GENERALIZATIONS OF SOME COMBINATORIAL INEQUALITIES OF H. J. RYSER[1]

BY
Marvin Marcus and William R. Gordon

1. Introduction and results

In a recent interesting paper, H. J. Ryser obtained the following results [1].

Let H be a nonnegative hermitian matrix of rank e and order v with eigenvalues $\lambda_1, \cdots, \lambda_v$, where $\lambda_1 \geqq \cdots \geqq \lambda_e > \lambda_{e+1} = \cdots = \lambda_v = 0$. Let h be an integer, $h > 1$, such that $e \leqq h \leqq v$, and define k and λ by

$$\text{trace }(H) = kh, \qquad \lambda_h \leqq k + (h-1)\lambda \leqq \lambda_1 .$$

Define the matrix B of order h by

$$B = (k - \lambda)I + \lambda J,$$

where I is the identity matrix and J is the matrix all of whose entries are 1's. Let

$$B_0 = B \dotplus 0,$$

where the matrix B_0 of order v is the direct sum of the matrix B of order h and the zero matrix of order $(v - h)$. Let

$$k^* = \text{trace }(H)/v, \qquad \mu = \textstyle\sum_{i=1}^{v} \sum_{j=1}^{v} h_{ij}, \qquad \lambda^* = ((\mu/v) - k^*)/(v-1).$$

Define the matrix B^* of order v by

$$B^* = (k^* - \lambda^*)I + \lambda^* J.$$

Finally let $C_r(A)$ denote the r^{th} compound matrix of A, and let $P_r(A)$ denote the r^{th} induced power matrix of A (for definitions of C_r and P_r see [1]). Then we have

Theorem 1. *The matrices H and B_0 satisfy*

$$\text{trace }(C_r(H)) \leqq \text{trace }(C_r(B_0)) \qquad (1 \leqq r \leqq v).$$

Equality holds for $r = 1, h+1, \cdots, v$. If $k + (h-1)\lambda \neq 0$ and equality holds for an r, $1 < r \leqq h$, or $k + (h-1)\lambda = 0$ and equality holds for an r, $1 < r < h$, then there exists a unitary U such that $H = U^{-1}B_0 U$.

Theorem 2. *The matrices H and B^* satisfy*

$$\text{trace }(C_r(H)) \leqq \text{trace }(C_r(B^*)) \qquad (1 \leqq r \leqq v).$$

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Equality holds for $r = 1$. *If* $k^* + (v - 1)\lambda^* \neq 0$ *and equality holds for* $1 < r \leqq v$, *or if* $k^* + (v - 1)\lambda^* = 0$ *and equality holds for an* r, $1 < r < v$, *then* $H = B^*$.

THEOREM 3. *The matrices* H *and* B_0 *satisfy*

$$\text{trace } (P_r(H)) \geqq \text{trace } (P_r(B_0)) \qquad (1 \leqq r \leqq v).$$

Equality holds for $r = 1$. *If equality holds for an* $r > 1$, *then there exists a unitary* U *such that* $H = U^{-1}B_0 U$.

THEOREM 4. *The matrices* H *and* B^* *satisfy*

$$\text{trace } (P_r(H)) \geqq \text{trace } (P_r(B^*)) \qquad (1 \leqq r \leqq v).$$

Equality holds for $r = 1$. *If equality holds for an* $r > 1$, *then* $H = B^*$.

Ryser applies some of these results to certain combinatorial problems such as the determination of necessary and sufficient conditions for both the existence of complete finite oriented graphs and the existence of solutions to the v, k, λ problem.

In what follows we state and prove generalizations of Ryser's results to a fairly wide class of concave and convex functions.

Throughout the remainder of this paper we assume that all vectors mentioned have nonnegative coordinates.

Let f be a nonnegative concave symmetric function on v-tuples of nonnegative reals. Suppose that whenever $\theta a + (1 - \theta)b \in G_f = \{x : f(x) > 0\}$, $0 < \theta < 1$, then $f(\theta a + (1 - \theta)b) = \theta f(a) + (1 - \theta)f(b)$ if and only if a and b are proportional ($a \sim b$); then f is called *strictly concave*. Similarly, if f is convex and satisfies this last condition, then f is called strictly convex. If A is a matrix with eigenvalues $\mu_1 \geqq 0, \cdots, \mu_v \geqq 0$, let $f(A)$ denote $f(\mu_1, \cdots, \mu_v)$. Then our main results are the following.

