|_{\dot{K}^{r\alpha(z_0+\rho e^{i\theta}),p(z_0+\rho e^{i\theta})/r}_{q(z_0+\rho e^{i\theta})/r}(\mathbb{R}^n)}\right\}\frac{d\theta}{2\pi}\\
&= \int_0^{2\pi} \log\left\{\|f_1\|^{\gamma(z_0+\rho e^{i\theta})}_{L^1(\mathbb{R}^n)}\|\widetilde{M}(f(z_0+\rho e^{i\theta}))(\cdot)\|^r_{\dot{K}^{\alpha(z_0+\rho e^{i\theta}),p(z_0+\rho e^{i\theta})}_{q(z_0+\rho e^{i\theta})}(\mathbb{R}^n)}\right\}\frac{d\theta}{2\pi},
\end{aligned}
$$

where we use the fact that

$$
\left(\dot{K}^{r\alpha(z_0+\rho e^{i\theta}),p(z_0+\rho e^{i\theta})/r}_{q(z_0+\rho e^{i\theta})/r}(\mathbb{R}^n)\right)^* = \dot{K}^{-r\alpha(z_0+\rho e^{i\theta}),1/\gamma(z_0+\rho e^{i\theta})}_{1/\ell(z_0+\rho e^{i\theta})}(\mathbb{R}^n)
$$

(see Corollary 2.1 in [18]) and

$$
\begin{aligned}
&\|g(\cdot, z_0+\rho e^{i\theta})\|_{\dot{K}^{-r\alpha(z_0+\rho e^{i\theta}),1/\gamma(z_0+\rho e^{i\theta})}_{1/\ell(z_0+\rho e^{i\theta})}(\mathbb{R}^n)}\\
&\quad= \left\{\sum_{k=-\infty}^{\infty} 2^{-kr\alpha(z_0+\rho e^{i\theta})/\gamma(z_0+\rho e^{i\theta})}2^{kr\alpha(z_0+\rho e^{i\theta})/\gamma(z_0+\rho e^{i\theta})}\right.\\
&\qquad\times \|f_1(\cdot)^{\ell(z_0+\rho e^{i\theta})}\chi_k\|^{1/\gamma(z_0+\rho e^{i\theta})}_{L^{1/\ell(z_0+\rho e^{i\theta})}(\mathbb{R}^n)}\\
&\qquad\left.\times \|f_1\chi_k\|^{(\gamma(z_0+\rho e^{i\theta})-\ell(z_0+\rho e^{i\theta}))/\gamma(z_0+\rho e^{i\theta})}_{L^1(\mathbb{R}^n)}\right\}^{\gamma(z_0+\rho e^{i\theta})}\\
&\quad= \left\{\sum_{k=-\infty}^{\infty}\|f_1\chi_k\|_{L^1(\mathbb{R}^n)}\right\}^{\gamma(z_0+\rho e^{i\theta})}\\
&\quad= \|f_1\|^{\gamma(z_0+\rho e^{i\theta})}_{L^1(\mathbb{R}^n)}.
\end{aligned}
$$

Here and in what follows, $(B)^*$ denotes the dual space of the space B. Using the fact that $(\dot{K}^{r\alpha(z_0),p(z_0)/r}_{q(z_0)/r}(\mathbb{R}^n))^* = \dot{K}^{-r\alpha(z_0),1/\gamma(z_0)}_{1/\ell(z_0)}(\mathbb{R}^n)$ and Lemma 2.1, we obtain

$$
\begin{aligned}
&\log\|\widetilde{M}(f(z_0))(\cdot)\|^r_{\dot{K}^{\alpha(z_0),p(z_0)}_{q(z_0)}(\mathbb{R}^n)}\\
&\quad= \log\sup\int_{\mathbb{R}^n} g(x, z_0)(\widetilde{M}(f(z_0)))^r(x)\,dx\\
&\quad\le \frac{r}{2\pi}\int_0^{2\pi}\log\|\widetilde{M}(f(z_0+\rho e^{i\theta}))(\cdot)\|_{\dot{K}^{\alpha(z_0+\rho e^{i\theta}),p(z_0+\rho e^{i\theta})}_{q(z_0+\rho e^{i\theta})}(\mathbb{R}^n)}\,d\theta,
\end{aligned}
$$

where the supremum is taken over all the simple and positive functions f_1 of the above form such that $\|f_1\|_{L^1(\mathbb{R}^n)} \le 1$.

This finishes the proof of Theorem 2.1. □

We now turn to the interpolation of Herz-type Hardy spaces.