THEOREM 5. *If* f *is concave (convex), then the matrices* H *and* B_0 *satisfy*

$$f(H) \leqq (\geqq) f(B_0).$$

If f *is strictly concave (convex), and if* $(\lambda_1, \cdots, \lambda_v) \in G_f$, *then equality holds if and only if* H *and* B_0 *have the same eigenvalues. If* f *is strictly concave (convex), and if for some integer* z, G_f *is the set of (nonnegative) vectors with at least* z *positive coordinates, and if* $k + (h - 1)\lambda \neq 0$ *and* $z \leqq h$, *or* $k + (h - 1)\lambda = 0$ *and* $z < h$, *then* $f(H) = f(B_0)$ *if and only if* H *and* B_0 *have the same eigenvalues.*

THEOREM 6. *If* f *is concave (convex), then the matrices* H *and* B^* *satisfy*

$$f(H) \leqq (\geqq) f(B^*).$$

If f *is strictly concave (convex), and if* $(\lambda_1, \cdots, \lambda_v) \in G_f$, *then equality holds if and only if* $H = B^*$. *If* f *is strictly concave (convex), and if for some integer*

z, G_f is the set of (nonnegative) vectors with at least z positive coordinates, and if $k^* + (v-1)\lambda^* \neq 0$ and $z \leqq v$, or $k^* + (v-1)\lambda^* = 0$ and $z < v$, then $f(H) = f(B^*)$ if and only if $H = B^*$.

II. Proofs

An n-square matrix A is *doubly stochastic* if the elements of A are nonnegative and $\sum_{i=1}^{n} a_{ij} = 1$ $(j = 1, \cdots, n)$ and $\sum_{j=1}^{n} a_{ij} = 1$ $(i = 1, \cdots, n)$. Birkhoff's theorem [2] states that the set of n-square doubly stochastic matrices is equal to the convex hull of the set of n-square permutation matrices.

Let f be the concave function defined in Section I. The following lemmas can be modified in the obvious way when f is convex.

LEMMA 1. *Let $a^{(1)}, \cdots, a^{(p)}$ be nonzero v-tuples of nonnegative reals. If $\sum_{j=1}^{p} \theta_j = 1$ and $0 < \theta_j$ $(j = 1, \cdots, p)$, then*

$$f\left(\sum_{j=1}^{p} \theta_j a^{(j)}\right) \geqq \sum_{j=1}^{p} \theta_j f(a^{(j)}).$$

If f is strictly concave and if $\sum_{j=1}^{p} \theta_j a^{(j)} \in G_f$, then equality holds if and only if $a^{(1)} \sim \cdots \sim a^{(p)}$.

Proof. Clearly $f(\sum_{j=1}^{p} \theta_j a^{(j)}) \geqq \sum_{j=1}^{p} \theta_j f(a^{(j)})$.

Suppose that f is strictly concave, that $\sum_{j=1}^{p} \theta_j a^{(j)} \in G_f$, and that $f(\sum_{j=1}^{p} \theta_j a^{(j)}) = \sum_{j=1}^{p} \theta_j f(a^{(j)})$. Now

$$\begin{aligned} f\left(\sum_{j=1}^{p} \theta_j a^{(j)}\right) &= f\left(\theta_i a^{(i)} + (1-\theta_i) \sum_{j\neq i} (1-\theta_i)^{-1}\theta_j a^{(j)}\right) \\ &\geqq \theta_i f(a^{(i)}) + (1-\theta_i) f\left(\sum_{j\neq i} (1-\theta_i)^{-1}\theta_j a^{(j)}\right) \\ &\geqq \theta_i f(a^{(i)}) + (1-\theta_i) \sum_{j\neq i} (1-\theta_i)^{-1}\theta_j f(a^{(j)}) \\ &= \sum_{j=1}^{p} \theta_j f(a^{(j)}) \qquad (i = 1, \cdots, p). \end{aligned}$$

Therefore, since we are assuming equality, we have that

$$f\left(\theta_i a^{(i)} + (1-\theta_i) \sum_{j\neq i} (1-\theta_i)^{-1}\theta_j a^{(j)}\right) = \theta_i f(a^{(i)}) + (1-\theta_i) f\left(\sum_{j\neq i} (1-\theta_i)^{-1}\theta_j a^{(j)}\right)$$

for $(i = 1, \cdots, p)$, and so, since f is strictly concave,

$$a^{(i)} \sim \sum_{j\neq i} (1-\theta_i)^{-1}\theta_j a^{(j)} \qquad (i = 1, \cdots, p).$$

Thus there exist nonzero numbers μ_i $(i = 1, \cdots, n)$ such that

$$\mu_i a^{(i)} = \sum_{j\neq i} (1-\theta_i)^{-1}\theta_j a^{(j)} \qquad (i = 1, \cdots, n).$$

Therefore

$$\mu_i(1-\theta_i) a^{(i)} = \sum_{j\neq i} \theta_j a^{(j)},$$

$$\mu_i(1-\theta_i) a^{(i)} + \theta_i a^{(i)} = \sum_{j=1}^{p} \theta_j a^{(j)} \qquad (i = 1, \cdots, n).$$

Hence $a^{(i)} \sim \sum_{j=1}^{p} \theta_j a^{(j)}$ $(i = 1, \cdots, n)$, and so $a^{(1)} \sim \cdots \sim a^{(p)}$.