THEOREM 2.2. *Let* $B(\theta) = H\dot{K}_{q(\theta)}^{\alpha(\theta),p(\theta)}(\mathbb{R}^n)$. *If there exist* $p_0, p_1 \in (0,\infty)$, $q_0 > 1$ *and* $\alpha_1, \alpha_0 > 0$ *such that*

$$\begin{cases} \max\{p_1, 1\} \geq p(\theta) \geq p_0 > 0 \\ q(\theta) \geq q_0 > 1 \\ \alpha_1 \geq \alpha(\theta) \geq n(1 - 1/q(\theta)) \\ \mathrm{Int}\{\alpha(\theta) + n/q(\theta)\} + 1 - (\alpha(\theta) + n/q(\theta)) \geq \alpha_0 > 0, \end{cases} \tag{2.6}$$

then

$$B(z) = H\dot{K}_{q(z)}^{\alpha(z),p(z)}(\mathbb{R}^n),$$

where $\mathrm{Int}\{x\}$ *denotes the maximum integer* $\leq x$, $\alpha(z) = \int_T \alpha(\theta) P_z(\theta)\, d\theta$,

$$\frac{1}{p(z)} = \int_T \frac{1}{p(\theta)} P_z(\theta)\, d\theta \quad \textit{and} \quad \frac{1}{q(z)} = \int_T \frac{1}{q(\theta)} P_z(\theta)\, d\theta.$$

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\widehat{\varphi}(0) \neq 0$. We define

$$\widetilde{M}(x, f) = \sup_{t>0} |f * \varphi_t(x)|$$

for every tempered distribution f. By Definition 1.2, $f \in H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ if and only if $\widetilde{M}(\cdot, f) \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \|\widetilde{M}(\cdot, f)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$. It is easy to see that $\widetilde{M}$ is a log-subharmonic operator associated with the family $\{B(\theta)\}_{\theta\in T}$ and obviously satisfies the conditions of Theorem 2.1. Thus, by Theorem 2.1, we obtain

$$\|f\|_{H\dot{K}_{q(z)}^{\alpha(z),p(z)}(\mathbb{R}^n)} = \|\widetilde{M}(\cdot, f)\|_{\dot{K}_{q(z)}^{\alpha(z),p(z)}(\mathbb{R}^n)} \leq c\|f\|_z$$

for every $f \in \beta$. Thus, $N_z = \{0\}$ and in order to prove Theorem 2.2, we only need to show that

$$H\dot{K}_{q(z_0)}^{\alpha(z_0),p(z_0)}(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n) \subseteq B(z_0),$$

for every $z_0 \in \Delta$, since the left-hand side is dense in the Herz-type Hardy spaces considered here (see [23]).

Let $h \in H\dot{K}_{q(z_0)}^{\alpha(z_0),p(z_0)}(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$ and $m \in \mathbb{Z}_+$ such that

$$m > \mathrm{Int}\{\alpha_1 + n(1/q_0 - 1)\}.$$

Then using the atomic decomposition proved in [23], we can write

$$h(x) = \sum_{k=-\infty}^{\infty} \lambda_k a_k(x) + \sum_{k=-\infty}^{\infty} \mu_k b_k(x),$$

where $\int_{\mathbb{R}^n} a_k(x)x^\gamma\,dx = 0 = \int_{\mathbb{R}^n} a_k(x)x^\beta\,dx$, for $\gamma, \beta \in \mathbb{Z}_+^n$, $|\gamma|, |\beta| \le m$,

$$\operatorname{supp} a_k \subset \widetilde{C}_k = \{x\colon\ 2^{k-1} - 2^k\varepsilon \le |x| \le 2^k + 2^k\varepsilon\}$$

with ε small enough, $\operatorname{supp} b_k \subset (\widetilde{C}_k \cup \widetilde{C}_{k+1})$, $\|a_k\|_{L^{q(z_0)}(\mathbb{R}^n)} \le |B(0, 2^{k+1})|^{-\alpha(z_0)/n}$, $\|b_k\|_{L^{q(z_0)}(\mathbb{R}^n)} \le |B(0, 2^{k+2})|^{-\alpha(z_0)/n}$,

$$0 \le \lambda_k \le c2^{k\alpha(z_0)} \sum_{j=k-1}^{k+1} \|f\chi_k\|_{L^{q(z_0)}(\mathbb{R}^n)}$$

and

$$0 \le \mu_k \le c2^{k\alpha(z_0)} \sum_{j=k-1}^{k+2} \|G(f)\chi_k\|_{L^{q(z_0)}(\mathbb{R}^n)}.$$

Here $G(f)$ is the grand maximal function of f in Fefferman and Stein's sense (see [11]), defined by

$$G(f)(x) = \sup_{\varphi\in\mathcal{A}_N}\ \sup_{\{0<t<\infty,\ |x-y|<t\}} |(f*\varphi_t)(y)|,$$

and

$$\mathcal{A}_N = \{\varphi \in \mathcal{S}(\mathbb{R}^n)\colon\ \sup_{|\gamma|,\ |\beta|\le N} |x^\gamma D^\beta\varphi(x)| \le 1\}$$

with $N \in \mathbb{N}$ and $N > \alpha_0 + n + 1$. Moreover, using $|f(x)\chi_k(x)| \le cG(f)(x)\chi_k(x)$ and the results in [25], we obtain

$$\begin{aligned}
&\left\{\sum_{k=-\infty}^{\infty} \left(\lambda_k^{p(z_0)} + \mu_k^{p(z_0)}\right)\right\}^{1/p(z_0)} \\
&\quad \le c\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha(z_0)p(z_0)}\|f\chi_k\|_{L^{q(z_0)}(\mathbb{R}^n)}^{p(z_0)}\right. \\
&\qquad\quad \left. + \sum_{k=-\infty}^{\infty} 2^{k\alpha(z_0)p(z_0)}\|G(f)\chi_k\|_{L^{q(z_0)}(\mathbb{R}^n)}^{p(z_0)}\right\}^{1/p(z_0)} \\
&\quad \le c\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha(z_0)p(z_0)}\|G(f)\chi_k\|_{L^{q(z_0)}(\mathbb{R}^n)}^{p(z_0)}\right\}^{1/p(z_0)} \\
&\quad \le c\|f\|_{H\dot{K}_{q(z_0)}^{\alpha(z_0),p(z_0)}(\mathbb{R}^n)},
\end{aligned}$$

where c is a constant independent of f.

Let $Q_z(\theta)$ be the conjugate Poisson kernel and define $H_z(\theta) = P_z(\theta) + iQ_z(\theta)$, that is, the Herglotz kernel for the unit disk. Let $\widetilde{\alpha}(z) = \int_T \alpha(\theta)H_z(\theta)\,d\theta$,

$$\frac{1}{\widetilde{p}(z)} = \int_T \frac{1}{p(\theta)}H_z(\theta)\,d\theta \quad\text{and}\quad \frac{1}{\widetilde{q}(z)} = \int_T \frac{1}{q(\theta)}H_z(\theta)\,d\theta,$$

so that $\widetilde{\alpha}(z)$, $1/\widetilde{p}(z)$ and $1/\widetilde{q}(z)$ are analytic in the unit disk and their real parts are $\alpha(z)$, $1/p(z)$ and $1/q(z)$ respectively. We can also require that $\widetilde{\alpha}(z_0) = \alpha(z_0)$, $1/\widetilde{p}(z_0) = 1/p(z_0)$ and $1/\widetilde{q}(z_0) = 1/q(z_0)$.

We then consider

$$\begin{aligned} F(x,z) &= \sum_{k=-\infty}^{\infty} (\lambda_k)^{p(z_0)/\widetilde{p}(z)} |B(0,2^{k+1})|^{\alpha(z_0)q(z_0)/(n\widetilde{q}(z))-\widetilde{\alpha}(z)/n} \\ &\quad \times \|a_k\|_{L^{q(z_0)}(\mathbb{R}^n)}^{q(z_0)/\widetilde{q}(z)} \|a_k\|^{-1}_{L^{\widetilde{q}(z)}(\mathbb{R}^n)} a_k(x) \\ &\quad + \sum_{k=-\infty}^{\infty} (\mu_k)^{p(z_0)/\widetilde{p}(z)} |B(0,2^{k+2})|^{\alpha(z_0)q(z_0)/(n\widetilde{q}(z))-\widetilde{\alpha}(z)/n} \\ &\quad \times \|b_k\|_{L^{q(z_0)}(\mathbb{R}^n)}^{q(z_0)/\widetilde{q}(z)} \|b_k\|^{-1}_{L^{\widetilde{q}(z)}(\mathbb{R}^n)} b_k(x). \end{aligned}$$

Some comments are necessary regarding the definition of $F(x,z)$. We have to show that this function is a limit of functions in $\mathcal{G}$. With the given definitions, this is quite obvious, except, perhaps, for the terms $\|a_k\|^{-1}_{L^{\widetilde{q}(z)}(\mathbb{R}^n)}$ and $\|b_k\|^{-1}_{L^{\widetilde{q}(z)}(\mathbb{R}^n)}$, which we have to show are analytic functions. We only do this for $\|a_k\|^{-1}_{L^{\widetilde{q}(z)}(\mathbb{R}^n)}$. For $\|b_k\|^{-1}_{L^{\widetilde{q}(z)}(\mathbb{R}^n)}$, this can be done similarly. We can suppose that a_k is not a.e. zero on $\widetilde{C}_k$. Then $\|a_k\|_{L^{\widetilde{q}(z)}(\mathbb{R}^n)}$ means

$$\|a_k\|_{L^{\widetilde{q}(z)}(\mathbb{R}^n)} = \exp\left\{ \frac{1}{\widetilde{q}(z)} \log \int_{\widetilde{C}_k \cap \{x\in\widetilde{C}_k:\ |a_k(x)|\neq 0\}} |a_k(x)|^{\widetilde{q}(z)}\,dx \right\},$$

where log refers to the principal branch of the logarithm function. That the logarithm of the integral exists is a consequence of the fact that the integral is never zero for every z belonging to the convex set Δ (see Theorem 2.16 in [1]). Thus, except for the terms $a_k(x)$ and $b_k(x)$, all the functions involved in the definition of $F(x,z)$ are analytic. Obviously $F(z_0) = h(x)$. Now, we estimate $\|F(\cdot,\theta)\|_{H\dot{K}_{q(\theta)}^{\alpha(\theta),p(\theta)}(\mathbb{R}^n)}$. Let