LEMMA 2. *If H is a nonnegative hermitian matrix with eigenvalues λ_1, $\lambda_2 \cdots, \lambda_n$, and if $\{x_1, \cdots, x_n\}$ is an orthonormal set of vectors, then*

$$f(\lambda_1, \cdots, \lambda_n) \leqq f((Hx_1, x_1), \cdots, (Hx_n, x_n)).$$

If f is strictly concave, and if $(\lambda_1, \cdots, \lambda_n) \in G_f$, then equality holds if and only if $((Hx_i, x_j))$ is a diagonal matrix.

Proof. The proof of the inequality is a familiar one [3]. However it is brief enough to include here and makes the discussion of the case of equality easier. If $H = 0$, the result is trivially true; hence assume $H \neq 0$. Let $K = ((Hx_i, x_j))$. Then $K = XH\bar{X}^T$, where X is the unitary matrix whose row vectors are, in order, $x_1, \cdots, x_n$. Thus H and K have the same eigenvalues $\lambda_1, \cdots, \lambda_n$. Let $u_1, \cdots, u_n$ be an orthonormal set of eigenvectors of H corresponding to $\lambda_1, \cdots, \lambda_n$ respectively. Let $\lambda = (\lambda_1, \cdots, \lambda_v)$, and let $\alpha_1, \cdots, \alpha_v$ be the main diagonal elements of K. Then

$$\begin{aligned}\alpha_i = (Hx_i, x_i) &= (H\textstyle\sum_{s=1}^{n} (x_i, u_s)u_s, \sum_{t=1}^{n} (x_i, u_t)u_t) \\ &= (\textstyle\sum_{s=1}^{n} (x_i, u_s)\lambda_s u_s, \sum_{t=1}^{n} (x_i, u_t)u_t) \\ &= \textstyle\sum_{j=1}^{n} |(x_i, u_j)|^2 \lambda_j \qquad (i = 1, \cdots, n).\end{aligned}$$

Let $S = (s_{ij}) = (|(x_i, u_j)|^2)$. Then clearly S is doubly stochastic, and $\alpha_i = (S_i, \lambda)$ where S_i is the i^{th} row of S $(i = 1, \cdots, n)$.

By Birkhoff's theorem there exist permutation matrices $P^{(1)}, \cdots, P^{(p)}$ such that $S = \sum_{j=1}^{p} \theta_j P^{(j)}$, where $\sum_{j=1}^{p} \theta_j = 1$ and $0 < \theta_j < 1$ $(j = 1, \cdots, p)$. Thus $\alpha_i = (S_i, \lambda) = ((\sum_{j=1}^{p} \theta_j P^{(j)})_i, \lambda)$. Therefore

$$f((Hx_1, x_1), \cdots, (Hx_n, x_n)) = f(S\lambda) = f(\textstyle\sum_{j=1}^{p} \theta_j P^{(j)}\lambda).$$

But f is concave, and so, by Lemma 1, $f(\sum_{j=1}^{p} \theta_j P^{(j)}\lambda) \geqq \sum_{j=1}^{p} \theta_j f(P^{(j)}\lambda)$. But $P^{(j)}$ is a permutation matrix, and f is a symmetric function, and therefore $f(P^{(j)}\lambda) = f(\lambda)$ $(j = 1, \cdots, p)$. Thus

$$f((Hx_1, x_1), \cdots, (Hx_n, x_n)) \geqq f(\lambda_1, \cdots, \lambda_n).$$

Suppose now that f is strictly concave, that $(\lambda_1, \cdots, \lambda_n) \in G_f$, and that $f((Hx_1, x_1), \cdots, (Hx_n, x_n)) = f(\lambda_1, \cdots, \lambda_n)$. This means that

$$f(\textstyle\sum_{j=1}^{p} \theta_j P^{(j)}\lambda) = \sum_{j=1}^{p} \theta_j f(P^{(j)}\lambda),$$

and so, by Lemma 1, $\theta_1 P^{(1)}\lambda \sim \cdots \sim \theta_p P^{(p)}\lambda$. Thus, setting $P^{(1)} = P$, we have that $\theta_i P^{(i)}\lambda = d_i P\lambda$ for some number d_i $(i = 1, \cdots, p)$. No d_i can be zero; otherwise all the d_i would be zero, and hence $\sum_{j=1}^{p} \theta_j P^{(j)}\lambda$ would be zero. This would imply that $(\lambda_1, \cdots, \lambda_n) = 0$ which would contradict the assumption that $H \neq 0$. Thus

$$\textstyle\sum_{j=1}^{p} \theta_j P^{(j)}\lambda = S\lambda = dP\lambda,$$

where $d = \sum_{i=1}^{p} d_i$. But $JS\lambda = J\lambda = J\,dP\lambda = dJ\lambda$, and so $d = 1$. Therefore $S\lambda = P\lambda$. Thus $\alpha_i = \lambda_{\sigma(i)}$ $(i = 1, \cdots, n)$ where σ is the permutation corresponding to the permutation matrix P. Since

$$\sum_{i=1}^{n} \alpha_i^2 = \sum_{i=1}^{n} \lambda_i^2 = \operatorname{trace}(K\bar{K}^T) = \sum_{i=1}^{n} \alpha_i^2 + \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} |k_{ij}|^2,$$

it follows that $K = \operatorname{diag}(\alpha_1, \cdots, \alpha_n)$, i.e., $K = ((Hx_i, x_j))$ is a diagonal matrix.

LEMMA 3. *Let $a = (a_1, \cdots, a_v)$, and let n be an integer, $1 \leqq n \leqq v$. Then*

$$f(a) \leqq f\big((1/n)\sum_{j=1}^{n} a_j, \cdots, (1/n)\sum_{j=1}^{n} a_j, a_{n+1}, \cdots, a_v\big).$$

If f is strictly concave, and if $a \in G_f$, then equality holds if and only if $a_1 = \cdots = a_n$.

Proof. The result is trivially true if $a = 0$, so assume henceforth that $a \neq 0$. Let $P = P_n \dotplus I_{v-n}$, where P_n is the permutation matrix corresponding to the full n-cycle and I_{v-n} is the $(v-n)$-order identity matrix.

Since f is symmetric,

$$\begin{aligned} f(a) &= f(Pa) = (1/n)\sum_{j=1}^{n} f(P^j a) \leqq f\big((1/n)\sum_{j=1}^{n} P^j a\big) \\ &= f\big((1/n)\sum_{j=1}^{n} a_j, \cdots, (1/n)\sum_{j=1}^{n} a_j, a_{n+1}, \cdots, a_v\big). \end{aligned}$$

Suppose now that f is strictly concave, that $a \in G_f$, and that

$$f(a) = f\big((1/n)\sum_{j=1}^{n} a_j, \cdots, (1/n)\sum_{j=1}^{n} a_j, a_{n+1}, \cdots, a_v\big).$$

Then by Lemma 1, $Pa \sim \cdots \sim P^n a = a$. Thus $Pa = \mu a$ for some number μ. Now $\mu \neq 0$; otherwise then $Pa = 0$, $a = 0$, contradicting the hypothesis that $a \neq 0$. Hence we may assume $\mu \neq 0$, and so $a_t = \mu a_{t-1 \pmod n}$ $(t = 1, \cdots, n)$. Therefore $a_1 = \mu^n a_1$. If $a_1 = 0$, then

$$a_2 = \mu a_1 = 0, \qquad a_3 = \mu a_2 = 0, \qquad \cdots, \qquad a_n = \mu a_{n-1} = 0,$$

i.e., $a_1 = \cdots = a_n = 0$. Similarly we can show that if $a_i = 0$ $(1 < i \leqq n)$, then $a_1 = \cdots = a_n = 0$. Therefore suppose that no $a_i = 0$ for $1 \leqq i \leqq n$. Then $\mu^n = 1$, and, since $Pa \geqq 0$, $a \geqq 0$, and neither is zero, it follows that $\mu = 1$. Therefore $a_1 = \cdots = a_n$.

We prove Theorems 5 and 6 when f is concave; the arguments for f convex are of course identical.