$$\begin{aligned} a_k(x,z) &= |B(0,2^{k+1})|^{\alpha(z_0)q(z_0)/(n\widetilde{q}(z))-\widetilde{\alpha}(z)/n} \\ &\quad \times \|a_k\|_{L^{q(z_0)}(\mathbb{R}^n)}^{q(z_0)/\widetilde{q}(z)} \|a_k\|^{-1}_{L^{\widetilde{q}(z)}(\mathbb{R}^n)} a_k(x) \end{aligned}$$

and

$$\begin{aligned} b_k(x,z) &= |B(0,2^{k+2})|^{\alpha(z_0)q(z_0)/(n\widetilde{q}(z))-\widetilde{\alpha}(z)/n} \\ &\quad \times \|b_k\|_{L^{q(z_0)}(\mathbb{R}^n)}^{q(z_0)/\widetilde{q}(z)} \|b_k\|^{-1}_{L^{\widetilde{q}(z)}(\mathbb{R}^n)} b_k(x). \end{aligned}$$

Then

$$\begin{aligned} \|a_k(\cdot,\theta)\|_{L^{q(\theta)}(\mathbb{R}^n)} &\le c|B(0,2^{k+1})|^{\alpha(z_0)q(z_0)/(nq(\theta))-\alpha(\theta)/n} \|a_k\|_{L^{q(z_0)}(\mathbb{R}^n)}^{q(z_0)/q(\theta)} \\ &\le c_1 |B(0,2^{k+1})|^{-\alpha(\theta)/n}. \end{aligned}$$

Similarly, we can show

$$\|b_k(\cdot,\theta)\|_{L^{q(\theta)}(\mathbb{R}^n)} \le c_1|B(0,2^{k+1})|^{-\alpha(\theta)/n}.$$

Thus $\{c_1\}^{-1}a_k(x,\theta)$ and $\{c_1\}^{-1}b_k(x,\theta)$ are central $(\alpha(\theta), q(\theta))$-atoms (see the definition in [23]). Moreover, by a similar computation to the proof of Theorem 1 in [23] (pp. 664–666) (or see the proof of Theorem 2.1 in [22]), we have

$$\begin{aligned}\|F(\cdot,\theta)\|_{H\dot{K}^{\alpha(\theta),p(\theta)}_{q(\theta)}(\mathbb{R}^n)} &= \|\widetilde{M}(\cdot,F)\|_{\dot{K}^{\alpha(\theta),p(\theta)}_{q(\theta)}(\mathbb{R}^n)}\\ &\le c(\theta)\left\{\sum_{k=-\infty}^{\infty}\left((\lambda_k)^{p(z_0)}+(\mu_k)^{p(z_0)}\right)\right\}^{1/p(\theta)}\\ &\le c(\theta)\|f\|^{p(z_0)/p(\theta)}_{H\dot{K}^{\alpha(z_0),p(z_0)}_{q(z_0)}(\mathbb{R}^n)};\end{aligned}$$

see [23] for the details. From (2.6), we can deduce that $\log^+ c(\theta) \in L^1(T)$ and therefore, $h \in B(z_0)$.

This finishes the proof of Theorem 2.2. □

Remark 2.2. It is well known that if $1 < q(\theta) < \infty$, $-n/q(\theta) < \alpha(\theta) < n(1-1/q(\theta))$ and $0 < p(\theta) < \infty$, then

$$H\dot{K}^{\alpha(\theta),p(\theta)}_{q(\theta)}(\mathbb{R}^n) = \dot{K}^{\alpha(\theta),p(\theta)}_{q(\theta)}(\mathbb{R}^n);$$

see [21] and [23].

Using this and the computational technique in the proof of Theorem 2.1 in [21], a similar proof to that of Theorem 2.2 gives us:

THEOREM 2.3. *Let* $B(\theta) = H\dot{K}^{\alpha(\theta),p(\theta)}_{q(\theta)}(\mathbb{R}^n)$. *If there exist* p_0, $p_1 \in (0,\infty)$, $q_0 > 1$, $\alpha_i > 0$, $i = 0, 1, 2, 3$ *such that when* $\alpha(\theta) \in [n(1-1/q(\theta)),\infty)$, (2.6) *holds and when* $\alpha(\theta) \in (-n/q(\theta), n(1-1/q(\theta)))$,

$$\begin{cases}\max\{p_1, 1\} \ge p(\theta) \ge p_0 > 0,\\ q(\theta) \ge q_0 > 1,\\ n > \alpha_3 \ge \alpha(\theta) + n/q(\theta) \ge \alpha_2 > 0,\end{cases}$$

then

$$B(z) = H\dot{K}^{\alpha(z),p(z)}_{q(z)}(\mathbb{R}^n),$$

where $\alpha(z) = \int_T \alpha(\theta)P_z(\theta)\,d\theta$,

$$\frac{1}{p(z)} = \int_T \frac{1}{p(\theta)}P_z(\theta)\,d\theta \quad \textit{and} \quad \frac{1}{q(z)} = \int_T \frac{1}{q(\theta)}P_z(\theta)\,d\theta.$$

As a simple corollary of Theorem 2.3, we can characterize the intermediate spaces obtained by the complex interpolation method for the couples of Herz-type Hardy spaces as follows. See [15] (pp. 619–623) or [3] (pp. 87–88) for the definition of the spaces $[\cdot, \cdot]_\theta$.