Proof of Theorem 5. Let $u_1, \cdots, u_v$ be an orthonormal set of eigenvectors of H corresponding to $\lambda_1, \cdots, \lambda_v$ respectively. Since

$$\lambda_h \leqq k + (h-1)\lambda \leqq \lambda_1,$$

there is a unit vector x_1 in the space spanned by $u_1, \cdots, u_h$ such that $(Hx_1, x_1) = k + (h-1)\lambda$. Choose orthonormal $x_2, \cdots, x_h$ in the intersection of the space spanned by $u_2, \cdots, u_h$ and the orthogonal complement of x_1; let $x_{h+1} = u_{h+1}, \cdots, x_v = u_v$. Then by Lemma 2

$$\begin{aligned} f(H) &= f(\lambda_1, \cdots, \lambda_v) \\ &\leqq f((Hx_1, x_1), \cdots, (Hx_v, x_v)) \\ &= f(k + (h-1)\lambda, (Hx_2, x_2), \cdots, (Hx_v, x_v)) \\ &= f(k + (h-1)\lambda, (Hx_2, x_2), \cdots, (Hx_h, x_h), 0, \cdots, 0). \end{aligned}$$

By Lemma 3

$$\begin{aligned} f(k + (h-1)\lambda, &(Hx_2, x_2), \cdots, (Hx_h, x_h), 0, \cdots, 0) \\ \leqq f(k + (h-1)\lambda, &(h-1)^{-1}\textstyle\sum_{j=2}^{h} (Hx_j, x_j), \cdots, \\ &(h-1)^{-1}\textstyle\sum_{j=2}^{h} (Hx_j, x_j), 0, \cdots, 0). \end{aligned}$$

Now

$$\begin{aligned} (k - \lambda) &= \text{trace } (H)/h - \lambda \\ &= h^{-1}(\textstyle\sum_{j=1}^{v}(Hx_j, x_j)) - \lambda \\ &= h^{-1}(k + (h-1)\lambda + \textstyle\sum_{j=2}^{h}(Hx_j, x_j) - \lambda h) \\ &= h^{-1}(k - \lambda + \textstyle\sum_{j=2}^{h}(Hx_j, x_j)). \end{aligned}$$

Therefore

$$h^{-1}(h-1)(k-\lambda) = h^{-1}\textstyle\sum_{j=2}^{h}(Hx_j, x_j),$$

and so

$$(h-1)^{-1}\textstyle\sum_{j=2}^{h}(Hx_j, x_j) = k - \lambda.$$

Thus

$$f(H) \leqq f(k + (h-1)\lambda, k - \lambda, \cdots, k - \lambda, 0, \cdots, 0).$$

But $B_0 = ((k - \lambda)I + \lambda J) \dotplus 0$, and so

$$k + (h-1)\lambda, \quad k - \lambda, \quad \cdots, \quad k - \lambda, \quad 0, \cdots, 0$$

are the eigenvalues of B_0. Thus $f(H) \leqq f(B_0)$.

Suppose now that f is strictly concave, that $(\lambda_1, \cdots, \lambda_v) \in G_f$, and that $f(H) = f(B_0)$. Then

$$f(\lambda_1, \cdots, \lambda_v) = f((Hx_1, x_1), \cdots, (Hx_v, x_v)),$$

and so by Lemma 2, $((Hx_i, x_j))$ is a diagonal matrix. Since we are assuming that $(\lambda_1, \cdots, \lambda_v) \in G_f$, it follows (in the case of equality) that

$$\begin{aligned} (k + (h-1)\lambda, &(h-1)^{-1}\textstyle\sum_{j=2}^{h}(Hx_j, x_j), \cdots, \\ &(h-1)^{-1}\textstyle\sum_{j=2}^{h}(Hx_j, x_j), 0, \cdots, 0) \in G_f. \end{aligned}$$

Since in the case of equality

$$\begin{aligned} f(k + (h-1)\lambda, &(h-1)^{-1}\textstyle\sum_{j=2}^{h}(Hx_j, x_j), \cdots, \\ &(h-1)^{-1}\textstyle\sum_{j=2}^{h}(Hx_j, x_j), 0, \cdots, 0) \\ &= f(k + (h-1)\lambda, (Hx_2, x_2), \cdots, (Hx_h, x_h), 0, \cdots, 0), \end{aligned}$$

it follows by Lemma 3 that

$$(Hx_2, x_2) = \cdots = (Hx_h, x_h) = k - \lambda.$$

Thus $((Hx_i, x_j))$ and B_0 have the same eigenvalues, and therefore, since H and $((Hx_i, x_j))$ are unitarily similar, it follows that H and B_0 have the same eigenvalues.

Suppose now that f is strictly concave, that there exists an integer z such that G_f is the set of x with at least z positive coordinates, and suppose that $f(H) = f(B_0)$.

We have just discussed the case in which $(\lambda_1, \cdots, \lambda_v) \in G_f$. Therefore let us assume that $(\lambda_1, \cdots, \lambda_v) \notin G_f$. This means that

$$f(\lambda_1, \cdots, \lambda_v) = f(k + (h-1)\lambda, k - \lambda, \cdots, k - \lambda, 0, \cdots, 0) = 0.$$

If $k + (h-1)\lambda \neq 0$ and $z \leqq h$, then $k - \lambda = 0$; otherwise

$$(k + (h-1)\lambda, k - \lambda, \cdots, k - \lambda, 0, \cdots, 0)$$

would have $h \geqq z$ positive coordinates. This would mean that it would belong to G_f, which is a contradiction. Therefore $\lambda = k = \text{trace}\,(H)/h$.

Now $\lambda_h \leqq k + (h-1)\lambda = kh \leqq \lambda_1$. But $kh = \text{trace}\ (H) = \sum_{j=1}^{v} \lambda_j$. Thus $\lambda_h \leqq \sum_{j=1}^{v} \lambda_j \leqq \lambda_1$. But $\lambda_j \geqq 0$ $(j = 1, \cdots, v)$, and therefore it follows that $\lambda_2 = \cdots = \lambda_v = 0$ and $k + (h-1)\lambda = \lambda_1$. Therefore $((Hx_i, x_j))$ and B_0 have the same eigenvalues, and consequently H and B_0 have the same eigenvalues.

If $k + (h-1)\lambda = 0$ and $h > z$, then it follows that $(k - \lambda) = 0$ and hence $\lambda_1 = \cdots = \lambda_v = 0$, i.e., H and B_0 are both the zero matrix.

Proof of Theorem 6. Theorem 5 with $h = v$ implies that $f(H) \geqq f(B^*)$.

Suppose now that f is strictly concave, that $(\lambda_1, \cdots, \lambda_v) \in G_f$, and that $f(H) = f(B^*)$. Let $x_1 = (1/\sqrt{v}, \cdots, 1/\sqrt{v})$. Then $(x_1, x_1) = 1$ and

$$(Hx_1, x_1) = \mu/v = k^* + (v-1)\lambda^*.$$

Complete x_1 to an orthonormal basis $\{x_1, \cdots, x_v\}$. Then, by the argument in the proof of Theorem 5, $((Hx_i, x_j))$ is a diagonal matrix whose eigenvalues are $k^* + (v-1)\lambda^*, k^* - \lambda^*, \cdots, k^* - \lambda^*$, and in fact

$$((Hx_i, x_j)) = \text{diag}\,(k^* + (v-1)\lambda^*, k^* - \lambda^*, \cdots, k^* - \lambda^*).$$

Now $(B^*x_1, x_1) = (k^* - \lambda^*) + \lambda^*(Jx_1, x_1) = k^* + (v-1)\lambda^*$, and

$$\begin{aligned}(B^*x_i, x_j) &= (k^* - \lambda^*)\delta_{ij} + \lambda^*(Jx_i, x_j)\\ &= k^* - \lambda^* \quad \text{if} \quad i = j \neq 1,\\ &= 0 \qquad\qquad \text{if} \quad i \neq j.\end{aligned}$$

Thus

$$((B^*x_i, x_j)) = \text{diag}\,(k^* + (v-1)\lambda^*, k^* - \lambda^*, \cdots, k^* - \lambda^*) = ((Hx_i, x_j))$$

and $H = B^*$.

Suppose now that f is strictly concave, that there exists an integer z such that G_f is the set of x with at least z positive coordinates, and that $f(H) = f(B^*)$. We have just considered the case in which $(\lambda_1, \cdots, \lambda_v) \in G_f$. Therefore assume that $(\lambda_1, \cdots, \lambda_v) \notin G_f$. This means that

$$f(\lambda_1, \cdots, \lambda_v) = f(k^* + (v-1)\lambda^*, k^* - \lambda^*, \cdots, k^* - \lambda^*) = 0.$$

If $k^* + (v-1)\lambda^* \neq 0$ and $z \leqq v$, then $k^* - \lambda^* = 0$; otherwise

$$(k^* + (v-1)\lambda^*, k^* - \lambda^*, \cdots, k^* - \lambda^*)$$

would have $v \geqq z$ positive coordinates and would therefore belong to G_f, which is a contradiction. Thus $k^* = \lambda^*$. Now clearly

$$\text{trace } (H) = \text{trace } (B^*) = k^* + (v-1)\lambda^* = \textstyle\sum_{i=1}^{v} \lambda_i = \mu/v.$$

It follows, since H is nonnegative hermitian, that $\lambda_v \leqq \mu/v \leqq \lambda_1$. Thus $\lambda_v \leqq \sum_{i=1}^{v} \lambda_i \leqq \lambda_1$. Once again $\lambda_i \geqq 0$ for $i = 1, \cdots, v$. Consequently $\lambda_1 = k^* + (v-1)\lambda^*$ and $\lambda_2 = \cdots = \lambda_v = 0$. Thus

$$((Hx_i, x_j)) = ((B^* x_i, x_j)),$$

and so $H = B^*$. If $k^* + (v-1)\lambda^* = 0$ and $z < v$, then clearly $k^* - \lambda^* = 0$. Thus H and B^* are both equal to the zero matrix. This completes the proof.