THEOREM 2.4. *Let* $0 < p_0, p_1 \le \infty$, $1 < q_0, q_1 < \infty$, $\theta \in (0, 1)$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then

$$\left[H\dot{K}_{q_0}^{n(1/p_0-1/q_0),p_0}(\mathbb{R}^n), H\dot{K}_{q_1}^{n(1/p_1-1/q_1),p_1}(\mathbb{R}^n)\right]_\theta = H\dot{K}_q^{n(1/p-1/q),p}(\mathbb{R}^n).$$

Remark 2.3. When $0 < p_0, p_1 < 1$, Theorem 2.4 has been obtained by García-Cuerva and Herrero in [15]. When $1 < p_0 = p_1 < \infty$ and $1 < q_0 = q_1 < \infty$, we obtain the result in [3], that is

$$\left[L^{p_0}(\mathbb{R}^n), L^{p_1}(\mathbb{R}^n)\right]_\theta = L^p(\mathbb{R}^n),$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$.

3. Interpolation for mixed couples of Herz spaces and Herz-type Hardy spaces

In this section, we treat the complex interpolation of some mixed couples of Herz spaces and Herz-type Hardy spaces, which are not covered by the above theorems. Our main theorem in this section is as follows.

THEOREM 3.1. *Let* $1 < q_0, q_1 < \infty$, $\theta \in (0, 1)$, $\alpha = (1-\theta)n(1-1/q_0) - \theta n/q_1$, $1/p = 1-\theta$ *and* $1/q = (1-\theta)/q_0 + \theta/q_1$. *Then*

$$\begin{aligned}\left[H\dot{K}_{q_0}^{n(1-1/q_0),1}(\mathbb{R}^n), \dot{K}_{q_1}^{-n/q_1,\infty}(\mathbb{R}^n)\right]_\theta &= \dot{K}_q^{n(1/p-1/q),p}(\mathbb{R}^n)\\ &= \left[\dot{K}_{q_0}^{n(1-1/q_0),1}(\mathbb{R}^n), \dot{K}_{q_1}^{-n/q_1,\infty}(\mathbb{R}^n)\right]_\theta.\end{aligned}$$

Proof. We first prove the first equality. To do this, let us first prove that

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \subseteq \left[H\dot{K}_{q_0}^{n(1-1/q_0),1}(\mathbb{R}^n), \dot{K}_{q_1}^{-n/q_1,\infty}(\mathbb{R}^n)\right]_\theta, \tag{3.1}$$

where we write $\alpha = n(1/p - 1/q)$. If $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, by the proof of Theorem 1 in [23] (see also the proof of Theorem 2.1 in [22]), we can decompose f into

$$f(x) = \sum_{k=-\infty}^{\infty} \lambda_k a_k(x) + \sum_{k=-\infty}^{\infty} \mu_k b_k(x),$$

where $\int_{\mathbb{R}^n} a_k(x)\,dx = 0 = \int_{\mathbb{R}^n} b_k(x)\,dx$,

$$\operatorname{supp} a_k \subset \widetilde{C}_k = \{x\colon 2^{k-1} - 2^k\varepsilon \le |x| \le 2^k + 2^k\varepsilon\}$$

with ε small enough, $\operatorname{supp} b_k \subset (\widetilde{C}_k \cup \widetilde{C}_{k+1})$, $\|a_k\|_{L^q(\mathbb{R}^n)} \le |B(0, 2^{k+1})|^{-\alpha/n}$, $\|b_k\|_{L^q(\mathbb{R}^n)} \le |B(0, 2^{k+2})|^{-\alpha/n}$,

$$0 \le \lambda_k \le c2^{k\alpha} \sum_{j=k-1}^{k+1} \|f\chi_k\|_{L^q(\mathbb{R}^n)}$$

and $0 \le \mu_k \le c2^{k\alpha} \sum_{j=k-1}^{k+2} \|G(f)\chi_k\|_{L^q(\mathbb{R}^n)}$. Here $G(f)$ is the grand maximal function of f in Fefferman and Stein's sense (see the proof of Theorem 2.2 and also [11]). Moreover

$$\begin{aligned}\left\{\sum_{k=-\infty}^{\infty} (\lambda_k^p + \mu_k^p)\right\}^{1/p} &\le c\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)}\right.\\ &\qquad \left. + \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|G(f)\chi_k\|^p_{L^q(\mathbb{R}^n)}\right\}^{1/p}\\ &\le c\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|G(f)\chi_k\|^p_{L^q(\mathbb{R}^n)}\right\}^{1/p}\\ &\le c\|G(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \le c\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)},\end{aligned}$$

because $\alpha = n(1/p - 1/q)$ and $1 < p, q < \infty$; in the last step, we used Theorem 2.1 in [21] (see also Theorem 5.1 in [18]).