III. Applications

In this section we describe the region G_f for various choices of f, indicate how Ryser's results follow from ours, and exhibit two applications of the results.

Let $f(a) = (E_r(a)/E_{r-p}(a))^{1/p}$ for $0 < p \leqq r$, where $E_r(a)$ is the r^{th} elementary symmetric function, i.e.,

$$E_r(a) = \sum_{\substack{i_1 + \cdots + i_v = r \\ 0 \leqq i_j \leqq 1}} a_1^{i_1} \cdots a_v^{i_v}.$$

If a has fewer than r positive coordinates, then clearly $E_r(a) = 0$, and so $f(a) = 0$. If a has at least r positive coordinates, then $E_r(a) > 0$, and certainly $E_{r-p}(a) > 0$. Thus in this case G_f is the set of nonnegative vectors with at least r positive coordinates. It has been shown [4] that f is concave for nonnegative vectors a and strictly concave. Since

$$\text{trace } (C_r(H)) = E_r(\lambda_1, \cdots, \lambda_v),$$

Theorems 5 and 6 with $f(a) = (E_r(a)/E_{r-p}(a))^{1/p}$ (with $p = r$) imply Ryser's results, Theorems 1 and 2, respectively.

Now let $f(a) = (h_r(a))^{1/r}$, where $h_r(a)$ is the completely symmetric function of degree r, i.e.,

$$h_r(a) = \sum_{\substack{i_1 + \cdots + i_v = r \\ 0 \leqq i_j}} a_1^{i_1} \cdots a_v^{i_v}.$$

If a has at least one positive coordinate, then clearly $f(a) > 0$, and if $a = 0$, then $f(a) = 0$. Thus in this case G_f is the set of nonnegative vectors with at least one positive coordinate. It is known [5] that $h_r^{1/r}(a)$ is convex for nonnegative a and is strictly convex. Since trace $(P_r(H)) = h_r(\lambda_1, \cdots, \lambda_v)$, once again Theorems 5 and 6 with $f(a) = (h_r(a))^{1/r}$ imply Ryser's results, Theorems 3 and 4 respectively.

Let $f(a) = T_n^{1/n}(a)$, where

$$T_n(a) = \sum_{\substack{i_1+\cdots+i_v=n\\ 0\leqq i_j}} \delta_{i_1} \cdots \delta_{i_v} a_1^{i_1} \cdots a_v^{i_v}$$

where

$$\begin{aligned} \delta_i &= \binom{k}{i} && \text{if} \quad k > 0, \\ &= (-1)^i\binom{k}{i} && \text{if} \quad k < 0, \end{aligned}$$

and k is any number, provided that if k is positive and not an integer then $n < k + 1$. Suppose k is a positive integer, and let

$$\begin{aligned} m &= [n/k] && \text{if} \quad k \text{ divides } n, \\ &= [n/k] + 1 && \text{if} \quad k \text{ does not divide } n. \end{aligned}$$

(Here $[x]$ denotes the greatest integer in x.) If $\delta_{i_j} = \binom{k}{i_j}$ is to be defined, then i_j can not be greater than k. This means that no exponent in a term $\delta_{i_1} \cdots \delta_{i_k} a_1^{i_1} \cdots a_v^{i_v}$ can be greater than k. On the other hand $i_1, \cdots, i_v$ must sum to n. Thus if k divides n, then the minimum number of the exponents which must be positive is $[n/k]$, and these exponents will each be k. If k does not divide n, then the minimum number of the exponents which must be positive is $[n/k] + 1$, and of these $[n/k]$ will be k and one will be between 0 and k. Thus if k is a positive integer, then $f(a) = 0$ if a has fewer than m positive coordinates, and $f(a) > 0$ if a has at least m positive coordinates. Therefore if k is a positive integer, then G_f is the set of nonnegative vectors with at least m positive coordinates. Suppose now that k is positive but not an integer. Recall that in this case we restrict n to being less than $k + 1$. Thus if a has at least one positive coordinate, say a_t, then $T_n(a)$ will be positive since the choice $(i_1, \cdots, i_t, \cdots, i_v) = (0, \cdots, 0, n, 0, \cdots, 0)$ assures us that there is at least one positive term in the summation. So in the case k positive and k not an integer, G_f is the set of nonnegative vectors with at least one positive coordinate. Suppose now that k is negative. Then $\binom{k}{t}$ is defined for all nonnegative integers t. Clearly $(-1)^t \binom{k}{t}$ is positive for all nonnegative t. Thus if a has at least one positive coordinate, say a_t, then $T_n(a)$ is positive since, as before, the choice $(i_1, \cdots, i_t, \cdots, i_v) = (0, \cdots, 0, n, 0, \cdots, 0)$ assures us that at least one of the terms in $T_n(a)$ is positive. Thus in the case that k is negative, G_f is the set of vectors with at least one positive coordinate. Whiteley shows [5] that $T_n^{1/n}$ is convex for $k < 0$ and concave for $k > 0$. Clearly $T_n^{1/n}$ is symmetric. It can be shown that $T_n^{1/n}$ is strictly convex for $k < 0$ and strictly concave for $k > 0$. It is interesting to note that for $k = 1$, $T_n(a) = E_n(a)$, and for $k = -1$,