Set $\alpha(z) = (1-z)n(1-1/q_0) - zn/q_1$, $1/p(z) = 1 - z$ and $1/q(z) = (1-z)/q_0 + z/q_1$. We define

$$\begin{aligned}F(x,z) &= \sum_{k=-\infty}^{\infty} (\lambda_k)^{p/p(z)}|B(0,2^{k+1})|^{\alpha q/(nq(z)) - \alpha(z)/n}\\ &\quad\times\left[|a_k(x)|^{q/q(z)-1}a_k(x) - P_k(x,z)\right]\\ &\quad+\sum_{k=-\infty}^{\infty} (\mu_k)^{p/p(z)}|B(0,2^{k+2})|^{\alpha q/(nq(z)) - \alpha(z)/n}\\ &\quad\times\left[|b_k(x)|^{q/q(z)-1}b_k(x) - Q_k(x,z)\right],\end{aligned}$$

where

$$P_k(x,z) = \frac{1}{|\widetilde{C}_k|}\int_{\widetilde{C}_k} |a_k(x)|^{q/q(z)-1}a_k(x)\,dx$$

and

$$Q_k(x,z) = \frac{1}{|\widetilde{C}_k \cup \widetilde{C}_{k+1}|}\int_{\widetilde{C}_k\cup\widetilde{C}_{k+1}} |b_k(x)|^{q/q(z)-1}b_k(x)\,dx.$$

Obviously $F(\cdot,\theta) = f(x)$. Next, we need to verify that $F(\cdot, it) \in H\dot{K}_{q_0}^{n(1-1/q_0),1}(\mathbb{R}^n)$ and $F(\cdot, 1+it) \in \dot{K}_{q_1}^{-n/q_1,\infty}(\mathbb{R}^n)$. For the former, let

$$a_k(x,z) = |B(0,2^{k+1})|^{\alpha q/(nq(z))-\alpha(z)/n}\left[|a_k(x)|^{q/q(z)-1}a_k(x) - P_k(x,z)\right]$$

and

$$b_k(x,z) = |B(0,2^{k+2})|^{\alpha q/(nq(z))-\alpha(z)/n}\left[|b_k(x)|^{q/q(z)-1}b_k(x) - Q_k(x,z)\right].$$

Notice that $\int_{\mathbb{R}^n} a_k(x,it)\,dx = 0 = \int_{\mathbb{R}^n} b_k(x,it)\,dx$, $\operatorname{supp} a_k(\cdot,it) \subseteq B_{k+1}$ and

$$\operatorname{supp} b_k(\cdot,it) \subseteq B_{k+2}.$$

Moreover,

$$\begin{aligned}\|a_k(\cdot,it)\|_{L^{q_0}(\mathbb{R}^n)} &\le c|B(0,2^{k+1})|^{\alpha q/(nq_0)-(1-1/q_0)}\|a_k^{q/q_0}\|_{L^{q_0}(\mathbb{R}^n)}\\ &\le c|B(0,2^{k+1})|^{-(1-1/q_0)}\end{aligned}$$

and

$$\|b_k(\cdot,it)\|_{L^{q_0}(\mathbb{R}^n)} \le c|B(0,2^{k+1})|^{-(1-1/q_0)}.$$

Thus, $a_k(\cdot,it)$ and $b_k(\cdot,it)$ are central $(n(1-1/q_0),1)$-atoms up to an absolute constant. Moreover

$$\begin{aligned}\|F(\cdot,it)\|_{H\dot{K}_{q_0}^{n(1-1/q_0),1}(\mathbb{R}^n)} &\le c\sum_{k=-\infty}^{\infty}\left(|\lambda_k|^{p/p(it)} + |\mu_k|^{p/p(it)}\right)\\ &= c\sum_{k=-\infty}^{\infty}\left(|\lambda_k|^{p} + |\mu_k|^{p}\right) \le c\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)},\end{aligned}$$

by [23] and [25]. Thus, $F(\cdot,it) \in H\dot{K}_{q_0}^{n(1-1/q_0),1}(\mathbb{R}^n)$. Similarly we can prove that $F(\cdot,1+it) \in \dot{K}_{q_1}^{-n/q_1,\infty}(\mathbb{R}^n)$. This proves (3.1).

Next, we turn to the proof of the reverse inclusion of (3.1). Notice that by the hypotheses, all relative spaces considered here are Banach spaces. Let

$$f \in \left[H\dot{K}_{q_0}^{n(1-1/q_0),1}(\mathbb{R}^n), \dot{K}_{q_1}^{-n/q_1,\infty}(\mathbb{R}^n)\right]_\theta.$$

Then, by the definition of the space $[\cdot,\cdot]_\theta$, there is an analytic function

$$g \in \mathcal{F}\left[H\dot{K}_{q_0}^{n(1-1/q_0),1}(\mathbb{R}^n), \dot{K}_{q_1}^{-n/q_1,\infty}(\mathbb{R}^n)\right]_\theta$$

such that $g(\theta) = f(x)$; see [3] (pp. 87–88) and [15]. Write $h_k(x,\theta) = g(\theta)\chi_k(x)$ and take $0 < r < 1$; using Lemma 2.21 in [15] (see also [5] and [16]), we obtain

$$|h_k(x,\theta)|^r \le (a_k(x))^{1-\theta}(b_k(x))^\theta,$$

where

$$a_k(x) = \frac{1}{1-\theta}\int_{\mathbb{R}} |h_k(x, it)|^r \mu_0(\theta, t)\, dt$$

and

$$b_k(x) = \frac{1}{\theta}\int_{\mathbb{R}} |h_k(x, 1+it)|^r \mu_1(\theta, t)\, dt.$$