$$T_n(a) = h_n(a).$$

Thus Theorems 5 and 6 can be specialized directly to the functions $T_n^{1/n}$.

Ryser applies Theorem 2 to the problem of determining whether or not a finite oriented graph is complete. A graph consists of a nonnull set V of objects called points and a set W of objects called lines, the two sets having no elements in common. With each line there are associated just two distinct points, called its endpoints. The line is said to join its endpoints. Isolated points, i.e., points having no lines associated with them, are permitted, and two or more lines may join the same endpoints. If the number of lines joining distinct point pairs is the same for each such pair, then G is complete. G is finite if both V and W are finite. G is oriented if each line is assigned a direction in one of the two possible ways. Let G be a finite oriented graph. Let $P_1, \cdots, P_v$ be the points, and let $L_1, \cdots, L_w$ be the lines of G. Let $p_{ij} = 1$ if P_i is the initial point of the directed line L_j; let $p_{ij} = -1$ if P_i is the terminal point of the directed line L_j; and let $p_{ij} = 0$ if P_i is not an endpoint of L_j. Then $P = (p_{ij})$ defines a matrix of size v by w. This matrix is called the incidence matrix of G. The matrix PP^T is nonnegative symmetric. As before let

$$\text{trace }(H) = k^*v, \qquad JHJ = \mu J, \qquad \mu = (k^* + (v-1)\lambda^*)v.$$

Then

$$k^* = 2w/v, \qquad \lambda^* = -2w/v(v-1).$$

The matrix $B = (k^* - \lambda^*)I + \lambda^* J$ is of order v, and G is complete if and only if $PP^T = B$.

We have

THEOREM 7. *If f is a nonnegative symmetric concave function, then the incidence matrix P of an oriented graph G of v points and w lines satisfies*

$$f(PP^T) \leqq f(B).$$

If f is strictly concave, and if $(\lambda_1, \cdots, \lambda_v)$, the vector of eigenvalues of PP^T, belongs to G_f, then equality holds if and only if G is complete.

This follows as a consequence of Theorem 6. Ryser's result is obtained by taking $f = E_r^{1/r}$.

Let v elements $x_1, \cdots, x_v$ be arranged into v sets $s_1, \cdots, s_v$ such that every set contains exactly k distinct elements and such that every pair of sets has exactly λ elements in common, $0 < \lambda < k < v$. Such an arrangement is called a v, k, λ configuration. It turns out that every v, k, λ configuration satisfies $\lambda = k(k-1)/(v-1)$. For such a configuration, let $a_{ij} = 1$ if x_j is an element of s_i, and let $a_{ij} = 0$ if x_j is not an element of s_i. The v by v

matrix $A = (a_{ij})$ of 0's and 1's is the incidence matrix of the v, k, λ configuration. Define the matrix B of order v by

$$B = (k - \lambda)I + \lambda J.$$

It is clear that if $0 < \lambda < k < v$, then a v, k, λ configuration exists if and only if there exists a 0, 1 matrix A of order v such that $AA^T = B$.

Then we may generalize Ryser's results.

THEOREM 8. *Let Q be a 0, 1 matrix of order v, containing exactly kv 1's. Let $\lambda = k(k-1)/(v-1)$ and $B = (k - \lambda)I + \lambda J$, where $0 < \lambda < k < v$. If f is concave nonnegative symmetric, then*

$$f(QQ^T) \leqq f(B).$$

If f is strictly concave, and if $(\lambda_1, \cdots, \lambda_v)$, the vector of eigenvalues of QQ^T, belongs to G_f, then equality holds if and only if Q is the incidence matrix of a v, k, λ configuration.

Once again Ryser proves Theorem 8 for the choice $f = E_r^{1/r}$.

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UNIVERSITY OF CALIFORNIA, SANTA BARBARA
UNIVERSITY, CALIFORNIA
THE UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, CANADA