Here $\mu_0(\theta, t)$ and $\mu_1(\theta, t)$ are positive measurable functions and

$$\frac{1}{1-\theta}\int_{\mathbb{R}} \mu_0(\theta, t)\, dt = 1 = \frac{1}{\theta}\int_{\mathbb{R}} \mu_1(\theta, t)\, dt.$$

Therefore,

$$\begin{aligned} \| |h_k(\cdot, \theta)|^r \|_{L^{q/r}(\mathbb{R}^n)} &\le \| (a_k)^{1-\theta} (b_k)^{\theta} \|_{L^{q/r}(\mathbb{R}^n)} \\ &\le \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|h_k(\cdot, it)\|^r_{L^{q_0}(\mathbb{R}^n)} \mu_0(\theta, t)\, dt \right)^{1-\theta} \\ &\quad \times \left(\frac{1}{\theta}\int_{\mathbb{R}} \|h_k(\cdot, 1+it)\|^r_{L^{q_1}(\mathbb{R}^n)} \right)^{\theta}. \end{aligned}$$

Thus

$$\begin{aligned} \|f\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} &= \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|g(\theta)\chi_k\|^p_{L^q(\mathbb{R}^n)} \right\}^{1/p} \\ &= \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \| |h_k(\cdot, \theta)|^r \|^{p/r}_{L^{q/r}(\mathbb{R}^n)} \right\}^{1/p} \\ &\le \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|h_k(\cdot, it)\|^r_{L^{q_0}(\mathbb{R}^n)} \mu_0(\theta, t)\, dt \right)^{(1-\theta)p/r} \right. \\ &\quad \left. \times \left(\frac{1}{\theta}\int_{\mathbb{R}} \|h_k(\cdot, 1+it)\|^r_{L^{q_1}(\mathbb{R}^n)} \mu_1(\theta, t)\, dt \right)^{p\theta/r} \right\}^{1/p} \\ &\le c \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p + knp\theta/q_1} \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|h_k(\cdot, it)\|^r_{L^{q_0}(\mathbb{R}^n)} \mu_0(\theta, t)\, dt \right)^{(1-\theta)p/r} \right\}^{1/p} \\ &= c \left\{ \sum_{k=-\infty}^{\infty} \left[\frac{1}{1-\theta}\int_{\mathbb{R}} \left(2^{kn(1-1/q_0)} \|h_k(\cdot, it)\|_{L^{q_0}(\mathbb{R}^n)}\right)^r \mu_0(\theta, t)\, dt \right]^{1/r} \right\}^{1/p} \\ &\le c \left\{ \frac{1}{1-\theta}\int_{\mathbb{R}} \|g(it)\|_{\dot{K}^{n(1-1/q_0),1}_{q_0}(\mathbb{R}^n)} \mu_0(\theta, t)\, dt \right\}^{1/p} \\ &\le c \sup_{t\in\mathbb{R}} \|g(it)\|^{1-\theta}_{H\dot{K}^{n(1-1/q_0),1}_{q_0}(\mathbb{R}^n)} \le c < \infty. \end{aligned}$$

That is, $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ which completes the proof of the first equality.

For the proof of the second equality, notice that

$$H\dot{K}_{q_0}^{n(1-1/q_0),1}(\mathbb{R}^n) \subset \dot{K}_{q_0}^{n(1-1/q_0),1}(\mathbb{R}^n);$$

by (3.1), we immediately obtain

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \subset \left[\dot{K}_{q_0}^{n(1-1/q_0),1}(\mathbb{R}^n), \dot{K}_{q_1}^{-n/q_1,\infty}(\mathbb{R}^n)\right]_\theta.$$

The reverse inclusion can be proved by a similar procedure to the one used in proving the reverse inclusion in (3.1). We omit the details.

This finishes the proof of Theorem 3.1. □

Remark 3.1. Observe that the proof of the first part of this theorem can also be done in a way similar to the proof of Theorem 2.2. We preferred to give a more elementary proof here in the case of couples of spaces.

A more general theorem can be proved, using a method similar to the one used to prove Theorem 3.1.

THEOREM 3.2. *Let* $1 < q_0, q_1 < \infty, 0 < p_0, p_1 \le \infty, -n/q_1 < \alpha_1 < n(1 - 1/q_1), \theta \in (0,1), \alpha = (1-\theta)n(1-1/q_0) + \theta\alpha_1, 1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $1/q = (1-\theta)/q_0 + \theta/q_1$. *Then*

$$\begin{aligned}\left[H\dot{K}_{q_0}^{n(1-1/q_0),p_0}(\mathbb{R}^n), \dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)\right]_\theta &= \dot{K}_q^{\alpha,p}(\mathbb{R}^n) \\ &= \left[\dot{K}_{q_0}^{n(1-1/q_0),p_0}(\mathbb{R}^n), \dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)\right]_\theta.\end{aligned}$$

Acknowledgement. We would like to express our deep thanks to Professor Guido Weiss for his continuing encouragement and support. We also like to thank the Mathematics Department of Washington University in St. Louis for their hospitality while this work was developed.

REFERENCES

1. R. B. Ash. *Complex variables*, Academic Press, New York, 1971.
2. A. Baernstein II and E. T. Sawyer, *Embedding and multiplier theorems for* $H^p(\mathbb{R}^n)$, Mem. Amer. Math. Soc., vol. 59, no. 318, Amer. Math. Soc., Providence, R.I., 1985.
3. J. Bergh and J. Löfström, *Interpolation spaces*, Springer-Verlag, New York, 1976.
4. A. Beurling, *Construction and analysis of some convolution algebras*, Ann. Inst. Fourier Grenoble **14** (1964), 1–32.
5. A. P. Calderón, *Intermediate spaces and interpolation: The complete method*, Studia Math. **24** (1964), 113–190.
6. A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution, II*, Adv. in Math. **24** (1977), 101–171.
7. Y. Z. Chen and K. S. Lau, *On some new classes of Hardy spaces*, J. Funct. Anal. **84** (1989), 255–278.
8. R. Coifman, M. Cwikel, R. Rochberg, Y. Sagher and G. Weiss, *A theory of complex interpolation for families of Banach spaces*, Adv. in Math. **43** (1982), 203–229.

9. ________, *Complex interpolation for families of Banach spaces*, Proc. Symposia Pure Math., vol. 35, Part 2, Amer. Math. Soc., Providence, R.I., 1979, pp. 269–289.
10. P. L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.
11. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.
12. H. G. Feightinger, *An elementary approach to Wiener's third Tauberian theorem for the Euclidean n-spaces*, Proceedings of Conference at Cortona 1984, Symposia Mathematica, vol. 29, Academic Press, New York, 1987, pp. 267–301.
13. T. M. Flett, *Some elementary inequalities for integrals with applications to Fourier transforms*, Proc. London Math. Soc. (3) **29** (1974), 538–556.
14. J. García-Cuerva, *Hardy spaces and Beurling algebras*, J. London Math. Soc. (2) **39** (1989), 499–513.
15. J. García-Cuerva and M.-J. L. Herrero, *A theory of Hardy spaces associated to Herz spaces*, Proc. London Math. Soc. (3) **69** (1994), 605–628.
16. E. Hernández, *Topics in complex interpolation*, Ph.D. thesis, Washington University, St. Louis, Missouri, 1981.
17. ________, *Intermediate spaces and the complex method of interpolation for families of Banach spaces*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. Serie IV **XIII** (1986), 245–266.
18. E. Hernández and D. Yang, *Interpolation of Herz spaces and applications*, Math. Nachr., to appear.
19. C. Herz, *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms*, J. Math. Mech. **18** (1968), 283–324.
20. S. Janson and P. W. Jones, *Interpolation between H^p spaces: The complex method*, J. Funct. Anal. **48** (1982), 58–80.
21. X. Li and D. Yang, *Boundedness of some sublinear operators on Herz spaces*, Illinois J. Math. **40** (1996), 484–501.
22. S. Lu and D. Yang, *The local versions of $H^p(\mathbb{R}^n)$ spaces at the origin*, Studia Math. **116** (1995), 103–131.
23. ________, *The weighted Herz-type Hardy spaces and its applications*, Sci. in China (Ser. A) **38** (1995), 662–673.
24. ________, *The decomposition of the weighted Herz spaces and its application*, Sci. in China (Ser. A) **38** (1995), 147–158.
25. ________, *Some characterizations of weighted Herz-type Hardy spaces and its applications*, Acta Math. Sinica (New Ser.) **13** (1997), 45–58.
26. ________, *Oscillatory singular integrals on Hardy spaces associated with Herz spaces*, Proc. Amer. Math. Soc. **123** (1995), 1695–1709.
27. A. Tabacco Vignati, *Interpolation of quasi-Banach spaces*, Ph.D. thesis, Washington University, St. Louis, Missouri, 1986.
28. ________, *Complex interpolation for families of quasi-Banach spaces*, Indiana Univ. Math. J. **37** (1988), 1–21.
29. ________, *Some techniques for the characterization of intermediate spaces*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. Serie IV **17** (1990), 323–341.
30. ________, *Complex interpolation of infinite families of Hardy classes*, Rend. Sem. Mat. Univers. Pol. Tor. **47** (1989), 71–86.
31. A. Tabacco Vignati and M. Vignati, *Equivalent interpolation methods for families of quasi-Banach lattices*, Boll. U. M. I. (7) **4-b** (1990), 1–11.

Eugenio Hernández, Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain
ehernan@ccuam3.sdi.uam.es

Dachun Yang, Department of Mathematics, Beijing Normal University, 100875 Beijing, The People's Republic of China
dcyang@bnu.edu.cn